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EXTRA COUNTABLY COMPACT SPACES

BY

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ABSTRACT A completely regular Hausdorff space is *extra countably compact* if every infinite subset of βX has an accumulation point in X. It is a theorem of Comfort and Waiveris that if X either an F-space or realcompact (topologically complete), then there is a set $\{P_\xi : \xi < 2^c\}$ of extra countably compact (countably compact) subspaces of βX such that $P_\xi \cap P_{\xi'} = X$, for $\xi < \xi' < 2^c$. Comfort and Waiveris conjecture that in all three cases, the spaces may be chosen pairwise non-homeomorphic. We prove this conjecture, using $\mathscr{D}$-limits and weak P-points. We also give a partial solution to another question asked by Comfort and Waiveris.

0. **Introduction.** The main purpose of this paper is to answer some questions of W. W. Comfort and C. Waiveris in [CW], which Comfort asked at the Moscow Conference in June, 1979.

By a space we mean a completely regular Hausdorff space.

DEFINITIONS. A space X is countably compact provided that every infinite subset of X has an accumulation point in X. If $X \subset Y$, then X is extra countably compact in Y if every infinite subset of Y has an accumulation point in X. If X is extra countably compact in βX, we say that X is extra countably compact.

Comfort and Waiveris prove the following theorem:

THEOREM [CW, Th. 2.6; W, Cor. 2.4 and Th. 2.8]. *Let X be either an F-space or a realcompact space. Then there is a set $\{P_\xi : \xi < 2^c\}$ of extra countably compact subspaces of βX such that $P_\xi \cap P_{\xi'} = X$, for $\xi < \xi' < 2^c$.* [CW, Th. 3.4; W, Th. 2.14]. *If X is topologically complete then the spaces P_ξ can be constructed to be countably compact. An example is given,* [CW, Ex. 3.2; W. Th. 2.10] *assuming the existence of a measurable cardinal, of a topologically complete space which is not an intersection of extra countably compact spaces.*

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Comfort and Waiveris conjecture [CW, 4.4] that in all three cases, the spaces P_ξ may be chosen pairwise non-homeomorphic, and prove this in certain special cases, most notably X separable metric [CW, Th. 4.4.4]. We show that the conjecture is true, using $\mathcal{D}$-limits and weak P-points.

1. **Preliminaries.** Let $(x_n : n<\omega)$ be a sequence in a space X, $x \in X$, $\mathcal{D} \in \beta(\omega)\setminus\omega$. Then $x = \mathcal{D} - \lim_{n<\omega} x_n$ provided that if $x \in 0$ open, then $\{n : x_n \in 0\} \in \mathcal{D}$. $\mathcal{D}$-limits were first defined in print by A. Bernstein [B, Def. 3.1]. For facts and applications of $\mathcal{D}$-limits, see [B], [GS] and [S_1]. See also [K_1] for some very powerful applications of this concept.

If $A \subset Y$, then acc A is the set of accumulation points of A in Y; that is, acc $A = \{p \in Y : |U \cap A| \geq \omega$ for every neighborhood U in Y of $p\}$.

Clearly a $\mathcal{D}$-limit of a sequence of distinct points $(x_n : n<\omega)$ is an accumulation point of the set $\{x_n : n<\omega\}$.

For any set X, $[X]^\omega = \{A \subset X : |A| = \omega\}$.

For a point $\mathcal{D} \in \beta(\omega)\setminus\omega$, $\mathcal{D}$ is called a weak P-point if $\mathcal{D} \notin$ acc A, for any $A \in [\beta(\omega)\setminus\omega]^\omega$.

Our construction works in ZFC because of the following theorem of K. Kunen.

THEOREM [K_2]. *There exist 2^c weak P-points in $\beta(\omega)\setminus\omega$.*

Let $\mathcal{D}_1, \mathcal{D}_2 \in \beta(\omega)\setminus\omega$. Then $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same type if there exists $f : \beta(\omega) \to \beta(\omega)$ a homeomorphism such that $f(\mathcal{D}_1) = \mathcal{D}_2$. Types of ultrafilters were apparently first defined by W. Rudin [R], and developed by Frolik [Fk]. Basic facts about types may be found in [CN]. Since $|t| = c$ for any type t, Kunen's theorem yields the existence of 2^c distinct weak P-point types, which is the result we need.

For some other applications of weak P-points to problems involving countably compact spaces, see [V_1], [S_1], and [S_2].

If $A \subset \beta X$, then $\bar{A}$ denotes the closure in βX of A; that is, $\bar{A} = \text{cl}_{\beta X} A$.

The word countable will always mean countably infinite.

A homeomorphism will always be an onto function.

A space X is an F-space if every cozero-set of X is C^*-embedded in X. It is known [GJ, 14N] that if X is an F-space, then βX is an F-space and every countable subset of βX is C^*-embedded in βX. This is the only property of F-spaces required here, in the sense that this property (every countable subset of βX is C^*-embedded) may replace the hypothesis that X is an F-space in the Theorem from [CW] as well as in our Theorem 2.3 [CW].

A space X is realcompact if and only if for every $p \in BX\setminus X$ there exists a zero set Z of βX such that $p \in Z \subset \beta X\setminus X$ [GJ]. The following easy lemma will be useful to us later. Note that there is something to prove, in the sense that the union of two realcompact spaces is not necessarily realcompact [GJ, 5I; V_2, Ex. 6.4].

LEMMA 1.1. *Let X be realcompact and $A \in [\beta X]^{\omega}$. Then $X \cup A$ is realcompact.*

Wis Comfort has pointed out that if X is realcompact and A is a Lindelöf subset of βX, then $X \cup A$ is realcompact.

For a space X, νX denotes the Hewitt realcompactification of X.

A space X is topologically complete if and only if X can be embedded as a closed subset of a product of metric spaces. The following lemma is the only fact that we need to know about topologically complete spaces [CW, Lemma 3.3; Ka; W, Th. 2.12].

LEMMA 1.2. *Let X be topologically complete, $A \in [X]^{\omega}$, and $p \in (\operatorname{acc} A) \setminus X$. Then $p \notin \nu X$.*

For more information about these kinds of spaces see [CW] or [GJ]. We will borrow freely from the results in [CW] and [W] for lemmas and useful facts. The reader will surely find it useful to have a copy of at least one of these papers to use as a reference. The reader is warned that the notation $\mathscr{D}(X)$ does not mean the same thing in the two papers and we will use the notation $D(X)$.

2. **The construction.**

We begin with a lemma.

LEMMA 2.1. *Let X be a space. Let $D(X)$ be a collection of C^*-embedded discrete subsets of X such that if $A, B \in D(X)$ and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$, then $\bar{A} \cap \bar{B} = \emptyset$. Let $\mathscr{U}$ and $\mathscr{V}$ be weak P-points of $\beta(\omega) \setminus \omega$ of different types. Let $A, B \in D(X)$ with $A = \{a_n : n < \omega\}$ and $B = \{b_n : n < \omega\}$ and let $x, y \in \beta X$ with $x = \mathscr{U}\text{-}\lim_{n<\omega} a_n$ and $y = \mathscr{V}\text{-}\lim_{n<\omega} b_n$. Then $x \neq y$.*

Proof. Case (i). There exist $A_1 \subset A$ and $B_1 \subset B$ with $A_1 = B_1$ and $x = \mathscr{U}\text{-}\lim_{a_n \in A_1} a_n$ and $y = \mathscr{V}\text{-}\lim_{b_n \in B_1} b_n$. We do not assume that $a_n = b_n$, for any $n \in N$. Then $A_1 = B_1$ is homeomorphic to ω and C^*-embedded in X, and hence in βX. Thus $\bar{A}_1 \approx \beta\omega$ so $x \neq y$, since $\mathscr{U}$ and $\mathscr{V}$ have different types.

Case (ii). There exists $A_1 \subseteq A \cap B$ with $x = \mathscr{U}\text{-}\lim_{a_n \in A_1} a_n$ and $y = \mathscr{V}\text{-}\lim_{b_n \in B \setminus A_1} b_n$, or there exists $B_1 \subseteq A \cap B$ with $y = \mathscr{V}\text{-}\lim_{b_n \in B_1} b_n$ and $x = \mathscr{U}\text{-}\lim_{a_n \in A \setminus B_1} a_n$. The two possibilities are handled similarly. Assuming the former, then since B is C^*-embedded in βX and B is homeomorphic to ω, we have $\bar{A}_1 \setminus A_1$ and $\overline{(B \setminus A_1)} \setminus (B \setminus A_1)$ are disjoint. Since $x \in \bar{A}_1 \setminus A_1$ and $y \in \overline{(B \setminus A_1)} \setminus (B \setminus A_1)$ we have $x \neq y$.

Case (iii). $\bar{A} \cap \bar{B} = \emptyset$. This is obvious.

Case (iv). $\{n : a_n \in \bar{B} \setminus B\} \in \mathscr{U}$ or $\{n : b_n \in \bar{A} \setminus A\} \in \mathscr{V}$. Let us assume that $M = \{n : a_n \in \bar{B} \setminus B\} \in \mathscr{U}$. Suppose that $x = y$. Identify $b_n = n$. Then $x = \mathscr{V}\text{-}\lim_{n<\omega} n$, i.e., $x = \mathscr{V}$. Then $\mathscr{V} = \mathscr{U}\text{-}\lim_{n \in M} a_n$ in $\beta(\omega) \setminus \omega$, but $\mathscr{V}$ is a weak P-point, which is a contradiction.

Case (v). $\bar{A} \cap B = \bar{B} \cap A = \emptyset$. Then $\bar{A} \cap \bar{B} = \emptyset$ by hypothesis. □

REMARK. This lemma and particularly the proof of case (iv) were partially inspired by a result of K. Kunen [K_3, Lemma 3.1].

The following lemma is useful in the case that X is realcompact to allow us to assume that the countable discrete set A for which $(\text{acc } A) \cap X = \emptyset$ is C-embedded in $X \cup A$.

LEMMA 2.2 [CW, lemma 2.5; W, lemma 2.5]. *Let X be realcompact and A a countable discrete subset of βX for which $(\text{acc } A) \cap X = \emptyset$. Then there is an infinite subset B of A such that B is C-embedded in $X \cup B$.*

THEOREM 2.3. *Let X be an F-space or a realcompact space which is not extra countably compact. Then there exists a set $\{X_1 : i \in I\}$ of extra countably compact spaces with $X \subset X_i \subset \beta X$, $\forall i$ and*

(i) $|I| = 2^c$,

(ii) $X_i \cap X_j = X$, *for* $i \neq j$,

(iii) X_i *and* X_j *are not homeomorphic, for* $i \neq j$.

Proof. If X is an F-space, let $D(\beta X) = \{A \subset \beta X: A \text{ is discrete countable}\}$. If X is not an F-space but is realcompact then let $D(\beta X) = \{A \subset \beta X: A$ is discrete countable, $(\text{acc } A) \cap X = \emptyset$ and A is C-embedded in $X \cup A\}$.

Then $D(\beta X)$ satisfies the hypothesis of Lemma 2.1 [CW, lemma 2.4; W, proof of Cor. 2.4 for F-spaces, lemma 2.6 for realcompact spaces].

Now let I be the set of all weak P-point types in $\beta(\omega) \setminus \omega$. Then $|I| = 2^c$ by Kunen's Theorem and let $\mathcal{D}_i$ be of type i. Let $X_i = X \cup \{x \in \beta X : x = \mathcal{D}_i\text{-}\lim_{n<\omega} a_n$ with $\{a_n : n < \omega\} \in D(\beta X)\}$. Then $i \neq j$ implies that $X_i \cap X_j = X$ by Lemma 2.1. To see that each X_i is extra countably compact is easy if X is an F-space because X_i contains a $\mathcal{D}_i$-limit point of each countable discrete subset of βX. If X is realcompact and $A \in [\beta X]^\omega$ for which $(\text{acc } A) \cap X = \emptyset$, then by Lemma 2.2, there exists $B \in [A]^\omega$ and $B \in D(\beta X)$. Then B has a $\mathcal{D}_i$-limit point in X_i which is an accumulation point of A.

Given $i \neq j$ and a bijection $f: X_i \to X_j$, we show that f is not a homeomorphism.

Since X is not extra countably compact, there is $A \in D(\beta X)$ for which $(\text{acc } A) \cap X = \emptyset$. We can find $B \subset X_i$, $B \subset \bar{A} \setminus A$, $B = \{b_n : n < \omega\}$ is discrete and $B \in D(\beta X)$. Such B exists because $|(\text{acc } A) \cap X_i| = 2^c$ [CW, Rem. 2.7]. Now let B be a countable discrete subset of $(\text{acc } A) \cap X_i$. Then since $B \subset \bar{A}$ and $(\text{acc } A) \cap X = \emptyset$, we have $(\text{acc } B) \cap X = \emptyset$. In the case that X is realcompact, Lemma 2.2 guarantees that B can be chosen to be C-embedded in $X \cup B$. Thus $B \in D(\beta X)$.

(i). Let X be an F-space. We claim that the sequence $(b_n : n < \omega)$ has no $\mathcal{D}_j$-limit point in X_i. For let $x = \mathcal{D}_j\text{-}\lim_{n<\omega} b_n$. Note that $x \notin X$. So if $x \in X_i$, then $x = \mathcal{D}_i\text{-}\lim_{n<\omega} x_n$, for some $\{x_n : n < \omega\} \in D(\beta X)$. But Lemma 2.1 says that this is impossible. If f were a homeomorphism then $\{f(b_n) : n < \omega\}$ would belong to

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$D(\beta X)$ and thus would have a $\mathcal{D}_j$-limit point in X_j. As having a $\mathcal{D}_j$-limit is a topological property, f clearly is not a homeomorphism. Note that we need to have removed the restriction $(\text{acc } A) \cap X = \varnothing$ in the definition of $D(\beta X)$ because it is possible that the sequence $(f(b_n): n < \omega)$ has an accumulation point in X. With the restriction $(\text{acc } A) \cap X = \varnothing$ removed, we can be sure we have adjoined a $\mathcal{D}_j$-limit for the sequence $(f(b_n): n < \omega)$ in X_j.

(ii) Let X be realcompact.

We know that B has a $\mathcal{D}_i$-limit point in X_i and it would suffice to show that $f(B) = (f(b_n): n < \omega)$ has no $\mathcal{D}_i$-limit point in X_j. The problem is that $f(B)$ might have a $\mathcal{D}_i$-limit point in X.

We distinguish between two cases.

(a) There exists $C = \{C_n : n < \omega\} \in [B]^\omega$ and $f(C)$ has no accumulation point in X. If necessary replace C by a smaller infinite subset so that $f(C)$ is C-embedded in $X \cup f(C)$. Then $(c_n : n < \omega)$ has a $\mathcal{D}_i$-limit point in X_i and $(f(c_n): n < \omega)$ has no $\mathcal{D}_i$-limit point in X_j, arguing as above.

(b) not case a). Let $Y = X \cup f(B)$. By Lemma 1.1, Y is realcompact. By not case a), every infinite subset of $f(B)$ has an accumulation point in Y, so $\text{cl}_Y f(B)$ is pseudocompact [GS, 4.5], and also realcompact, hence compact.

Then $f(B)$ has a $\mathcal{D}_j$-limit point in Y, and hence in X_j. But $B = (b_n : n < \omega)$ has no $\mathcal{D}_j$-limit point in X_i, again arguing as above.

THEOREM 2.4. *Let X be a topologically complete space which is not countably compact. Then there exists a set of countably compact subspaces $\{P_i : i \in I\}$ of βX such that*

(i) $|I| = 2^c$,

(ii) $i \neq j$ implies $P_i \cap P_j = X$,

(iii) $i \neq j$ implies P_i and P_j are not homeomorphic.

Proof. There is from Theorem 2.3 a set $\{X_i : i \in I\}$ of extra countably compact spaces such that $i \neq j$ implies $X_i \cap X_j = \nu X$. Following [CW] set $Y = \bigcup\{\bar{A} : A \in [X]^\omega\}$. In [W] this set is denoted by $\omega^+ X$. Lemma 1.2 says that $Y \cap (\nu X \setminus X) = \varnothing$. Set $P_i = Y \cap X_i$ for each $i \in I$, and $|I| = 2^c$. It is not difficult to show that $P_i \cap P_j = X$ for $i \neq j$ and that each P_i is countably compact; for the details see the proof of [CW, Th. 3.4; W, Th. 2.14].

Now let $f: P_i \to P_j$ be a bijection; we will show that f is not a homeomorphism. Since X is not countably compact, there exists $B \in [X]^\omega$ with no accumulation point in X. Then $(\text{acc } B) \cap \nu X = \varnothing$. By Lemma 2.2 we may assume that B is C-embedded in X and hence in νX. Of course $\beta(\nu X) = \beta X$ so $B \in D(\beta X)$ in the construction of the set $\{X_i : i \in I\}$ which is based on νX. The proof proceeds similarly to the proof of the case X realcompact in Theorem 2.3. We distinguish between two cases.

(a) There exists $C \in [B]^\omega$ and $f(C)$ has no accumulation point in X. Then we may assume that $f(C)$ is C-embedded in $\nu X \cup f(C)$. Then C has a $\mathcal{D}_i$-limit

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point in P_i but $f(C)$ has no $\mathcal{D}_i$-limit point in P_j, since $f(C)$ has no $\mathcal{D}_i$-limit point in X_j.

(b) not case a). Let $Z = \nu X \cup f(B)$. Then Z is realcompact. By not case a), every infinite subset of $f(B)$ has an accumulation point in X and hence in Z. Thus $\text{cl}_Z f(B)$ is pseudocompact, and realcompact, hence compact. Since $f(B) \subset P_j = Y \cap X_j$, $f(B) \subset Y$. Now $f(B)$ is countable and Y is $\aleph_0$-bounded [Wo], i.e., if $A \in [Y]^\omega$, $\text{cl}_Y A$ is compact. Thus $\text{cl}_Y f(B)$ is compact. Also $Y \cap Z \subset Z \cup f(B)$, since $Y \cap (\nu X \setminus X) = \emptyset$. Then we have $\text{cl}_Y f(B) = \text{cl}_Z f(B) \subset Y \cap Z \subset X \cup f(B) \subset P_j$. Thus $\text{cl}_{P_j} f(B)$ is compact. Thus $f(B)$ has a $\mathcal{D}_j$-limit point in P_j, but B has no $\mathcal{D}_j$-limit point in P_i, since B has no $\mathcal{D}_j$-limit point in X_i.

3. **Related matters.** In section 4 of [CW] there is a rather large and interesting collection of ideas and questions concerning intersections of extra countably compact spaces and also intersections of countably compact spaces. We contribute this section in response to those ideas and questions.

The following result [CW, 4.6a; W. Th. 2.11] is due to M. Henriksen.

THEOREM 3.1. *A space X can be written as an intersection of extra countably compact subspaces of βX if and only if no sequence of βX converges to a point of $\beta X \setminus X$.*

The analogous statement for countably compact spaces is false. I am grateful to Eric van Douwen for some interesting and helpful discussions and for informing me of the following result of his and D. Burke [BvD], which is a very nice example and which completely answered our question.

EXAMPLE 3.2 [BvD]. There is a space X such that no sequence from X converges to a point in $\beta X \setminus X$, but X cannot be expressed as an intersection of countably compact subspaces of βX.

The example has a point $p \in \beta X \setminus X$ such that p belongs to every countably compact subset of βX that contains X.

The following theorem generalizes Theorem 3.1 with the same proof:

THEOREM 3.3. *Let X be dense in a compact space Y. Then X can be expressed as an intersection of extra countably compact subspaces of Y if and only if no sequence from Y converges to a point in $Y \setminus X$.*

The following is taken from 4.5 of [CW].

Denote by $(\delta(\alpha))\mathscr{C}(\alpha)$ the class of spaces X for which there is a family $\{P(i): i \in I\}$ of (extra) countably compact subspaces of βX such that $|I| < \alpha$ and $X = \bigcap_{i \in I} P(i)$.

For what pairs (α, α') or cardinal numbers are the inclusions $\mathscr{C}(\alpha) \subset \mathscr{C}(\alpha')$ and $\delta(\alpha) \subset \delta(\alpha')$ valid?

These are apparently hard questions and we have results only in some very

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special cases. Theorem 3.1 suggests that sequential compactness (every sequence has a convergent subsequence) might be fruitful avenue of approach, in light of the fact that it is consistent that a compact space of cardinality less than 2^c is sequentially compact. That this is a theorem in certain models was first shown in [Fr] by Franklin assuming the Continuum Hypothesis, then in [MS] by Malyhin and Šapirovskiĭ assuming Martin's Axiom, and then in [ST] by Szymański and Turzański assuming $P(c)$. See [vDF] for a unified approach to the theory of convergent sequences in compact spaces. In the sequel we will assume $P(c)$ in most of our results.

LEMMA 3.4. *Assume $P(c)$. Let X be dense in Y compact. If X can be expressed as an intersection of extra countably compact subspaces of Y, then for any $A \in [Y]^\omega$ for which $(\text{acc } A) \cap X = \emptyset$, we have $|\text{cl}_Y A| = 2^c$.*

Proof. By Theorem 3.3 no sequence of Y converges to a point in $Y \setminus X$. Let $A \in [Y]^\omega$ for which $(\text{acc } A) \cap X = \emptyset$. If $|\text{cl}_Y A| < 2^c$, the $\text{cl}_Y A$ is sequentially compact. Then there is a sequence from A which converges to a point which must be in $Y \setminus X$, since $(\text{acc } A) \cap X = \emptyset$, which is a contradiction.

COROLLARY 3.5. *Assume $P(c)$. Suppose that X can be expressed as an intersection of extra countably compact subspaces of βX. Then for any $A \in [\beta X]^\omega$ for which $(\text{acc } A) \cap X = \emptyset$, $|\bar{A}| = 2^c$.*

Some additional hypothesis such as $P(c)$ is necessary in Lemma 3.4. Consistent examples exist of a compact space X of cardinality c with no nontrivial convergent sequences. Such spaces were constructed assuming the Partition Hypothesis (PH) by Fedorčuk in [Fe] and assuming the Open Splitting Hypothesis (OH) by van Douwen and Fleissner in [vDF]. Observe that X may chosen separable, since for any $A \in [X]^\omega$, $|\text{cl}_X A| = c$, which follows from Lemma 6.1 of [vD].

EXAMPLE 3.6. Assume OS or PH. Then there exists a countable X dense in Y compact, X is an intersection of extra countably compact subspaces of Y, but $|\text{cl}_Y X| = c$.

Proof. Let Y be compact of cardinality c with no nontrivial convergent subsequences, and X a countable dense subset. Then obviously $|\text{cl}_Y X| = c$. By Theorem 3.3, X is an intersection of extra countably compact subspaces of Y.

Lemma 3.4 and Example 3.6 yield the following independence results:

THEOREM 3.7. *The following statement is independent of ZFC:*
There exists a countable X dense in Y compact, $|Y| = c$, and X is an intersection of extra countably compact subspaces of Y.

Assuming $P(c)$ and applying Corollary 3.5, we can respond positively to part of Question 4.5 of [CW] in the case that $|\beta X| \leq \leq 2^c$.

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THEOREM 3.8. *Assume $P(c)$. Suppose X is an intersection of extra countably compact subspaces of βX. If $|\beta X|\le 2^c$, then X is an intersection of 2 extra countably compact subspaces of βX in 2^c different ways. In other words, there exists a set $\{P_\xi:\xi<2^c\}$ of extra countably compact subspaces of βX such that if $\xi<\xi'<2^c$, then $P_\xi\cap P_{\xi'}=X$.*

Proof. Corollary 3.5 shows that $|\beta X|\ge 2^c$. Since by hypothesis $|\beta X|\le 2^c$, $|\beta X|=2^c$. We will inductively construct $(P_\xi:\xi<2^c)$.

Now let $i:2^c\to 2^c\times 2^c$ be a bijection. Write $[\beta X]^\omega=(A_\delta:\delta<2^c)$. For $\xi<2^c$, $i(\xi)=(\alpha,\delta)$ and assume that for $\xi'<\xi$, $i(\xi')=(\alpha',\delta')$ and we have chosen $x_{\xi'}=x_{\alpha',\delta'}\in \text{acc } A_{\delta'}$ and if $\rho<\xi'$ then $x_\rho\ne x_{\xi'}$.

If $(\text{acc } A_\delta)\cap X\ne\varnothing$, let $x_\xi\in(\text{acc } A_\delta)\cap X$.

Otherwise, by Corollary 3.5, $|\bar{A}_\delta|=2^c$. Find $x_\xi=x_{\alpha,\delta}\in(\text{acc } A_\delta)\setminus\{x_{\xi'}:\xi'<\xi\}$.

Now let $P_\alpha=X\cup\{x_{\alpha,\delta}:A_\delta\in[\beta X]^\omega\}$. Then each P_α is extra countably compact and $\alpha<\alpha'<2^c$ implies $P_\alpha\cap P_{\alpha'}=X$.

We do not know if the spaces P_α may be chosen pairwise nonhomeomorphic nor whether or not the analogous statement is true for countable compactness.

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