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ON COMPLETE INTERSECTIONS OVER AN ALGEBRAICALLY NON-CLOSED FIELD

BY

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ABSTRACT. We give a criterion in order that an affine variety defined over any field has a complete intersection (c.i.) embedding into some affine space. Moreover we give an example of a smooth real curve C all of whose embeddings into affine spaces are c.i.; nevertheless it has an embedding into $\mathbb{R}^3$ which cannot be realized as a c.i. by polynomials.

Introduction. Let k denote an infinite field and V an irreducible affine d-dimensional variety over k with structure sheaf $\mathcal{O}_V$. Let $k[V]$ (resp. $\Gamma_V = \Gamma(V, \mathcal{O}_V)$) denote the ring of polynomial (resp. global regular) functions defined on V. After Mohan Kumar [6] $k[V]$ is called *abstract complete intersection* (A.C.I.) if V can be embedded as a complete intersection (c.i.) in some affine space, i.e. if there exists an integer m such that $k[V] \simeq k[X_1, \ldots, X_m]/\boldsymbol{a}$ where the ideal $\boldsymbol{a}$ is generated by ht $\boldsymbol{a} = m - d$ polynomials. The main results of [6], provided Fract $(k[V])$ is separable over k, are:

The ring $k[V]$ is an A.C.I. if and only if the module of 1-differentials $\Omega^1_{k[V]/k}$ has a free resolution of length $\leqq 1$.

Assuming that $k[V]$ is an A.C.I., then any embedding of V in an m-dimensional affine space is a c.i. if $m \leqq d + 2$ or $m \geqq 2d + 2$ (the last assuming further that V is smooth).

Our goal is to investigate what kind of information can be deduced from the [6] results when dealing with the rings Γ_V. We have been led to this question also because of a recent paper of Bochnak and Kucharz on complete intersection real varieties where it is proved that:

A compact orientable algebraic manifold $V \subset \mathbb{R}^n$ is an algebraic c.i. (i.e. Γ_V is a c.i. in $\Gamma_{\mathbb{R}^n}$) if codim $V \leqq 2$ or if V is connected and dim $V = 2$.

Bochnak and Kucharz also raise the question whether for an algebraic c.i. $V \subset \mathbb{R}^n$ is $\mathbb{R}[V]$ a c.i. in $\mathbb{R}[\mathbb{R}^n]$, here we give a negative answer to this question (cf 3.3).

More precisely, in section 1 we introduce the notations. In section 2 we prove some results, concerning the property of A.C.I., for the rings Γ_V and compare the notion of

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A.C.I. for $k[\mathbf{V}]$ and for $\Gamma_{\mathbf{V}}$. In section 3 we give two meaningful examples. The first is a smooth affine curve such that:

(i) its ring of global regular functions is a c.i. in "every embedding of it into an affine space",

(ii) an affine representation of it is not an A.C.I.

The second is an affine smooth curve whose "distinguished" complexification is not an A.C.I.

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1. **Preliminaries and Notations**. Throughout this paper k will denote an infinite field which is not algebraically closed and $\bar{k}$ an algebraic closure of it. Moreover a variety will be always assumed to be affine and irreducible.

If R is a k-algebra, we denote by Max R the subspace of Spec R consisting of all the maximal ideals of R.

Let V be an algebraic subvariety of k^n, we denote by I_V (resp. $\boldsymbol{J}_V$) the ideal of V in $k[X_1, \ldots, X_n]$ (resp. Γ_{k^n}) and we have $\Gamma_V \simeq \Gamma_{k^n}/\boldsymbol{J}_V \simeq N_V^{-1}k[V]$ where $k[V] \simeq k[X_1, \ldots, X_n]/\boldsymbol{I}_V$ and $N_V = \{f \in k[V] | f(x) \neq 0 : x \in V\}$.

We recall that an abstract affine k-variety $(X, \mathcal{O}_X)$ is a topological space X plus a sheaf of k-valued functions on it which is isomorphic, as ringed space, to an algebraic subvariety of some k^n with its structure sheaf.

In order to distinguish an abstract affine k-variety from its embedded models, we will use boldface characters to denote the former. But, when no confusion can arise we will neglect this caution.

Let $\mathbf{V}$ be an abstract affine k-variety, to each embedding $\phi: \mathbf{V} \to k^n$ it corresponds a finitely generated k-algebra $P = k[X_1, \ldots, X_n]/\boldsymbol{I}_{\phi(\mathbf{V})} \simeq k[\mathbf{V}]$ such that $\Gamma_{\mathbf{V}} \simeq N_P^{-1}P$ where $N_P = \{f \in P | f(x) \neq 0 \ \forall\, x \in \phi(\mathbf{V})\}$. Moreover, any embedded model $V \subset k^n$ of an abstract affine k-variety $\mathbf{V}$ determines a completion $\tilde{V} \subset \bar{k}^n$ and it turns out that $\bar{k}[\tilde{V}] \simeq k[V] \otimes_k \bar{k}$. If $\tilde{V} \subset \bar{k}^n$ is the completion of $V \subset k^n$, then $\tilde{V}$ is defined over k, that is $\boldsymbol{I}_{\tilde{V}} \subset \bar{k}[X_1, \ldots, X_n]$ is generated by $\boldsymbol{I}_{\tilde{V}} \cap k[X_1, \ldots, X_n]$. If $k = \mathbb{R}$ and $V \subset \mathbb{R}^n$ this is equivalent to saying that $\tilde{V} \subset \mathbb{C}^n$ has a real structure on it (i.e. an antiholomorphic involution $\sigma : \tilde{V} \to \tilde{V}$ for which $(\tilde{V})^\sigma \simeq V$), and furthermore $\tilde{V}$ is called complexification of V.

For general reference we quote [9]. In particular we refer to [5] where it is proved that an affine k-variety $\mathbf{V}$ is determined up to isomorphism by $\Gamma_{\mathbf{V}}$ (as $\mathbf{V} \simeq \operatorname{Max} \Gamma_{\mathbf{V}}$) and the category $\mathcal{A}_k$ of k-algebras that are rings of global regular functions of some affine k-variety is characterized (cf. [5] Prop. 2.1. and Thm. 2.2).

Finally, in order to have the proper terminology at our disposal we recall that if $\mathbf{V}$ is an affine curve over k, its different completions have been compared in [4] whose results can be summarized as follows. It is convenient (and it is possible) to consider completions having their possible singularities concentrated in their rational parts. We will always (tacitly) assume this property for completions as well as for affine representations. Given two completions $\tilde{V}_1$, $\tilde{V}_2$ of a curve $\mathbf{V}$, we say that $\tilde{V}_1$ precedes

$\tilde{V}_2$ (in symbols $\tilde{V}_1 \prec \tilde{V}_2$) if there is an open immersion $\tilde{V}_2 \to \tilde{V}_1$ (this is equivalent to say that there is a flat ring homomorphism between the involved affine representations). The relation $\prec$ is a partial ordering, with minimal elements, on the set of all completions of $\mathbf{V}$. If all the minimal elements of $\prec$ are isomorphic, i.e. if $\prec$ has a minimum element, we say that this is the "distinguished" completion of $\mathbf{V}$ (such a completion in [4] was unhappily called canonical).

2. **Results**.

DEFINITION 2.1. *Let A be an integral domain in $\mathcal{A}_k$. Then A is called A.C.I. if there exists an integer n such that $A \simeq \Gamma_{k^n}/\boldsymbol{a}$ with $\boldsymbol{a} \subset \Gamma_{k^n}$ a c.i. ideal.*

REMARK 2.2. *By mimicking [6] if A is a domain in $\mathcal{A}_k$, the expression "embedding of A into an affine space" means embedding of the affine k-variety in some affine space k^n.*

PROPOSITION 2.3. *Let A be a domain in $\mathcal{A}_k$. Then A is an A.C.I. if and only if there exists an affine representation of it which is an A.C.I.*

PROOF. Assume first that A is an A.C.I. Let $A \simeq \Gamma_{k^n}/\boldsymbol{a}$ with $\boldsymbol{a} \subset \Gamma_{k^n}$ generated by $r = \operatorname{ht} \boldsymbol{a}$ elements, say $\boldsymbol{a} = (a_1, \ldots, a_r)$ where $a_i = F_i/G_i$ with $F_i, G_i \in k[X_1, \ldots, X_n]$ and $G_i^{-1}(0) = \emptyset$, $i = 1, \ldots, r$. Let $V \subset k^n$ be the corresponding embedded variety and consider the isomorphism $\phi : \Gamma_{k^n/\boldsymbol{a}} \xrightarrow{\sim} N_V^{-1}(k[X_1, \ldots, X_n]/\boldsymbol{I}_V)$ with $\boldsymbol{I}_V$ and N_V as in §1. Let $\{H_1, \ldots, H_s\}$ be any system of generators for $\boldsymbol{I}_V \subset k[X_1, \ldots, X_n]$. Clearly $\phi^{-1}(H_j) \in \boldsymbol{a}$ for each $j \in \{1, \ldots, s\}$, thus $\phi^{-1}(H_j) = \sum_1^r{}_i\, b_{ij} a_i$ where $b_{ij} = F_{ij}/G_{ij}$ with F_{ij}, G_{ij} as above. It turns out that $\phi(G_i)$, $\phi(G_{ij}) \in N_V$ for each $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$. Therefore:

$$P = (k[X_1, \ldots, X_n]/\boldsymbol{I}_V)\,[1/\phi(G_1), \ldots, 1/\phi(G_r), 1/\phi(G_{11}), \ldots, 1/\phi(G_{rs})]$$

$$\simeq k[X_1, \ldots, X_n, Y_1, \ldots, Y_r, T_{11}, \ldots, T_{rs}]/(\phi(F_1), \ldots, \phi(F_r),$$

$$Y_1\phi(G_1) - 1, \ldots, T_{rs}\phi(G_{rs}) - 1)$$

is an affine representation of A as $\phi(F_i) \in \boldsymbol{I}_V$ and

$$H_j = \sum_1^r{}_i\, \phi(b_{ij})\phi(a_i) = \sum_1^r{}_i\, \phi(F_{ij})T_{ij}\phi(F_i)Y_i.$$

Moreover,

$$\operatorname{ht}(\phi(F_1), \ldots, \phi(F_r), Y_1\phi(G_1) - 1, \ldots, T_{rs}\phi(G_{rs}) - 1) = r(2 + s)$$

that is P is an A.C.I.

Conversely, if A has an affine representation P which is an A.C.I., then $P \simeq k[X_1, \ldots, X_n]/\boldsymbol{b}$ with $\boldsymbol{b} \subset k[X_1, \ldots, X_n]$ a c.i. ideal. As $A = N_P^{-1}P \simeq \Gamma_{k^n}/\boldsymbol{b}^e$ and $\boldsymbol{b}^e \subset \Gamma_{k^n}$ is a c.i. ideal, our contention is fully proved.

THEOREM 2.4. *Let A be a domain in $\mathcal{A}_k$ and assume that* Fract (A) *is separable over k. Then A is an A.C.I. if and only if the module of* 1*-differentials $\Omega^1_{A/k}$ has a free resolution of length $\leqq 1$. Assuming further that A is smooth over k and of dimension d, if A is an A.C.I., then any embedding of A into an affine space k^m with $m \geqq 2d + 2$ is a c.i.*

PROOF. It runs exactly as in the corresponding parts of the theorem at p. 534 on [6].

COROLLARY 2.5. *If $\mathbf{V}$ is an abstract curve and $\Gamma_{\mathbf{V}}$ is an A.C.I., then every model of $\mathbf{V}$ embedded in an m-dimensional affine space with $m \geqq 4$ is a c.i.*

REMARK 2.6. The Mohan Kumar theorem says further that any embedding of an A.C.I. finitely generated k-algebra into an affine space is a c.i. if the codimension is $\leqq$ 2. Altogether it says in particular that, for a finitely generated, smooth, 1-dimensional k-algebra which is an A.C.I., every embedding into an affine space is a c.i. As we do not know whether the projective finitely generated modules over the rings Γ_{k^n} $(n \in N)$ are free, we can not apply the Mohan Kumar proof in its fullness to our situation. So in particular we do not know whether for A.C.I. curves over an algebraically non-closed field every embedded model is a c.i. or not since we lose control of the 2-codimensional embeddings.

We can state for curves the following partial results:

PROPOSITION 2.7. *Let $\mathbf{C}$ be an A.C.I. real affine curve which is smooth and compact in the usual topology. Then every embedding of $\mathbf{C}$ into an affine space is a c.i.*

PROOF. It comes out immediately by combining the Bochnak–Kucharz result quoted in the introduction with (2.5).

PROPOSITION 2.8. *Let $\mathbf{C}$ be an affine curve over some k and let P be an affine representation of $\mathbf{C}$ which is an A.C.I. Then every affine representation Q of $\mathbf{C}$ such that $P \triangleleft Q$ is an A.C.I.*

PROOF. By the Mohan Kumar theorem $\Omega^1_{P/k}$ has a free resolution of length $\leqq 1$, say $0 \to P^r \to P^s \to \Omega^1_{P/k} \to 0$ (*). The ring Q is a flat extension of P, thus tensoring (*) over P with Q we get $0 \to Q^r \to Q^s \to \Omega^1_{Q/k} \to 0$ which is a free resolution of length $\leqq 1$ for Q. Thus Q is an A.C.I.

COROLLARY 2.9. *If $\mathbf{C}$ is an affine curve with distinguished affine representation which is an A.C.I., then every affine representation of $\mathbf{C}$ is an A.C.I.*

REMARK 2.9. If a conjecture of Renschuch holds, then the above corollary fails when we do not confine ourselves to considering completions with their possible singularities concentrated in the rational parts. In fact, if we consider the affine representation of the real line

$$P = \mathbb{R}[(t^2 + 1)(t^2 + 4),\, t(t^2 + 1)(t^2 + 4),\, t^3 + 7t] \simeq \mathbb{R}[X, Y, Z]/\boldsymbol{a}$$

and if

$$f_1 = 2Z^2 - YZ + X^2 - 22X + 72, \quad f_2 = XYZ - X^3 - 2Y^2 + 4X^2,$$

$$f_3 = Y^2Z - 2X^2Z - X^2Y + 18XY$$

then the homogenized polynomials ${}^hf_1, {}^hf_2, {}^hf_3$ are a Gjunter basis for the homogenized ideal ${}^h\boldsymbol{a}$. From the Renschuch conjecture (cf [8] after Satz 6) it follows that $\boldsymbol{a}$ cannot be generated by less than 3 elements, that is P is not an A.C.I. (cf [6] loc. cit.). It is worth pointing out that the distinguished affine representation for the real line is $Q = \mathbb{R}[X]$ and $P \not< Q$. Yet $P \otimes_{\mathbb{R}} \mathbb{C}$ is the coordinate ring of a complex curve having two non-real ordinary nodes and moreover Q is not a flat ring extension of P (cf [4] ex. 1.1 and rem. 2.3.).

EXAMPLE 3.1. *Let $\mathscr{C}$ be a complete non singular complex curve of genus 3 having a real structure σ with real points $(\mathscr{C})^\sigma$ on it. Suppose further that $\mathscr{C}$ is non-hypelliptic. The canonical embedding of $\mathscr{C}$ is a quartic plane smooth projective curve γ and it carries on it a real structure that we can assume induced from the complex conjugation on $\mathbb{P}^2_{\mathbb{C}}$. Let ω_γ be the canonical sheaf, as γ has 28 bitangents and $\omega_\gamma \simeq \mathcal{O}_\gamma(1)$, there are only 28 pairs of points $P, Q \in \gamma$ such that $|2P + 2Q| = |\omega_\gamma|$. Thus we can choose a point $A \in \gamma$ such that $A \neq \bar{A}$ and $|2A + 2\bar{A}| \neq |\omega_\gamma|$. Let now $\tilde{\phi} : \mathscr{C} \to \mathbb{P}^5_{\mathbb{C}}$ be the embedding given by $|4A + 4\bar{A}|$ and let $\tilde{C} \subset \mathbb{C}^5$ be the affine complex curve characterized by $\tilde{\phi}(\mathscr{C})$.* Let $\boldsymbol{P} \subset \mathbb{C}[X_1, X_2, X_3, X_4, X_5] = A$ *be the ideal of $\tilde{C}$* and let $\boldsymbol{p} = \boldsymbol{P} \cap \mathbb{R}[X_1, X_2, X_3, X_4, X_5]$ *it is $A/\boldsymbol{P} \simeq \mathbb{R}[X_1, \ldots, X_5]/\boldsymbol{p} \otimes_{\mathbb{R}} \mathbb{C}$ since $\tilde{C}$ inherits a real structure. Thus, if $\boldsymbol{p}$ were generated by four elements also $\boldsymbol{P}$ would be so. By mimicking Murthy's argument (cf.* [7]) *we show that $\boldsymbol{P}$ is not generated by four elements. If it was so, then* $\mathrm{Ext}^4_A(A/\boldsymbol{P}, A) \simeq A/\boldsymbol{P}$. *As* $\mathrm{Ext}^4_A(A/\boldsymbol{P}, A)$ *is the module of sections of the canonical sheaf on $\tilde{C}$ (cf. [1] 1.4.6) it would follow that $\omega_{\tilde{C}}$ is trivial. Thus there would exist a real effective divisor D on $\tilde{\phi}(\mathscr{C})$ with support at infinity, $D \in |\omega_{\tilde{\phi}(\mathscr{C})}|$. As* $\deg|\omega_{\tilde{\phi}(\mathscr{C})}| = 4$ *and D is real, necessarily it would correspond to $2A + 2\bar{A}$ on γ and this is a contradiction. Now we can project $\tilde{C}$ isomorphically into $\mathbb{C}^3$ and observe that its real part is a smooth real curve which is compact in the usual topology of $\mathbb{R}^3$. Thus, by the Bochnak–Kucharz theorem, the ring of global regular functions on it is a complete intersection.*

EXAMPLE 3.2. *We adjust to the real case Murthy's example. More precisely let $\mathscr{C}$ be a complete non-singular complex curve of genus 2 that has a real structure σ with real points $(\mathscr{C})^\sigma$ on it. Let K be a divisor in the canonical class. By* [3] (6.1) *and* (6.3) *we may assume that K is real and that the induced morphism $(\mathscr{C}, \sigma) \to (\mathbb{P}^1_{\mathbb{C}}, {}^-)$ is real and gives a double covering of $\mathbb{P}^1_{\mathbb{C}}$ with* 6 $(=2g + 2)$ *ramification points of which at least one, say Q, is real, then $2Q \equiv K$. Moreover we can find a real point P on $\mathscr{C}$ such that $2P \not\equiv K$. For, if $P \in \mathscr{C}$ is such that $[2P - 2Q] = 0 \in J(\mathscr{C})$ (the Jacobian variety of $\mathscr{C}$), then $[P - Q]$ has order* 2 *in $J(\mathscr{C})$ and there are only* 16 $(=2^{2g})$ *such points in $J(\mathscr{C})$. We can now repeat Murthy's argument in order to show that the embedded* *real affine curve $C \subset \mathbb{R}^3$ gotten from the embedding $\mathscr{C} \xrightarrow[|5P|]{} \mathbb{P}^3_{\mathbb{C}}$ is not a c.i. Observe that $C \subset \mathbb{R}^3$ is a smooth affine curve which is not compact in the usual topology. Hence, (cf. [4] Th. 4.3.) its complexification is the distinguished complexification of the involved real curve* **C** *and it is not A.C.I.*

REMARK 3.3. The affine real curve $\mathscr{C}$ of (3.1) has the property that is a c.i. in every embedding of it into an affine space as well as its affine representation $P = \mathbb{R}[X_1, X_2, X_3, X_4, X_5]/\boldsymbol{p}$ is not a c.i. This fact answers negatively to a question posed in [2] where it is asked whether for a variety $V \subset \mathbb{R}^n$, Γ_V c.i. would imply $\mathbb{R}[V]$ c.i. We also point out that the affine representation corresponding to the canonical embedding of $\mathscr{C}$ is a c.i.

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