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Generalized Commutativity in Group Algebras

Yu. A. Bahturin and M. M. Parmenter

Abstract. We study group algebras FG which can be graded by a finite abelian group Γ such that FG is β-commutative for a skew-symmetric bicharacter β on Γ with values in F^*.

1 Introduction

Given an algebra A over a field F graded by an Abelian group Γ and a bicharacter $\beta\colon \Gamma \times \Gamma \to F^*$ (see [BFM]) we say that A is β-commutative if for any $x \in A_g$, $y \in A_h$ we always have

$$xy = \beta(g, h)yx. \tag{1}$$

Examples include commutative algebras (with $\Gamma = \{e\}$ and $\beta(e, e) = 1$) and supercommutative algebras (with $\Gamma = \mathbb{Z}$ and $\beta(i, j) = (-1)^{ij}$ for all i, j).

In a wide variety of situations (see *e.g.* [BFM]) β-commutative algebras behave very much in the same way as ordinary commutative algebras. So it is interesting to find out which associative algebras A may be given a grading by an Abelian group Γ with a bicharacter β such that A becomes β-commutative. In this paper, we are primarily interested in the case where $A = FG$ is a group algebra of a group G and the grading is finite, that is $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ and $\operatorname{Supp} A = \{\gamma \mid A_\gamma \neq 0\}$ is finite.

An almost immediate consequence of our first result (Theorem 3.1) is that over any algebraically closed field F of characteristic 0, the group algebra FG of a finite group G can be made β-commutative. This is actually quite an easy fact, following from the results of [BSZ], and is closely related to the question of determining possible gradings on matrix rings. It contrasts dramatically with the situation where $\operatorname{char} F = p > 0$, since in that case the group algebra FG of a finite p-group G can be made β-commutative only when G is Abelian (Theorem 4.2). Of course, finite groups where p does not divide $|G|$ follow the pattern of the zero characteristic case.

It is easy to find examples of infinite groups G such that FG cannot be made β-commutative for any field F (*e.g.* nonabelian ordered groups, even when the grading is not finite). More interestingly, note that when an algebra A is β-commutative and the grading is finite, it follows from [BC] and [BZ] that A must be PI (since A_e is commutative and hence PI). Passman [P] has described all groups G such that FG is *PI*—if $\operatorname{char} F = 0$, then G must have a normal subgroup H of finite index such

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that H is Abelian while if char $F = p > 0$, then G must have a normal subgroup H of finite index such that the commutator subgroup H' of H is a finite p-group. So we can obtain from this many more families of groups where FG cannot be made β-commutative (assuming the grading is finite).

When F is an algebraically closed field of characteristic 0, the question of whether FG can be made β-commutative whenever FG is PI remains open. We offer some positive results (Corollary 5.3 of Theorem 5.2) in the case where G is infinite and is a split extension PB where P is an Abelian normal subgroup and B is a finite Abelian subgroup.

The following section of the paper contains some necessary definitions and general results about β-commutativity of associative algebras.

2 General Facts About β-Commutative Algebras

We recall that a function $\beta\colon \Gamma \times \Gamma \to F^*$ defined on ordered pairs of elements of a group Γ and taking invertible values in a commutative ring F is called a *skew-symmetric bicharacter* if for any $g, h, k \in \Gamma$ one has

$$
\begin{aligned}
\beta(gh, k) &= \beta(g, k)\beta(h, k) \\
\beta(g, hk) &= \beta(g, h)\beta(g, k) \\
\beta(g, h)\beta(h, g) &= 1
\end{aligned}
\tag{2}
$$

A simple consequence of (2) is that for any $g \in \Gamma$ we have $\beta(g, g) = \pm 1$. Then $\Gamma_+ = \{g \mid \beta(g, g) = 1\}$ is a subgroup of index at most 2 in Γ. Also $\beta(e, g) = \beta(g, e) = 1$ for any $g \in G$.

An algebra A is called Γ-*graded* if $A = \bigoplus_{g\in\Gamma} A_g$ where A_g is a subspace for any $g \in \Gamma$ and $A_g A_h \subseteq A_{gh}$ for any $g, h \in \Gamma$. Given a bicharacter $\beta\colon \Gamma \times \Gamma \to F^*$ and a Γ-graded algebra A we can make A into a so-called (Γ, β)-Lie algebra if we set

$$
[a, b]_\beta = ab - \beta(g, h)ba \tag{3}
$$

whenever $a \in A_g$, $b \in A_h$. The bracket can be extended to all $a, b \in A$ by linearity.

We then have the following identities satisfied by any homogeneous $a \in A_g$, $b \in A_h$, $c \in A_k$. For ease of notation, the subscript on the bracket will be suppressed.

$$
[a, b] + \beta(g, h)[b, a] = 0 \tag{4}
$$

$$
\beta(k, g)\big[[a, b], c\big] + \beta(g, h)\big[[b, c], a\big] + \beta(h, k)\big[[c, a], b\big] = 0 \tag{5}
$$

If $[x, y]_\beta = 0$ for all $x, y \in A$, then A is called β-*commutative*. It follows from $\beta(e, g) = \beta(g, e) = 1$ that the neutral component A_e of a β-commutative algebra is central.

As an example, we consider the Grassman algebra $\mathcal{G} = \mathcal{G}(V)$ of a vector space V. Then, graded by $\mathbb{Z}_2$, $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ where $\mathcal{G}_0$ is the span of all tensors of even degree and $\mathcal{G}_1$ the span of those of odd degree. If $\beta(i, j) = (-1)^{ij}$ then it is well known that $\mathcal{G}$ is β-commutative.

When V is finite dimensional, $\mathcal{G}$ is the sum of a field F and a nilpotent ideal. The algebra $A = \left\{ \left(\begin{smallmatrix} \lambda & \mu \\ 0 & 0 \end{smallmatrix} \right) \;\middle|\; \lambda, \mu \in F \right\}$ also has this property, yet it can never be made β-commutative. To see this, assume the contrary. First consider the case where $A = A_g$ for some $g \in \Gamma$. Since $A^2 \neq 0$, this forces $g^2 = g$ and hence $g = e$, contradicting the fact that A is not commutative. Next consider the possibility that $A = A_g \oplus A_h$ for some $g, h \in \Gamma$. Here either $gh \notin \operatorname{Supp} A$ or $gh = g$ or $gh = h$. In the last two cases we have either $g = e$ or $h = e$, and since A_e is central and $\dim A_g = \dim A_h = 1$, this means that A must be commutative. In the first case, $A_g A_h = A_h A_g = 0$. If $A_g = \left\langle \left(\begin{smallmatrix} \lambda & \mu \\ 0 & 0 \end{smallmatrix} \right) \right\rangle$ and $A_h = \left\langle \left(\begin{smallmatrix} \lambda_1 & \mu_1 \\ 0 & 0 \end{smallmatrix} \right) \right\rangle$, then $0 = \left(\begin{smallmatrix} \lambda\lambda_1 & \lambda\mu_1 \\ 0 & 0 \end{smallmatrix} \right) = \left(\begin{smallmatrix} \lambda_1\lambda & \lambda_1\mu \\ 0 & 0 \end{smallmatrix} \right)$. If $\lambda = 0$ then $\mu \neq 0$ and so $\lambda_1 = 0$, and similarly $\lambda_1 = 0$ implies $\lambda = 0$. This leads to $A_g = A_h$, again giving a contradiction.

3 Matrix Algebras

We start with the main general result about β-commutativity in matrix algebras.

Theorem 3.1 *Let A be a matrix algebra $M_n(F)$ over a field F with a primitive n-th root of unity. Then A is β-commutative for an appropriate bicharacter β on a finite Abelian group Γ.*

Proof We use a construction from [BSZ]. Let Γ be the direct product of two cyclic groups of order $n : \Gamma = \langle a \rangle \times \langle b \rangle$. Suppose that ξ is a primitive n-th root of unity in F. Choose the following two matrices in $A = M_n(F)$.

$$X_a = \begin{pmatrix} \xi^{n-1} & 0 & \cdots & 0 & 0 \\ 0 & \xi^{n-2} & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & \xi & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad Y_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{6}$$

It is easy to check that

$$X_a^n = Y_b^n = I_n, \quad X_a Y_b X_a^{-1} = \xi Y_b \tag{7}$$

As a consequence, it is not hard to see that the set of all elements of the form $X_a^i Y_b^j$, $0 \leq i, j < n$, is a basis of A.

Now for $g = a^i b^j$ set $A_g = \langle X_a^i Y_b^j \rangle_F$, $1 \leq i, j < n$. Then $A = \bigoplus_{g \in \Gamma} A_g$ and, thanks to (7), we also have

$$X_a^i Y_b^j X_a^{i'} Y_b^{j'} = \xi^{-ji'} X_a^{i+i'} Y_b^{j+j'} \tag{8}$$

It is immediate that we have a grading of A by $\Gamma = \langle a \rangle \times \langle b \rangle$. Define a bicharacter $\beta \colon \Gamma \times \Gamma \to F^*$ by setting

$$\beta(a^i b^j, a^{i'} b^{j'}) = \xi^{ij' - ji'}, \quad 0 \leq i, j, i', j' < n \tag{9}$$

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Then β is a skew-symmetric bicharacter of Γ.
Also, using (8) and (9) we calculate:

$$\begin{aligned} X_a^i Y_b^j X_a^{i'} Y_b^{j'} &= \xi^{-ji'} X_a^{i+i'} Y^{j+j'} \\ &= \xi^{-ji'} \xi^{ij'} X_a^{i'} Y_b^{j'} X_a^i Y_b^j \\ &= \beta(a^i b^j, a^{i'} b^{j'}) X_a^{i'} Y_b^{j'} X_a^i Y_b^j \end{aligned}$$

So if $u \in A_g$ and $v \in A_h$, we have $uv = \beta(g, h)vu$ as required. ■

Next we show that the restriction imposed above on the order of the matrix algebra is essential. We will use the following observation which is an easy consequence of results from [BSZ].

Proposition 3.2 *Let F be a field of characteristic $p > 0$ and $A = M_n(F)$, a matrix algebra of order n over F where p divides n. Then if A is graded by an Abelian group Γ, we must have* $\dim A_e > 1$.

Proof Say to the contrary $\dim A_e = 1$ (note the identity matrix is in A_e). Then by Lemma 4 in [BSZ], the grading is "fine" (*i.e.* the dimension of each homogeneous component equals 1) and each nonzero homogeneous element must be invertible. Hence $H = \operatorname{Supp} A$ is a subgroup, and the order of H is n^2. Since p divides n, we can choose an element $g \in H$ of order p. Let $A_g = \langle X_g \rangle$, and note that since the identity matrix I belongs to A_e, $X_g^p = \lambda I$ for some scalar λ. Now if Y is any nonzero homogeneous element in A, we must have $X_g Y X_g^{-1} = \alpha Y$ for some scalar α. Since X_g^p is central, we conclude that $\alpha^p = 1$ and hence $\alpha = 1$ (since $\operatorname{char} F = p$). Thus X_g is central, contradicting the fact that the center of A is $A_e \neq A_g$. ■

The argument just given can also be applied to $M_n(Q)$ whenever $n > 1$ is odd.

Since we saw in the previous section that β-commutativity of A implies A_e is central, this proposition immediately tells us that $M_n(F)$ can never be made β-commutative when $\operatorname{char} F = p$ and p divides n. More generally we obtain

Corollary 3.3 *Let $A = M_n(F)$ where* $\operatorname{char} F = p > 0$ *and p divides n. Then $M_n(F)$ cannot be made β-nilpotent for any bicharacter β on an Abelian group Γ.*

Proof Let the contrary be true and for some grading by an Abelian group Γ the full matrix algebra A be β-nilpotent. Let us consider $R = A_e$, the identity component of the grading. Since A has no nontrivial proper ideals, it follows from [ZS] that R has no nilpotent ideals. Therefore, R is semisimple. Since $\beta(e, e) = 1$ it follows that R is Lie nilpotent. A matrix algebra of any order greater than 2 always contains a two-dimensional non-nilpotent subalgebra spanned by E_{11} and E_{12}. Thus all the simple components of R are one-dimensional. It follows from Proposition 3.2 that R cannot be one-dimensional. Therefore R has the form $R = F\varepsilon_1 \oplus \cdots \oplus F\varepsilon_t$ where $t > 1$ and $\{\varepsilon_1, \ldots, \varepsilon_t\}$ is a system of pairwise orthogonal idempotents. It follows that all nonzero elements from R are semisimple. Now let us denote $X = \varepsilon_1 \in A_e$. Since $\beta(e, g) = 1$ for any $g \in G$ we have that the linear operator $\operatorname{ad} X$ given by $\operatorname{ad} X(Y) = [X, Y]$ for any (homogeneous) $Y \in A$ is nilpotent of index at most n^2. If k is a natural number such that $p^k > n^2$ then $(\operatorname{ad} X)^{p^k} = 0$. According to [Jac] if $\operatorname{char} F = p > 0$ then $(\operatorname{ad} X)^{p^k} = \operatorname{ad} X^{p^k}$. Thus X^{p^k} is a central element in $A = M_n(F)$, hence a scalar matrix. On the other hand, $X^{p^k} = \varepsilon_1^{p_k} = \varepsilon_1$ since this is an idempotent. So we have $\varepsilon_1 = \lambda E = \lambda\varepsilon_1 + \cdots + \lambda\varepsilon_t$, which is impossible if $t > 1$. The proof is complete. ■

Another example of a matrix algebra which cannot be made β-commutative is the algebra of all upper (or lower) triangular matrices of any given order $n \geq 3$. In this case, the base field can be arbitrary.

Proposition 3.4 *Let $A = T_n$ be the algebra of all upper triangular matrices with zeros on the main diagonal of order $n \geq 3$ over any field F. Then A cannot be made β-commutative for any bicharacter β.*

Proof Assume to the contrary that A is graded by an Abelian group Γ and is β-commutative for a bicharacter $\beta\colon \Gamma \times \Gamma \to F^*$. Since A has a graded basis B, A^2 is a subspace of A and $\dim A/A^2 = n - 1$, we can find $n - 1$ elements in B, say $b_1, b_2, \ldots, b_{n-1}$, which, together ith a basis of A^2, form a basis for A.

We can write

$$
\begin{aligned}
b_1 &= \alpha_{11}E_{12} + \alpha_{12}E_{23} + \cdots + \alpha_{1,n-1}E_{n-1,n} + b_1' \\
b_2 &= \alpha_{21}E_{12} + \alpha_{22}E_{23} + \cdots + \alpha_{2,n-1}E_{n-1,n} + b_2' \\
&\vdots \\
b_{n-1} &= \alpha_{n-1,1}E_{12} + \alpha_{n-1,2}E_{23} + \cdots + \alpha_{n-1,n-1}E_{n-1,n} + b_{n-1}'
\end{aligned}
\tag{10}
$$

where $b_i' \in A^2$, $1 \leq i \leq n-1$, and all α_{ij} belong to F (here E_{ij} are the matrix units).

Let σ be any permutation of $\{1, 2, \ldots, n-1\}$. Then the element $b_{\sigma(1)}b_{\sigma(2)} \cdots b_{\sigma(n-1)}$ is equal to $\alpha_{\sigma(1),1}\alpha_{\sigma(2),2} \cdots \alpha_{\sigma(n-1),n-1}E_{1n}$. If all of these elements are zero, then the determinant of the matrix of coefficients in (10) is zero, which contradicts $b_1, b_2, \ldots, b_{n-1}$ being linearly independent mod A^2. But then it follows from β-commutativity that none of these elements can be zero, and we conclude that $\alpha_{ij} \neq 0$ for all choices of i and j.

Now suppose that the degree of b_i is g_i for each i. What we have just proved implies that b_i^{n-1} is nonzero and has degree g_i^{n-1}. Similarly, if $j \neq i$ then $b_i^{n-2}b_j$ is also nonzero and has degree $g_i^{n-2}g_j$. But both b_i^{n-1} and $b_i^{n-2}b_j$ are scalar multiples of E_{1n}. We conclude that $g_i^{n-1} = g_i^{n-2}g_j$, and so $g_i = g_j$. Thus all b_i have the same degree, which we will call g.

Recall now that $\beta(g, g) = \pm 1$. If $\operatorname{char} F \neq 2$, we know that $b_i^2 = \beta(g, g)b_i^2$ and, since $b_i^2 \neq 0$ (because $n \geq 3$), this means that $\beta(g, g) = 1$. When $\operatorname{char} F = 2$, $\beta(g, g) = 1$ is immediate. It follows that $b_ib_j = \beta(g, g)b_jb_i = b_jb_i$ for all i, j. But A is generated as an algebra by $\{b_1, b_2, \ldots, b_{n-1}\}$ (since A is nilpotent) and we conclude that A must be commutative, a contradiction. ■

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4 Group Algebras of Finite Groups

Before proceeding to the main topic of this section, assume first that A_1 (resp. A_2) is β_1 (resp. β_2)-commutative when graded by Abelian group Γ_1 (resp. Γ_2). Then $A_1 \times A_2$ can be graded by $\Gamma_1 \times \Gamma_2$ in an obvious way (since $(a_1, a_2) = (a_1, 0) + (0, a_2)$) and if $\beta\colon (\Gamma_1 \times \Gamma_2) \times (\Gamma_1 \times \Gamma_2) \to F^*$ is defined by

$$\beta\big((g_1, g_2), (g_1', g_2')\big) = \beta_1(g_1, g_1')\beta_2(g_2, g_2')$$

then β is a bicharacter and $A_1 \times A_2$ is β-commutative. This has clear implications concerning finite dimensional semisimple algebras, and when applied to group algebras leads to

Corollary 4.1 *Let G be a finite group and F an algebraically closed field of characteristic 0 or $p > 0$ with $(|G|, p) = 1$. Then FG is β-commutative for an appropriate bicharacter β over a finite Abelian group Γ.*

The situation is very different when $\operatorname{char} F = p > 0$ and G is a finite p-group. To see this, we will need to recall some properties of the Brauer-Jennings-Zassenhaus basis of $\Delta(FG)$, the augmentation ideal of FG.

First assume $F = \mathrm{GF}(p)$, the field of p elements, in which case we shall just denote the augmentation ideal by Δ. Define the dimension subgroups D_i of G by

$$D_i = \{g \mid g - 1 \in \Delta^i\}$$

for each $i \geq 1$.

Whenever $\lambda \geq 1$, $\frac{D_\lambda}{D_{\lambda+1}}$ is an elementary abelian p-group [Jen]. Say its rank is d_λ, and let $\{g_{\lambda,1}, g_{\lambda,2}, \dots, g_{\lambda,d_\lambda}\}$ be a complete set of representatives in G of a minimal basis for $\frac{D_\lambda}{D_{\lambda+1}}$. Consider the set of all products of the form

$$\Pi_{i,\lambda}(g_{\lambda,i} - 1)^{\alpha_{\lambda,i}}$$

where $0 \leq \alpha_{\lambda,i} < p$ in all cases and where the terms in each product are written in order of increasing i and λ. Define the weight of such a product to be $\omega = \sum_{i,\lambda} \lambda\alpha_{\lambda,i}$.

Jennings [Jen] proved that for each $j \geq 1$, the set of all such products whose weight is $\geq j$ forms a basis for Δ^j. Note that this allows us to write down a basis for $\frac{\Delta^j}{\Delta^{j+1}}$ whenever $j \geq 1$.

Although Jennings' paper was concerned solely with the field $\mathrm{GF}(p)$, the products just described form a basis for $\Delta^j(FG)$ where F is any field of characteristic p. To see this, note that $\Delta^j(FG)$ is spanned by all products $(g_1 - 1)(g_2 - 1)\cdots(g_k - 1)$ where $k \geq j$ and $g_i \in G$ for each i. Since these elements are in Δ^j, we see that the Brauer-Jennings-Zassenhaus basis for Δ^j must span $\Delta^j(FG)$ as well. In particular this holds when $j = 1$, and since $\dim_F \Delta(FG) = \dim_{\mathrm{GF}(p)} \Delta = |G|$, elements of the Brauer-Jennings-Zassenhaus basis must still be linearly independent in FG. It follows that they form a basis for $\Delta(FG)$.

We will be primarily interested in the cases $j = 1$ and $j = 2$. When $j = 1$, we have that $\{(g_{1,1} - 1) + \Delta^2, \dots, (g_{1,d_1} - 1) + \Delta^2\}$ is a basis for Δ/Δ^2 where $\{\bar{g}_{1,1}, \dots, \bar{g}_{1,d_1}\}$

is a basis for G/D_2. When $j = 2$, $\{(g_{1,i}-1)(g_{1,j}-1)+\Delta^3 \mid i \le j\} \cup \{(g_{2,i}-1)+\Delta^3\}$ is a basis for Δ^2/Δ^3, except in the case $p = 2$ where "$i < j$" is needed in the first set.

Theorem 4.2 *If F is a field of characteristic p and G is a finite p-group, then FG is β-commutative for some bicharacter β if and only if G is Abelian.*

Proof For ease of notation, let Δ denote the augmentation ideal $\Delta(FG)$ in this argument.

Assume to the contrary that G is not Abelian and FG has been graded in such a way that it is β-commutative. Since FG has a graded basis, it follows that we can find noncommuting homogeneous elements x, y in FG such that

$$FG = \langle 1\rangle_F \oplus \Delta^2 \oplus \langle x\rangle_F \oplus \langle y\rangle_F \oplus \cdots$$

where the subspaces not shown are 1-dimensional and generated by homogeneous elements. Note that the nilpotence of Δ and the argument given at the end of Proposition 3.4 are used here.

We know that $xy = fyx$ for some $f \neq 0, 1$ in F. Using augmentation, it follows that $xy \in \Delta$ and hence either x or y must lie in Δ. We may assume $y \in \Delta$.

Using the Brauer-Jennings-Zassenhaus basis for Δ/Δ^2, and abusing notation by writing g_i instead of $g_{1,i}$ for each $1 \le i \le d_1$, we can write

$$\begin{aligned} x &= \gamma \cdot 1 + \beta_{11}(g_1-1) + \beta_{12}(g_2-1) + \cdots + \beta_{1d_1}(g_{d_1}-1) + T_1 \\ y &= \beta_{21}(g_1-1) + \beta_{22}(g_2-1) + \cdots + \beta_{2d_1}(g_{d_1}-1) + T_2 \end{aligned}$$

where $\gamma \in F$, $\beta_{ij} \in F$ for all i and j, and $T_i \in \Delta^2$ for each i. Note that since x and y form part of a basis of FG, along with 1 and a basis of Δ^2, we know that $\{\beta_{1j}\}$ and $\{\beta_{2j}\}$ each contains at least one nonzero entry.

Since $xy = fyx$, we have

$$\begin{aligned} &\gamma \sum_j \beta_{2j}(g_j-1) + \gamma T_2 + \sum_{i,j} \beta_{1i}\beta_{2j}(g_i-1)(g_j-1) \\ &\qquad = f\gamma \sum_j \beta_{2j}(g_j-1) + f\gamma T_2 + f \sum_{i,j} \beta_{1i}\beta_{2j}(g_j-1)(g_i-1) + T^1 \end{aligned}$$

where $T^1 \in \Delta^3$.

It follows that

$$\begin{aligned} &\sum_j \gamma(1-f)\beta_{2j}(g_j-1) + \gamma(1-f)T_2 \\ &\qquad + \sum_{i,j} \beta_{1i}\beta_{2j}\big((g_i-1)(g_j-1) - f(g_j-1)(g_i-1)\big) \in \Delta^3. \end{aligned}$$

Since $\{(g_j-1)+\Delta^2\}$ is a basis for Δ/Δ^2 and $\beta_{2j} \neq 0$ for some j, we conclude that $\gamma = 0$. In other words, both x and y are in Δ.

We now have

$$\sum_i \beta_{1i}\beta_{2i}(1-f)(g_i-1)^2+\sum_{i<j}\beta_{1i}\beta_{2j}\big((g_i-1)(g_j-1)-f(g_j-1)(g_i-1)\big)$$
$$+\sum_{i>j}\beta_{1i}\beta_{2j}\big((g_i-1)(g_j-1)-f(g_j-1)(g_i-1)\big)\in\Delta^3.$$

Hence,

$$\sum_i \beta_{1i}\beta_{2i}(1-f)(g_i-1)^2+\sum_{i<j}\beta_{1i}\beta_{2j}(1-f)(g_i-1)(g_j-1)$$
$$+\sum_{i<j}\beta_{1i}\beta_{2j}f(g_ig_j-g_jg_i)+\sum_{i>j}\beta_{1i}\beta_{2j}(1-f)(g_j-1)(g_i-1)$$
$$+\sum_{i>j}\beta_{1i}\beta_{2j}(g_ig_j-g_jg_i)\in\Delta^3.$$

Since $g_ig_j-g_jg_i=([g_i,g_j]-1)g_jg_i$ and $G_2\subseteq D_2$, the above can be rewritten (using the notation introduced earlier) as

$$\sum_i \beta_{1i}\beta_{2i}(1-f)(g_i-1)^2$$
$$+\sum_{i<j}(\beta_{1i}\beta_{2j}+\beta_{1j}\beta_{2i})(1-f)(g_i-1)(g_j-1)+\sum\varepsilon_i(g_{2,i}-1)\in\Delta^3$$

for suitable $\varepsilon_i\in F$, $1\le i\le d_2$.

When $p>2$, the above expression is just a linear combination of basis elements of Δ^2 modulo Δ^3. When $p=2$, the first term is a linear combination of $(g_i-1)^2=g_i^2-1$ and, since $g_i^2\in D_2$ in this case, the first term can be rewritten as $\sum\zeta_i(g_{2,i}-1)\bmod\Delta^3$ for suitable $\zeta_i\in F$. Again, this gives us a linear combination of basis elements of Δ^2 modulo Δ^3.

In any case, we can conclude that $\beta_{1i}\beta_{2j}+\beta_{1j}\beta_{2i}=0$ for all $i\neq j$.

If $\beta_{1i}=0$ and $\beta_{2i}\neq0$, this says that $\beta_{1j}=0$ for all j which is not the case. Applying a similar argument to the case where $\beta_{2i}=0$, we conclude that $\beta_{1i}=0\Leftrightarrow\beta_{2i}=0$.

Consider now the $p>2$ case. We know that $\beta_{1i}\neq0$ for some i, and we have just shown that $\beta_{2i}\neq0$ as well. But then $(g_i-1)^2$ has a nonzero coefficient in the above linear combination, which can't occur.

Now assume $p=2$. Again we know that $\beta_{1i}\neq0$ (and hence $\beta_{2i}\neq0$) for some i. If $\beta_{1j}\neq0$ for some $j\neq i$, then our equation says that $\beta_{2i}\beta_{1i}^{-1}=-\beta_{2j}\beta_{1j}^{-1}=\beta_{2j}\beta_{1j}^{-1}$. It follows that $y-(\beta_{2i}\beta_{1i}^{-1})x\in\Delta^2$, contradicting the fact that x and y are linearly independent modulo Δ^2. ∎

In contrast with the case of matrix algebras where it was shown that when char F divides n, $M_n(F)$ cannot be made β-nilpotent for any bicharacter β (see Corollary 3.3), it is easily seen that the group algebra FG of a finite p-group is β-nilpotent even when β is the trivial bicharacter.

The general situation when G is a finite group which is not a p-group but whose order is divisible by the characteristic of F is unclear, as the following two examples indicate.

Example 4.3 Let $F = \mathrm{GF}(3)$ and $G = S_3$, the symmetric group on 3 letters. Then $L = FG$ can be made β-commutative.

To see this, let $G = \langle x, y \mid x^3 = 1 = y^2, yx = x^2 y\rangle$ and let $H = \langle a, b \mid a^2 = 1 = b^2, ba = ab\rangle$, the Klein-4 group. We can grade FG by H if we let the homogeneous components be $L_1 = \langle 1, x + x^2\rangle_F$, $L_a = \langle x - x^2\rangle_F$, $L_b = \langle y, xy + x^2 y\rangle_F$, $L_{ab} = \langle xy - x^2 y\rangle_F$. Defining β by the rules $\beta(g, 1) = \beta(1, g) = \beta(g, g) = 1$ for all g, $\beta(g, h) = -1$ in all other cases, it is easy to check that FG is β-commutative.

Example 4.4 Again let $G = S_3$ and now let F be any field of characteristic 2. Then $L = FG$ cannot be made β-commutative.

To see why this is the case, assume the contrary (with FG graded by Γ). We will need to use the fact that the Jacobson radical $J(FG)$ of FG is just the ideal $(\sum_{g\in G} g) \cdot FG = (\sum_{g\in G} g) \cdot F$, and is therefore contained in the center of FG. It follows that if x is noncentral and homogeneous, then x cannot be nilpotent (if it were, then x would generate a nilpotent ideal in FG).

Since FG is not commutative, we can choose homogeneous elements which don't commute with each other. In other words, there exist $a, b \in \Gamma$ such that $\beta(a, b) \neq 1$ and such that we can find $x \in L_a$ and $y \in L_b$ with $xy \neq yx$.

Since x is not central, we know that $x^2 \neq 0$ by the earlier remark. Note that $x^2 \in L_{a^2}$, and that $L_{a^2} \neq L_a$ ($a \neq 1$ because L_1 is central) and $L_{a^2} \neq L_b$ ($\beta(a, a^2) = 1$ while $\beta(a, b) \neq 1$). Similarly we see that $y^2 \neq 0 \in L_{b^2}$, and that $L_{b^2} \neq L_b$ and $L_{b^2} \neq L_a$. In addition, $L_{b^2} \neq L_{a^2}$ ($\beta(a, b) \neq 1 \Rightarrow \beta(a, b^2) = \big(\beta(a, b)\big)^2 \neq 1$ because $\operatorname{char} F = 2$. Hence $\beta(a^2, b^2) \neq 1$). We also know from these arguments that $L_{a^2} \neq L_1$ and $L_{b^2} \neq L_1$.

Since x and y are homogeneous and $xy \neq yx$, we know that $xy \neq 0$. We also have $xy \in L_{ab}$, and clearly $L_{ab} \neq L_a$ and $L_{ab} \neq L_b$. In addition $L_{ab} \neq L_{a^2}$ and $L_{ab} \neq L_{b^2}$. Finally we note that $L_{ab} \neq L_1$ (otherwise $b = a^{-1}$, but $1 = \beta(a, 1) = \big(\beta(a, a)\big)\big(\beta(a, a^{-1})\big) = \beta(a, a^{-1})$ since $\operatorname{char} F = 2$).

So far we have shown that the six homogeneous components $L_1, L_a, L_b, L_{a^2}, L_{b^2}, L_{ab}$ are all nonzero and distinct. Since $\dim_F FS_3 = 6$, all other homogeneous components must be 0 and each of the six components listed must be of dimension 1 over F. In particular, $L_1 = F$.

Next consider $0 \neq x^3 \in L_{a^3}$. It follows from above that L_{a^3} must be one of the components just listed. Clearly, $L_{a^3} \neq L_{a^2}$ and $L_{a^3} \neq L_a$. Repeating earlier arguments, we see that $L_{a^3} \neq L_b\big(\beta(a, a^3) = 1\big)$, $L_{a^3} \neq L_{b^2}\big(\beta(a, b^2) \neq 1\big)$ and $L_{a^3} \neq L_{ab}$. So the only possibility is that $L_{a^3} = L_1$, in which case we must have $x^3 = f_1 \neq 0$ where $f_1 \in F$. An identical argument shows that $y^3 = f_2 \neq 0$ where $f_2 \in F$.

Now we have $(xy)^3 = [\beta(y, x)]^3 x^3 y^3 = [\beta(y, x)]^3 f_1 f_2$. In particular, this tells us that $(xy)^3 \neq 0$ and so $(xy)^2 \neq 0$. Now $(xy)^2 \in L_{a^2b^2}$, so $L_{a^2b^2}$ must be one of the six homogeneous components listed earlier. Clearly $L_{a^2b^2} \neq L_{a^2}$, $L_{a^2b^2} \neq L_{b^2}$ and $L_{a^2b^2} \neq L_{ab}$. In addition, $L_{a^2b^2} \neq L_a$ and $L_{a^2b^2} \neq L_b$ by arguments seen before. Finally

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note that $L_{a^2b^2} \neq L_1$ ($\beta(a^2, a^{-2}) = 1$ and $\beta(a^2, b^2) \neq 1$).

We have our contradiction. ■

5 Group Algebras of Infinite Groups

As mentioned in the introduction, it is not difficult to find examples of infinite groups G such that FG cannot be made β-commutative for any field F. In addition, if we restrict our attention to finite gradings then β-commutativity implies PI and this observation imposes strong restrictions on $G[P]$.

On the other hand, the techniques illustrated in Example 4.3 can be extended to many infinite groups.

Example 5.1 Assume F contains a primitive p-th root of unity ζ where p is prime and G contains an Abelian normal subgroup H of index p. Then $L = FG$ can be made β-commutative.

To see this, let $G = \langle H, g\rangle$ and $\Gamma = \langle x\rangle \times \langle y\rangle$ where $x^p = y^p = 1$. For $0 \leq i, j < p$, define the homogeneous component

$$L_{x^i y^j} = \langle \{a + \zeta^i a^g + \zeta^{2i} a^{g^2} + \cdots \zeta^{(p-1)i} a^{g^{p-1}} \mid a \in A\}\rangle_F \cdot g^j.$$

For example, $L_1 = \langle \{a + a^g + \cdots + a^{g^{p-1}} \mid a \in A\}\rangle_F$, the subspace generated by all conjugacy class sums of elements from A. Since $L_{x^i y^j} = (L_{x^i}) \cdot g^j$, $(L_{x^i})^g = L_{x^i}$ and $L_{x^i} L_{x^j} \subseteq L_{x^{i+j}}$ we see that this is a grading of FG. Moreover, FG is β-commutative if we define $\beta\colon \Gamma \times \Gamma \to F^*$ by $\beta(x^i y^j, x^{i'} y^{j'}) = \zeta^{i'j - ij'}$. ■

It is still open as to whether FG can be made β-commutative whenever FG is PI when F is algebraically closed of characteristic zero. The next theorem is stated in more general terms, but its corollary represents progress on this question.

Theorem 5.2 *Let A be a G-graded α-commutative algebra over a field F containing n different roots of 1 of degree n, α a bicharacter of G, B an Abelian group acting on A by G-graded semisimple automorphisms. Then the smash product $C = A\#B$ is β-commutative for an appropriate bicharacter β on the group $\tilde{G} = G \times \hat{B} \times B$, where $\hat{B}$ is the dual group of B, that is, the group of all homomorphisms from B into F.*

Proof We have

$$A = \bigoplus_{g \in G} A_g, \tag{11}$$

$$A_g = \bigoplus_{\chi \in \hat{B}} (A_g)_\chi, \tag{12}$$

where $V_\chi = \{v \in V \mid b * v = \chi(b)v \,\forall b \in B\}$, V being any vector space with an action of B. Because the action is semisimple the decomposition (12) holds. Now for any $(g, \chi, b) \in \tilde{G}$ we set

$$C_{(g,\chi,b)} = (A_g)_\chi \# b. \tag{13}$$

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We have to check that

$$C = \bigoplus_{(g,\chi,b)\in\tilde{G}} C_{(g,\chi,b)} \tag{14}$$

is a grading of C and that C is β-commutative where β is a bicharacter given by the formula

$$\beta\big((g,\chi,b),(h,\psi,d)\big) = \chi(d)^{-1}\psi(b)\alpha(g,h). \tag{15}$$

Let us recall that the multiplication in the smash product $C = A\#B$ is given by this formula:

$$(a\#b)(a'\#d) = \big(a(b * a')\big)\#(bd). \tag{16}$$

If we use (16) with $a \in (A_g)_\chi$, $a' \in (A_h)_\psi$ then we obtain that $b * a' = \psi(b)a'$ and so

$$(a\#b)(a'\#d) = \psi(b)(aa')\#(bd). \tag{17}$$

Now $aa' \in A_g A_h \subset A_{gh}$. Also for any $x \in B$ we have

$$x * (aa') = (x * a)(x * a') = \chi(x)a\psi(x)a' = (\chi\psi)(x)(aa'),$$

proving that the right hand side of (17) is an element of $C_{(gh,\chi\psi,bd)}$.

It remains to prove β-commutativity of the algebra obtained. Again let us look into (17) and compute $(a'\#d)(a\#b)$. The result will be

$$(a'\#d)(a\#b) = \chi(d)(a'a)\#(db) = \alpha(h,g)\chi(d)(aa')\#(bd).$$

Now we use (15) and (17) to obtain the following:

$$\begin{aligned}\beta\big((g,\chi,b),(h,\psi,d)\big)\,(a'\#d)(a\#b) &= \chi(d)^{-1}\psi(b)\alpha(g,h)\alpha(h,g)\chi(d)(aa')\#(bd)\\ &= \psi(b)(aa')\#(bd)\\ &= (a\#b)(a'\#d),\end{aligned} \qquad \blacksquare$$

as required.

Corollary 5.3 *Let a group S have the form of a split extension $S = PB$ where P is an Abelian normal subgroup and B is a finite Abelian subgroup of order n. If F is a field containing n roots of 1 of degree n then the group algebra FS can be made β-commutative for an appropriate bicharacter β on an Abelian group.*

Proof We can apply the previous theorem if we set $A = FP$, the group algebra of P. The group G is then trivial. The action of B on A is defined by the action of B on P by conjugation. The condition on the field ensures that A is a semisimple B-module. It is well known also that $FS = FP\#FB$. $\blacksquare$

To close this section, we will show very quickly why integral group rings are not of interest when studying β-commutativity.

Proposition 5.4 *If $\mathbb{Z}G$ is β-commutative, then G must be Abelian.*

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Proof Say $\mathbb{Z}G$ is β-commutative. Note that β can only take the values ± 1 in $\mathbb{Z}$.

Let $g, h \in G$. By writing g and h as sums of homogeneous terms, and using the fact that homogeneous elements must either commute or anti-commute, we conclude that $gh - hg = 2x$ for some $x \in \mathbb{Z}G$. But this can only happen if $gh - hg = 0$, *i.e.* G must be Abelian. ■

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Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, Newfoundland
A1C 5S7