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FOURTH-ORDER BOUNDARY VALUE PROBLEMS AT NONRESONANCE

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We establish under nonuniform nonresonance conditions an existence and uniqueness theorem for a linear, and the solvability for a nonlinear, fourth-order boundary value problem which occurs frequently in plate deflection theory.

1 INTRODUCTION

The linear fourth-order boundary value problem

$$(1)\qquad \begin{gathered} d^4y/dx^4 - f(x)y = g(x), \qquad 0 < x < 1, \\ y(0) = y_0, \quad y(1) = y_1, \quad y''(0) = \tilde{y}_0, \quad y''(1) = \tilde{y}_1 \end{gathered}$$

and its nonlinear version

$$(2)\qquad \begin{gathered} d^4y/dx^4 - F(x,y,y',y'',y''')y = G(x,y,y',y'',y'''), \qquad 0 < x < 1, \\ y(0) = y_0, \quad y(1) = y_1, \quad y''(0) = \tilde{y}_0, \quad y''(1) = \tilde{y}_1 \end{gathered}$$

occur frequently in plate deflection theory. Usmani [**4**] states an existence and uniqueness theorem for problem (1) under the condition $f(x) < \pi^4$ and in a recent communication [**5**] we observe that the existence and uniqueness theorem for problem (1) holds under the general condition $f(x) \neq k^4\pi^4$ for $k = 1, 2, \ldots$. This last condition restricts the problem to the so-called uniform nonresonance case. In Section 2 we establish an existence and uniqueness theorem for problem (1) under a nonuniform nonresonance condition which allows some "partial" resonance, that is, the occurence of $f(x) = k^4\pi^4$ on a subset of $[0,1]$. In Section 3 we apply the theorem obtained in Section 2 to establish a solvability theorem for the nonlinear problem (2), also under a nonuniform nonresonance condition which improves some known results (for example, Aftabizadeh [**1**]). Our argument below is a combination of the Fredholm alternative theorem and a modification of the method developed by Nkashama and Willem [**3**]. Throughout this paper all functions are assumed to be real and continuous.

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2 The linear problem

Let $\mathrm{Im}(F)$ denote the image of a function $F\colon [0,1] \to \mathbf{R}$ and $\mathrm{Int}(A)$ the interior of the set $A \subseteq \mathbf{R}$.

THEOREM 1. *Suppose that $f(x)$ satisfies*

$$f^{-1}(k^4\pi^4) \neq [0,1], \qquad k = 1,2,\ldots, \text{and} \tag{3a}$$

$$\{k^4\pi^4 : k = 1,2,\ldots\} \cap Int(Im(f)) = \emptyset. \tag{4a}$$

Then Problem (1) has a unique solution.

Remarks. (1) The uniform nonresonance condittion which can be stated as $\{k^4\pi^4 : k = 1,2,\ldots\} \cap \mathrm{Im}(f) = 0$ satisfies conditions (3a) and (4a).

(2) Condition (3a) is in fact necessary for the uniqueness and existence of a solution to problem (1), and this condition can be restated as

$$f(x) \not\equiv k^4\pi^4, \qquad k = 1,2,\ldots \tag{3b}$$

(3) Since, by the continuity of f, $\mathrm{Im}(f)$ is a closed interval, therefore condition (4a) is equivalent to the statement that either

$$f(x) \leqslant \pi^4 \text{ for all } x \in [0,1] \tag{4b}$$

or there is some integer $k \geqslant 1$ such that

$$k^4\pi^4 \leqslant f(x) \leqslant (k+1)^4\pi^4 \text{ for all } x \in [0,1]. \tag{4c}$$

(4a) is thus a condition which allows some "partial" resonance.

PROOF OF THEOREM 1: Let $G(x,s)$ be the Green function of the problem

$$u''(x) = h(x), \qquad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

Then we can convert Problem (1) into an integral equation over the space $C[0,1]$:

$$y - Ty = z \tag{5}$$

where

$$(Ty)(x) = \int_0^1 \int_0^1 G(x,s)G(s,t)f(t)y(t)\,dtds, \text{ and}$$
$$z(x) = y_0 + x(y_1 - y_0) + \int_0^1 G(x,s)[\tilde{y}_0 + s(\tilde{y}_1 - \tilde{y}_0) + \int_0^1 G(s,t)g(t)\,dt]\,ds.$$

Now it suffices to show that for any $z \in C[0,1]$, equation (5) is uniquely solvable in the space $C[0,1]$. Since $T : C[0,1] \to C[0,1]$ is a linear compact operator, by the well-known Fredholm alternatives (see, for example, Gilbarg and Trudinger [**2**, p71]) we see that it will be enough to prove that the only solution of equation

$$y - Ty = 0 \tag{6}$$

is the trivial solution y=0. We proceed as follows.

Convert equation (6) back into the boundary value problem

$$\begin{aligned} &d^4y/dx^4 - f(x)y = 0, \qquad 0 < x < 1, \\ &y(0) = y(1) = y''(0) = y''(1) = 0. \end{aligned} \tag{7}$$

Since $\{\sqrt{2}\sin j\pi x : j = 1, 2, \dots\}$ is a complete orthonormal basis of $L^2[0,1]$, we have, in $L^2[0,1]$,

$$y = \sqrt{2}\sum_{j=1}^{\infty} a_j \sin j\pi x,$$

$$d^4y/dx^4 = \sqrt{2}\sum_{j=1}^{\infty} a_j j^4\pi^4 \sin j\pi x.$$

Denote by $(.,.)$ the standard inner product defined on $L^2[0,1]$. Then the selfadjointness of the operator d^4/dx^4 gives the relation

$$\begin{aligned} 0 &= \left(d^4y/dx^4 - fy, y_2 - y_1\right) \\ &= \left(d^4y_2/dx^4 - fy_2, y_2\right) - \left(d^4y_1/dx^4 - fy_1, y_1\right) \end{aligned} \tag{8}$$

where the decomposition $y = y_1 + y_2$ is made such that

$$y_1 = 0, \quad y_2 = y, \qquad \text{if } f \text{ satisfies (4b);}$$

$$y_1 = \sqrt{2}\sum_{j=1}^{k} a_j \sin j\pi x \qquad \text{and}$$

$$y_2 = \sqrt{2}\sum_{j=k+1}^{\infty} a_j \sin j\pi x \qquad \text{if } f \text{ satisfies (4c).}$$

If f satisfies (4b), we have from (8) and the Parseval equality that

$$\begin{aligned} 0 &= \left(d^4y/dx^4, y\right) - (fy, y) \\ &\geqslant \sum_{j=1}^{\infty} a_j^2\left(j^4\pi^4 - \pi^4\right). \end{aligned}$$

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Therefore $a_j = 0$, $j = 2, 3, \ldots$. Hence, $y = \sqrt{2} a_1 \sin \pi x$. Inserting this into (7), we get

$$a_1 (\pi^4 - f(x)) \sin \pi x = 0, \qquad \text{for all } x \in [0, 1].$$

Now using (3b) we conclude that $a_1 = 0$, so $y = 0$. If f satisfies (4c), we have

$$\begin{aligned}(d^4 y_2/dx^4 - f y_2, y_2) &\geqslant (d^4 y_2/dx^4, y_2) - (k+1)^4 \pi^4 (y_2, y_2) \\ &= \sum_{j=k+1}^{\infty} a_j^2 \left(j^4 \pi^4 - (k+1)^4 \pi^4\right) \geqslant 0\end{aligned}$$

and

$$\begin{aligned}(d^4 y_1/dx^4 - f y_1, y_1) &\leqslant (d^4 y_1/dx^4, y_1) - k^4 \pi^4 (y_1, y_1) \\ &= \sum_{j=1}^{k} a_j^2 (j^4 \pi^4 - k^4 \pi^4) \leqslant 0.\end{aligned}$$

Substituting the above two inequalities into (8) we obtain

(9a) $$(d^4 y_2/dx^4 - f y_2, y_2) = 0,$$

(9b) $$\sum_{j=k+1}^{\infty} a_j^2 (j^4 \pi^4 - k^4 \pi^4) = 0,$$

(10a) $$(d^4 y_1/dx^4 - f y_1, y_1) = 0$$

(10b) $$\sum_{j=1}^{k} a_j^2 (j^4 \pi^4 - k^4 \pi^4) = 0.$$

Hence $a_j = 0$, $j \neq k, k+1$. Consequently $y_1 = \sqrt{2} a_k \sin k\pi x$ and $y_2 = \sqrt{2} a_{k+1} \sin (k+1)\pi x$. Inserting these expressions into (10a) and (9a) respectively and observing condition (3) we obtain $a_k = a_{k+1} = 0$, so again $y = 0$.

The proof of the theorem is now complete. ∎

3 The nonlinear problem

In this section we study the nonlinear problem (2). We use X to denote an arbitary point in $\mathbf{R}^4$. First we formulate (3)–(4) type conditions for the function $F(x, X)$.

(H) Suppose that F is a bounded function on $[0, 1] \times \mathbf{R}^4$ and define $a(x), b(x) \in L^\infty[0, 1]$ by

$$a(x) = \inf_X F(x, X), \quad b(x) = \sup_X F(x, X),$$

where the measurability of a and b is assumed. Assume further that either $b(x) \leqslant \pi^4$ a.e. or there is an integer k such that $k^4 \pi^4 \leqslant a(x) \leqslant b(x) \leqslant (k+1)^4 \pi^4$ a.e. and moreover, neither $a^{-1}(k^4 \pi^4)$ nor $b^{-1}(k^4 \pi^4)$ is a measure of 1.

THEOREM 2. *If $G(x, X)$ is a bounded function and function $F(x, X)$ satisfies hypothesis (H), then Problem (2) has at least one solution.*

PROOF: The proof uses Theorem 1 and the Schauder fixed point theorem. Define a map $T : C^3[0,1] \to C^3[0,1]$ by $u = Tw$, where u, w are related by

$$\text{(11)} \qquad \begin{aligned} &d^4u/dx^4 - F(x, w, w', w'', w''')u = G(x, w, w', w'', w'''), \qquad 0 < x < 1, \\ &u(0) = y_0, \quad u(1) = y_1, \quad u''(0) = \tilde{y}_0, \quad u''(1) = \tilde{y}_1. \end{aligned}$$

We see easily that $f(x) = F(x, w(x), w'(x), w''(x), w'''(x))$ satisfies conditions (3) and (4) so the map T is well-defined. First we show that the image of T, $\mathrm{Im}(T)$ say, is a bounded subset of $C^3[0,1]$.

Otherwise if $\mathrm{Im}(T)$ is not bounded, then there is a sequence $\{w_n\}$ in $C^3[0,1]$ such that $u_n = Tw_n$ satisfies

$$\text{(12)} \qquad |u_n|_{C^3[0,1]} \to \infty \text{ as } n \to \infty:$$

To simplify notation, in the following we denote by $|\bullet|_i$ the standard norm of the space $C^i[0,1]$. We shall see below that $\{|u_n|_0\}$ being bounded is equivalent to $\{|u_n|_4\}$ being bounded, thus we can assume from (12) that

$$\text{(13)} \qquad a_n = |u_n|_0 \to \infty \text{ as } n \to \infty.$$

In Problem (11), put $v_n = u_n/a_n$ and $f_n(x) = F(x, w_n(x), w_n'(x), w_n''(x), w_n'''(x))$. Since $\{f_n\}$ is a bounded sequence in $L^2[0,1]$, we may assume that $\{f_n\}$ is weakly convergent to some $f_0 \in L^2[0,1]$. Then from

$$\int_0^1 a(x)h(x)\, dx \leqslant \int_0^1 f_n(x)h(x)\, dx \leqslant \int_0^1 b(x)h(x)\, dx$$

for all $h \in L^\infty[0,1]$ with $h(x) \geqslant 0$ a.e., we see that

$$\text{(14)} \qquad a(x) \leqslant f_0(x) \leqslant b(x), \text{ a.e. on } [0,1].$$

On the other hand, since $\{v_n\}$ satisfies

$$\text{(15)} \qquad \begin{aligned} &d^4v_n/dx^4 - f_n(x)v_n = G(x, w_n, w_n', w_n'', w_n''')/a_n, \quad 0 < x < 1, \\ &v_n(0) = y_0/a_n, \quad v_n(1) = y_1/a_n, \quad v_n''(0) = \tilde{y}_0/a_n, \quad v_n''(1) = \tilde{y}_1/a_n, \end{aligned}$$

therefore $\{d^4v_n/dx^4\}$ is a bounded sequence in $C[0,1]$. Define

$$V_n = v_n'' - [v_n''(0) + x(v_n''(1) - v_n''(0))].$$

Then $\{V_n''\}$ is a bounded sequence in $C[0,1]$ and $V_n(0) = V_n(1) = 0$. The mean value theorem says that there is a point $\tilde{x}_n \in [0,1]$ such that $V_n'(\tilde{x}_n) = 0$. Hence the formula

$$V_n'(x) = \int_{\tilde{x}_n}^{x} V_n''(s)\,ds$$

implies that $\{V_n'\}$ is a bounded sequence in $C[0,1]$. Moreover,

$$V_n(x) = \int_0^x V_n'(s)\,ds$$

gives the boundedness of $\{V_n\}$ in $C[0,1]$. This shows that $\{V_n\}$, and hence $\{v_n''\}$, is a bounded sequence in $C^2[0,1]$. A similar argument proves that $\{v_n\}$ is a bounded sequence in $C^2[0,1]$. Consequently $\{v_n\}$ is a bounded sequence in $C^4[0,1]$. Now by the compact embedding $C^4[0,1] \to C^3[0,1]$ we can assume for convenience that $v_n \to v_0$ in $C^3[0,1]$ for some $v_0 \in C^3[0,1]$. Finally, letting $n \to \infty$ in (15) we easily conclude that $v_0'''(x)$ is absolutely continuous and v_0 satisfies

$$d^4v_0/dx^4 - f_0(x)v_0 = 0, \text{ a.e. on } [0,1],$$
$$v_0(0) = v_0(1) = v_0''(0) = v_0''(1) = 0.$$

We readily verify as was done in Section 2 that $v_0 = 0$. This contradicts the fact that $|v_0|_0 = 1$ since $v_n \to v_0$ in $C^3[0,1]$ and $|v_n|_0 = 1$, $n = 1, 2, \ldots$.

To use the Schauder fixed point theorem, it remains to show that T is completely continuous.

The compactness of T follows from the compactness of $\text{cl}(\text{Im}(T))$ where we use notation $\text{cl}(A)$ to denote the closure of a set A in an appropriate space. Let $u_n \in \text{Im}(T)$, then $\{u_n\}$ is a bounded sequence in $C^3[0,1]$. Assume $u_n = Tw_n$, $f_n = F(x, w_n, w_n', w_n'', w_n''')$ and $g_n = G(x, w_n, w_n', w_n'', w_n''')$. Then in Problem (11) the boundedness of $\{f_n\}$ and $\{g_n\}$ in $C[0,1]$ implies the boundedness of $\{d^4u_n/dx^4\}$ in $C[0,1]$. Therefore $\{u_n\}$ is a bounded sequence in $C^4[0,1]$. Again using the compact embedding $C^4[0,1] \to C^3[0,1]$ we conclude that there is a subsequence of $\{u_n\}$ which converges in $C^3[0,1]$. Hence $\text{cl}(\text{Im}(T))$ is compact.

Continuity follows from the fact that $w_n \to w_0$ in $C^3[0,1]$ implies that $u_n = Tw_n \to u_0 = Tw_0$ in $C^3[0,1]$. We shall argue by contradiction.

Suppose that $|u_n - u_0|_3 \not\to 0$. Then by going to a subsequence if necessary, we may assume that $|u_n - u_0|_3 \geqslant c > 0$, $n = 1, 2, \ldots$, for some constant c. The compactness of T says that there is a subsequence, which we still denote by $\{u_n\}$ for convenience, such that $u_n \to v_0$ in $C^3[0,1]$. Noting that $w_n \to w_0$ in $C^3[0,1]$ in the following

$$d^4u_n/dx^4 - F(x, w_n, w_n', w_n'', w_n''')u_n = G(x, w_n, w_n', w_n'', w_n'''), \qquad 0 < x < 1,$$
$$u_n(0) = y_0, \quad u_n(1) = y_1, \quad u_n''(0) = \tilde{y}_0, \quad u_n''(1) = \tilde{y}_1,$$

we see immediately that $v_0 \in C^4[0,1]$ and $v_0 = Tw_0$. But Theorem 1 says that $v_0 = u_0$, thus giving a contradiction.

Now we know that $T : C^3[0,1] \to C^3[0,1]$ is completely continuous and $\mathrm{Im}(T)$ is bounded. Let $M > 0$ be large so that

$$\mathrm{Im}(T) \subset B = \{u \in C^3[0,1] \colon |u|_3 \leqslant M\}.$$

Then T sends B into B, so T has at least one fixed point $y \in B$ by the Schauder fixed point theorem. This y is a solution to Problem 2. ∎

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