GENERALIZED DEGREE THEORY FOR SEMILINEAR OPERATOR EQUATIONS

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Abstract. In this paper, we construct a generalized degree theory of Browder-Petryshyn or Petryshyn type for a class of semilinear operator equations involving a Fredholm type mapping with infinite dimensional kernel.

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1. Introduction and Preliminaries. In this paper, we study the following semilinear operator equation

\[ Lx - Nx = f, \quad x \in X, f \in Y, \]

where \( X, Y \) are Banach spaces, and \( L : D(L) \subset X \to Y \) is a linear Fredholm type mapping with \( \dim(\text{Ker}(L)) = +\infty \), and \( N : \Omega \cap D(L) \to Y \) is a nonlinear mapping. This type of map equation has been extensively studied by Mawhin, Petryshyn and others for the case when \( \dim(\text{Ker}(L)) < +\infty \), see [8], [12] for references. By imposing some suitable conditions on \( X, L \) and \( Y \), we can apply Browder-Petryshyn’s degree and Petryshyn’s generalized degree theory to study such an equation. A generalized degree theory for \( L - N \) is defined in three ways by following Browder-Petryshyn and Petryshyn’s method or combining them with Mawhin’s method. First we recall some definitions.

Definition 1.1. [12] Let \( X \) be a real separable Banach space, \( (X_n)_{n=1}^{\infty} \) a sequence of finite dimensional subspaces of \( X \), and \( P_n : X \to X_n \) a projecton for \( n = 1, 2, \ldots \). If \( P_n x \to x \) as \( n \to \infty \), for all \( x \in X \), then \( \{X_n, P_n\} \) is called a projectionally complete scheme for \( X \).

Definition 1.2. [12] Let \( X, Y \) be two real separable Banach spaces, \( (X_n \subset X)_{n=1}^{\infty}, (Y_n \subset Y)_{n=1}^{\infty} \) two sequences of oriented finite dimensional subspaces such that \( \dim(X_n) = \dim(Y_n) \), and let \( Q_n : Y \to Y_n \) be a linear mapping of \( Y \) onto \( Y_n \) for \( n = 1, 2, \ldots \). If \( \lim_{n \to \infty} d(x, X_n) = 0 \), and \( (Q_n) \) is uniformly bounded, then we call \( \Gamma_A = \{X_n, Y_n, Q_n\} \) an admissible scheme for \( (X, Y) \); if \( Q_n y \to y \) for all \( y \in Y \), then we say \( \Gamma_0 = \{X_n, Y_n, Q_n\} \) is a projectionally complete scheme for \( (X, Y) \).
**Definition 1.3.** [12] Let $X, Y$ be real separable Banach spaces with a projectionally complete scheme $\Gamma_0 = \{X_n, Y_n, Q_n\}$, $D \subset X$, and $T : D \to Y$. Suppose that the following conditions are satisfied:

1. $Q_nT : D \cap X_n \to Y_n$ is continuous for $n = 1, 2 \ldots$;
2. For any bounded sequence $(x_{n_j} \in X_{n_j} \cap D)_{j=1}^{\infty}$ such that $Q_{n_j}Tx_{n_j} \to y$, there exists a subsequence $(x'_{n_j})$ such that $x'_{n_j} \to x \in D$ and $Tx = y$;
3. For any bounded sequence $(x_{n_j} \in X_{n_j} \cap D)_{j=1}^{\infty}$ such that $Q_{n_j}Tx_{n_j} \to y$, there exists $x \in D$ such that $Tx = y$.

Then $T$ is said to be $A$-proper with respect to $\Gamma_0$; if (2) is replaced by the following

1. For any bounded sequence $(x_{n_j} \in X_{n_j} \cap D)_{j=1}^{\infty}$ such that $Q_{n_j}Tx_{n_j} \to y$, there exists $x \in D$ such that $Tx = y$.

Then $T$ is said to be pseudo $A$-proper with respect to $\Gamma_0$.

**Definition 1.4.** Let $X, Y$ be two real Banach spaces, $L : D(L) \subseteq X \to Y$ is a linear mapping, and we say $L$ is a Fredholm mapping of index zero type if

1. $\text{Ker}(L) = \{x \in X : Lx = 0\}$, $\text{Im}(L) = \{x \in D(L)\}$ are closed in $H$;
2. $X = \text{Ker}(L) \oplus X_1$ for some subspace $X_1$ of $X$, $Y = Y_1 \oplus \text{Im}(L)$ for some subspace $Y_1$ of $Y$;
3. $\text{Ker}(L)$ is linearly homeomorphic to $\text{Coker}(L) = Y/\text{Im}(L)$.

**Remark 1.** Obviously, if $X$ is linearly homeomorphic to $Y$, $L = 0$ is a Fredholm mapping of index zero type, but not a Fredholm mapping of index zero. If $L$ is a Fredholm mapping of index zero, then $\dim(\text{Ker}(L)) = \dim(\text{Coker}(L)) < +\infty$, and so $\text{Ker}(L)$ is linearly homeomorphic to $\text{Coker}(L)$; thus $L$ is a Fredholm mapping of index zero type.

Now, assume that $L : D(L) \subset X \to Y$ is a Fredholm mapping of index zero type. Then there exist linear projections $P : X \to X$ and $Q : Y \to Y$ such that $\text{Im}(P) = \text{Ker}(L)$ and $\text{Im}(Q) = Y_1$.

Obviously, the restriction of $L_P$ of $L$ to $D(L) \cap \text{Ker}(P)$ is one to one and onto $\text{Im}(L)$, so its inverse $K_P : \text{Im}(L) \to D(L) \cap \text{Ker}(P)$ is defined. Let $J : \text{Ker}(L) \to Y_1$ be a linear homeomorphism, and set $K = K_P(I - Q)$.

**Proposition 1.5.** $L + \lambda JP : X \to Y$ is a bijective mapping for each $\lambda \neq 0$.

**Proof.** For each $\lambda \neq 0$, if $Lx + \lambda JPx = 0$, then $JPx = 0$, $Lx = 0$, so $x \in \text{Ker}(L)$, thus $x = 0$. On the other hand, for $y = y_1 + y_2 \in Y$, $y_1 \in Y_1, y_2 \in \text{Im}(L)$, put $x = \lambda^{-1}F^{-1}y_1 + KPy_2$, then $Lx + \lambda JPx = y$. Therefore $L + \lambda JP$ is bijective.

**Proposition 1.6.** Let $X, Y$ be real separable Banach spaces, and $(Y_n, Q_n)$ a projectionally complete scheme for $Y$, and let $L : D(L) \subset X \to Y$ be a Fredholm mapping of zero index type. Then for each $\lambda \neq 0$, there exists a projectionally complete scheme $\Gamma_{\lambda, L}$ for $(X, Y)$.

**Proof.** For each $\lambda \neq 0$, put $K_{\lambda} = L + \lambda JP$. By Proposition 1.5, $K_{\lambda}$ is bijective. Set $X_n = K_{\lambda}^{-1}X_n$ for $n = 1, 2 \ldots$. Obviously, we have $\dim(X_n) = \dim(Y_n)$, and $X = \bigcup_{n=1}^{\infty}X_n$. Thus $\Gamma_{\lambda} = \{X_n, Y_n, Q_n\}$ is a projectionally complete scheme for $(X, Y)$.

Petryshyn showed that if $L$ is a Fredholm mapping of index zero, then $L$ is $A$-proper with respect to $\Gamma_{1, L}$, see [12]. Here we have a similar result.

**Proposition 1.7.** Let $L : D(L) \subset X \to Y$ be a Fredholm mapping of zero index type, and assume that $X$ is reflexive. If $G \subset X$ is bounded closed convex, then $L : G \cap D(L) \to Y$ is pseudo $A$-proper with respect to $\Gamma_{\lambda, L}$ for each $\lambda \neq 0$. 

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Proof. For any sequence $x_{n_k} \in G \cap D(L) \cap X_{n_k}$ with $Q_{n_k}Lx_{n_k} \to y$, we may assume that $x_{n_k} \to x_0 \in G$ by taking a subsequence.

Notice that $Q_{n_k}(Lx_{n_k} + \lambda JPx_{n_k}) = Lx_{n_k} + \lambda JPx_{n_k}$, and $JPx_{n_k} \to JPx_0$, so we have

$$x_{n_k} = (L + JP)^{-1}(Q_{n_k}(Lx_{n_k} + JPx_{n_k})) \to (L + JP)^{-1}(y + JPx_0) = x_0.$$ 

Thus $x_0 \in D(L)$, and $Lx_0 = y$, so therefore $L$ is pseudo $A$-proper with respect to $\Gamma_{\lambda,L}$.

Definition 1.8. Let $X$ be a real separable Banach space and $\Gamma_0 = (X_n, P_n)$ a projectionally complete scheme for $X$, $Y$ a real Banach space, $L : D(L) \subseteq X \to Y$ a Fredholm mapping of zero index type, and let $N : D \subseteq X \to Y$ be a mapping.

1. If $I - P - (J^{-1}Q + KPQ)N$ is $A$-proper with respect to $\Gamma_0$, then we say $N$ is $L$-$A$-proper with respect to $\Gamma_0$;

2. If $I - P - (J^{-1}Q + KPQ)N$ is pseudo $A$-proper with respect to $\Gamma_0$, then we say $N$ is pseudo $L$-$A$-proper with respect to $\Gamma_0$;

3. A family of mappings $H(t, x) : [0, 1] \times D \to Y$ is called a homotopy of $L$-$A$-proper mappings with respect to $\Gamma_0$ if $H(t, \cdot)$ is an $L$-$A$-proper mapping with respect to $\Gamma_0$ for each $t \in [0, 1]$.

Proposition 1.9. Let $L : D(L) \subseteq X \to Y$ be a linear mapping with $\ker(L) = \{0\}$, and $\text{Im}(L) = Y$. Then the following conclusions hold

1. if $\Gamma_0 = (X_n, P_n)$ is a projectionally complete scheme for $X$, then $0$ is $L$-$A$-proper with respect to $\Gamma_0$;

2. if $(Y_n, Q_n)$ is a projectionally complete scheme for $Y$, and $L^\perp$ is continuous, then $L$ is $A$-proper with respect to $\Gamma_{1,L}$, where $\Gamma_{1,L}$ is constructed as in Proposition 1.6.

Proof. (1) We have $P = 0$, and $Q = 0$, and the identity mapping $I : X \to X$ is obviously $A$-proper with respect to $\Gamma_0$. Thus $0$ is $L$-$A$-proper with respect to $\Gamma_0$.

(2) Since $\ker(L) = \{0\}$, the mapping $K$ in the proof of Proposition 1.6 is just the mapping $L$, so $X_n = L^{-1}Y_n$. If $x_{n_k} \in X_{n_k}$ such that $Q_{n_k}Lx_{n_k} \to y$, then $Lx_{n_k} = Q_{n_k}Lx_{n_k} \to y$. Therefore we have $x_{n_k} \to L^{-1}y$. The conclusion holds.

Proposition 1.10. Let $L : D(L) \subseteq X \to Y$ be a Fredholm mapping of zero index type, $\Gamma_0 = (X_n, P_n)$ a projectionally complete scheme for $X$, $G \subseteq X$ a bounded closed convex subset, and $T : G \to Y$ a weakly continuous mapping, with $X$ reflexive. Then $T$ is $L$-pseudo $A$-proper with respect to $\Gamma_0$.

Proof. For any subsequence $x_{n_k} \in X_{n_k}$ such that $P_{n_k}(I - P - J^{-1}Q)T)x_{n_k} \to y$, we may assume that $x_{n_k} \to x_0 \in G$ by taking a subsequence, and so $(I - P)x_{n_k} \to x_0$, $J^{-1}Q^T x_{n_k} \to J^{-1}Q^T x_0$, and $K_{Q}T x_{n_k} \to K_{Q}T x_0$. Consequently, $(I - P - J^{-1}Q - K_{Q}T)x_0 = y$, so $T$ is $L$-pseudo $A$-proper with respect to $\Gamma_0$.

Proposition 1.11. Let $X, Y$ be real separable Banach spaces, and $(Y_n, Q_n)$ a projectionally complete scheme for $Y$. Let $L : D(L) \subseteq X \to Y$ be a Fredholm mapping of zero index type, $G \subseteq X$ a bounded closed subset, and $N : G \to Y$ a continuous compact mapping. Then $L + \lambda JP - N$ is $A$-proper with respect to $\Gamma_{\lambda,L}$ for each $\lambda > 0$.

Proof. For any sequence $x_{n_k} \in G \cap D(L) \cap X_{n_k}$ with $Q_{n_k}(L + \lambda JP - N)x_{n_k} \to y$, in view of the compactness of $N$, we may assume that $N\lambda_{n_k} \to y_0 \in Y$ by taking a subsequence.
Notice that $Q_{n_k}(Lx_{n_k} + \lambda JPx_{n_k}) = Lx_{n_k} + \lambda JPx_{n_k}$, so we have

$$x_{n_k} = (L + JP)^{-1}[Q_{n_k}(L + \lambda JP - N)x_{n_k} + Q_{n_k}Nx_{n_k}] \to (L + \lambda JP)^{-1}(y + y_0) = x_0.$$ 

Thus $x_0 \in D(L)$, and $N x_0 = y_0$, $(L + \lambda JP - N)x_0 = y$, and therefore $L$ is A-proper with respect to $\Gamma_{\lambda,L}$.

2. Generalized degree theory for $L\ominus N$. In this section, $X$, $Y$ are real separable Banach spaces, $L : D(L) \subseteq X \to Y$ is a Fredholm mapping of index zero type with $D(L)$ dense in $X$, and $N : \overline{\Omega} \subseteq X \to Y$ is a nonlinear mapping, and we consider the semilinear operator equation $Lx - Nx = 0$. We will apply Browder-Petryshyn and Petryshyn’s generalized degree theory to study such an equation in three different ways.

**Lemma 2.1.** Let $L : D(L) \subseteq X \to Y$ be a Fredholm mapping of index zero type, and $\Omega \subseteq X$ an open bounded subset, and let $N : \overline{\Omega} \to Y$ be a mapping. If $0 \notin (L - N)(\partial \Omega \cap D(L))$, then $0 \notin [I - P - (J^{-1}Q + K_{PQ})N](\partial \Omega)$.

**Proof.** Suppose the contrary i.e. suppose there exists $x_0 \in \partial \Omega$ such that $0 \in x_0 - Px_0 - (J^{-1}Q + K_{PQ})N x_0$. Since $J^{-1}QTx_0 \in \ker(L) = \text{Im}(P)$, $x_0 - Px_0 \in \ker(P)$, and $K_{PQ}T x_0 \in D(L) \cap \ker(P)$, we must have

$$J^{-1}QNx_0 = 0, \quad x_0 - Px_0 - K_{PQ}Nx_0 = 0.$$

Therefore we have

$$QN x_0 = 0, \quad x_0 -Px_0 - K_{PQ}N x_0 = 0, \text{ i.e. } Lx_0 - Nx_0 = 0,$$

which is a contradiction to $0 \notin (L - N)(\partial \Omega \cap D(L))$. □

Now, let $L : D(L) \subseteq X \to Y$ be a Fredholm mapping of index zero type, $\Gamma_0 = (X_n, P_n)$ a projectionally complete scheme for $X$ and $\Omega \subseteq X$ an open bounded subset, and let $N : \overline{\Omega} \to Y$ be an L-A-proper mapping with respect to $\Gamma_0$. Suppose $0 \notin (L - T)(\partial \Omega \cap D(L))$. By Lemma 2.1, $0 \notin [I - P - (J^{-1}Q + K_{PQ})N](\partial \Omega)$. Since $I - P - (J^{-1}Q + K_{PQ})N$ is an A-proper mapping with respect to $\Gamma_0$, the generalized degree $\deg(I - P - (J^{-1}Q + K_{PQ})N, \Omega, 0)$ is well defined, see [3], and we define

$$\deg_{\Gamma_0,J}(L - N, \Omega, 0) = \deg(I - P - (J^{-1}Q + K_{PQ})N, \Omega, 0), \quad (2.1)$$

which is called the generalized coincidence degree of $L$ and $N$ on $\Omega$.

**Theorem 2.2.** The generalized coincidence degree of $L$ and $N$ defined by (2.1) on $\Omega$ has the following properties.

1. If $\Omega_1$ and $\Omega_2$ are disjoint open subsets of $\Omega$ such that $0$ does not belong to $(L - N)(D(L) \cap \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then

$$\deg_{\Gamma_0,J}(L - N, \Omega, 0) \leq \deg_{\Gamma_0,J}(L - N, \Omega_1) + \deg_{\Gamma_0,J}(L - N, \Omega_2, 0).$$

2. If $H(t, x) : [0, 1] \times \overline{\Omega} \to Y$ is a homotopy of L-A-proper mappings with respect to $\Gamma_0$, and if $0 \neq Lx - H(t, x)$ for all $(t, x) \in [0, 1] \times \partial \Omega \cap D(L)$, then $\deg_{\Gamma_0,J}(L - H(t, \cdot), \Omega, 0)$ does not depend on $t \in [0, 1]$. 

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(3) If \( \deg_{\Gamma_0}(L - N, \Omega, 0) \neq \{0\} \), then \( 0 \in (L - N)(D(L) \cap \Omega) \).

(4) If \( L : D(L) \subseteq X \to Y \) is a linear mapping such that \( L^{-1} : Y \to D(L) \) is continuous, then \( \deg_{\Gamma_0, \mathcal{J}}(L, \Omega, 0) = \{1\} \) if \( 0 \in \Omega \).

(5) If \( \Omega \) is a symmetric neighbourhood of \( 0 \), and \( N : \overline{\Omega} \to Y \) is an odd \( L \)-A-proper mapping with respect to \( \Gamma_0 \) with \( 0 \notin (L - N)(\partial \Omega \cap D(L)) \), then \( \deg_{\Gamma_0, \mathcal{J}}(L - N, \Omega, 0) \) does not contain even numbers.

**Proof.** (1)–(3) follow directly from the definition and the properties of generalized degree.

(4) Since \( \ker(L) = \{0\} \), \( P = 0 \), \( Q = 0 \), the zero mapping is \( L \)-A-proper with respect to \( \Gamma_0 \). Thus \( \deg_{\Gamma_0, \mathcal{J}}(L, \Omega, 0) = \deg(I, \Omega, 0) = \{1\} \).

(5) Since \( N \) is odd, the mapping \( I - P - (J^{-1}Q + K_{PQ})N \) is odd. Thus \( \deg(I - P - (J^{-1}Q + K_{PQ})N, \Omega, 0) \) does not contain even numbers, and the conclusion follows by definition. \( \square \)

**Corollary 2.3.** Let \( L : D(L) \subseteq X \to Y \) be a linear mapping such that \( L^{-1} : Y \to D(L) \) is continuous, \( \Omega \subset X \) an open bounded subset with \( 0 \in \Omega \), and \( N : \overline{\Omega} \to Y \) a mapping such that \( (L - tN)_{t \in [0,1]} \) is a homotopy of \( L \)-A-proper mappings with respect to \( \Gamma_0 \). If \( Lx \notin tNx \) for all \((t, x) \in [0,1] \times \partial \Omega \cap D(L) \), then \( \deg(L - N, \Omega, 0) = 1 \).

In the following, let \( L : D(L) \subset X \to Y \) be a densely defined Fredholm mapping of zero index type. We assume that \( \Gamma_0 = (Y_n, Q_n) \) is a projectionally complete scheme for \( Y \), \( \Gamma_{\lambda, L} \) is as defined in Proposition 1.6, and \( L + \lambda JP - N \) is an \( A \)-proper map with respect to \( \Gamma_{\lambda, L} \) for \( \lambda \in (0, \lambda_0) \), where \( \lambda_0 > 0 \) is a constant. Suppose that \( 0 \notin (L - N)(D(L) \cap \partial \Omega) \). Then there exists \( \lambda_1 < \lambda_0 \) such that \( 0 \notin (L + \lambda JP - N)(D(L) \cap \partial \Omega) \) for all \( \lambda \in (0, \lambda_1) \). We define a generalized degree

\[
\deg(L - N, \Omega, 0) = \cap_{0 < \lambda < \lambda_1} \cup_{0 < \epsilon \leq \lambda} \deg(L + \epsilon JP - N, \Omega, 0),
\tag{2.2}
\]

where \( \deg(L + \epsilon JP - N, \Omega, 0) \) is the generalized degree for \( A \)-proper maps with respect to \( \Gamma_{\lambda, L} \), see [12].

Notice that if \( 0 \notin (L + \lambda JP - N)(D(L) \cap \partial \Omega) \) for all \( \lambda \in (0, \lambda_2) \), then it is easy to check that

\[
\cap_{0 < \lambda < \lambda_1} \cup_{0 < \epsilon \leq \lambda} \deg(L + \epsilon JP - N, \Omega, 0) = \cap_{0 < \lambda < \lambda_2} \cup_{0 < \epsilon \leq \lambda} \deg(L + \epsilon JP - N, \Omega, 0).
\]

Thus (2.2) is well defined.

**Remark.** A degree theory for uniform limits of \( A \)-proper maps has been defined by P. M. Fitzpatrick [5]. Since \( \Gamma_{\lambda, L} \) depends on \( \lambda \), and \( L + \lambda JP - N \) is an \( A \)-proper map with respect to \( \Gamma_{\lambda, L} \), \( L - N \) is slightly different to the uniform limits of \( A \)-proper maps. Of course, a slight generalization of the ideas in [5] could be applied here also.

**Theorem 2.4.** The generalized degree defined by (2.2) has the following properties.

(1) If \( \Omega_1 \) and \( \Omega_2 \) are two open subsets of \( \Omega \) such that \( \Omega_1 \cap \Omega_2 = \emptyset \), and \( 0 \notin (L - N)(D(L) \cap \Omega \setminus (\Omega_1 \cup \Omega_2)) \), then

\[
\deg(L - N, \Omega, 0) \subseteq \deg(L - N, \Omega_1) + \deg(L - N, \Omega_2, 0).
\]

(2) If \( H(t, x) : [0,1] \times \overline{\Omega} \to Y \) satisfies \( 0 \notin \cup_{t \in [0,1]}(L - H(t, \cdot))(D(L) \cap \partial \Omega) \), and \( \{L + \lambda JP - H(t, \cdot)\}_{t \in [0,1]} \) is a homotopy of \( A \)-proper maps with respect to \( \Gamma_{\lambda, L} \) for each

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\[ \lambda \in (0, \lambda_0), \text{ where } \lambda_0 > 0 \text{ is a constant, then } \deg(L - H(t, \cdot), \Omega, 0) \text{ does not depend on } t \in [0, 1]. \]

(3) If \( \deg_{\Gamma_\lambda}(L - N, \Omega, 0) \neq \{0\} \), then \( 0 \in (L - N)(D(L) \cap \Omega). \)

(4) If \( \Omega \) is a symmetric neighbourhood of 0, and \( N : \overline{\Omega} \to Y \) is an odd mapping such that \( L + \lambda JP - N \) is A-proper with respect to \( \Gamma_{\lambda, L} \) for each \( \lambda \in (0, \lambda_0) \), where \( \lambda_0 > 0 \) is a constant, and \( 0 \notin (L - N)(\partial \Omega \cap D(L)) \), then \( \deg(L - N, \Omega, 0) \) does not contain even numbers.

(5) \( \deg(L, \Omega, 0) \subseteq \{\pm 1\} \) if \( 0 \notin \Omega \).

**Proof.** (1). By assumption, there exists \( \lambda_0 > 0 \) such that

\[ 0 \notin (L + \lambda JP - N)(D(L) \cap \Omega \setminus (\Omega_1 \cup \Omega_2)) \]

for all \( \lambda \in (0, \lambda_0) \). If \( m \in \deg(L - N, \Omega, 0) \), then there exist \( \lambda_j \to 0^+, \lambda_j < \lambda_0, j = 1, 2, \ldots \), such that \( m \in \deg(L + \lambda_j JP - N, \Omega, 0) \). By Theorem 2.1 of [11], we have

\[ \deg(L + \lambda_j JP - N, \Omega, 0) \subseteq \deg(L + \lambda_j JP - N, \Omega_1, 0) + \deg(L + \lambda_j JP - N, \Omega_2, 0) \]

for \( j = 1, 2, \ldots \). Thus (1) follows from (2.2).

(2). Since \( 0 \notin \bigcup_{t \in [0, 1]} (L - H(t, \cdot))(D(L) \cap \partial \Omega) \), there exists \( \lambda_1 > 0 \) such that \( 0 \notin \bigcup_{t \in [0, 1]} (L + \lambda JP - H(t, \cdot))(\partial \Omega \cap D(L)) \) for \( \lambda \in (0, \lambda_1) \). By Theorem 2.1 of [11], \( \deg(L + \lambda JP - H(t, \cdot), \Omega, 0) \) does not depend on \( t \in [0, 1] \) for \( \lambda \in (0, \min(\lambda_0, \lambda_1)) \). So (2) follows from (2.2).

(3). If \( \deg_{\Gamma_\lambda}(L - N, \Omega, 0) \neq \{0\} \), then there exists \( 0 \neq m \in \deg_{\Gamma_\lambda}(L - N, \Omega, 0) \), so there exists \( \lambda_j \to 0^+ \) such that \( m \in \deg(L + \lambda_j JP - N, \Omega, 0) \). Therefore \( (L + \lambda_j JP - N) x \) has a solution in \( \Omega \cap D(L) \), \( j = 1, 2, \ldots \). By letting \( j \to \infty \), we obtain \( 0 \in (L - N)(D(L) \cap \Omega) \).

(4). We leave the proof to the reader.

(5). \( L + \lambda JP \) is A-proper with respect to \( \Gamma_{\lambda, L} \), and \( 0 \notin (L + \lambda JP)(\partial \Omega \cap D(L)) \) for all \( \lambda > 0 \). Since \( L + \lambda JP \) is bijective, \( \deg(L + \lambda JP, \Omega, 0) \subseteq \{\pm 1\} \) for all \( \lambda > 0 \). Thus we have

\[ \deg(L - N, \Omega, 0) \subseteq \{\pm 1\}. \]

**Theorem 2.5.** Let \( X, Y \) be real separable Banach spaces, and \( (Y_n, Q_n) \) a projectionally complete scheme for \( Y \), and let \( L : D(L) \subset X \to Y \) be a Fredholm mapping of zero index type, \( 0 \in \Omega \subset X \) a bounded subset, and \( N : \overline{\Omega} \to Y \) a continuous compact mapping. Suppose the following conditions are satisfied

(1) \( 0 \notin (L - N)(\partial \Omega \cap D(L)) \);
(2) \( 0 \notin Qn(\partial \Omega \cap D(L)) \).

Then \( \deg(L - N, \Omega, 0) = \deg(L - QN, \Omega, 0) \).

**Proof.** For each \( \lambda \in (0, \lambda_0) \), a similar proof to Proposition 1.11 shows that \( \{L + \lambda JP - tN - (1 - t)QN\}_{t \in [0, 1]} \) is a homotopy of A-proper maps with respect to \( \Gamma_{\lambda, L} \).

Now we claim that \( 0 \notin \bigcup_{t \in [0, 1]} (L - tN - (1 - t)QN)(D(L) \cap \partial \Omega) \).

If this is not true, then there exist \( t_j \in [0, 1] \) with \( t_j \to t_0, x_j \in \partial \Omega \cap D(L) \), such that \( Lx_j - t_j N x_j - (1 - t_j)QN x_j \to 0 \).

Case (1): if \( t_0 = 1 \), then \( Lx_j - N x_j \to 0 \), which is a contradiction to assumption (1).

Case (2): if \( t_0 \neq 1 \), then \( QLx_j - QN x_j \to 0 \), thus we have \( QN x_j \to 0 \) and \( x_j \in D(L) \), which is a contradiction to assumption (2).
By (2) of Theorem 2.4, we obtain $\deg(L - N, \Omega, 0) = \deg(L - QN, \Omega, 0)$. □

Finally, let $L : D(L) \subseteq X \rightarrow Y$ be a Fredholm mapping of index zero type, $\Gamma_0 = (X_n, P_n)$ a projectionally complete scheme for $X$, and $\Omega \subset X$ an open bounded subset, and let $N : \overline{\Omega} \rightarrow Y$ be a mapping such that $I - (L + \lambda JP)^{-1}(N + \lambda JP)$ is an A-proper map with respect to $\Gamma_0$ for some $\lambda > 0$. One can easily see that $0 \in Lx - Nx$ iff $0 \in (I - (L + \lambda JP)^{-1}(N + \lambda JP))x$. Assume that $0 \notin (L - N)(\partial\Omega \cap D(L))$. Then $0 \notin (I - (L + \lambda JP)^{-1}(N + \lambda JP))(\partial\Omega)$ for all $\lambda > 0$, and we define a generalized degree

$$\deg_{\Gamma_0}(L - N, \Omega, 0) = \cup_{0 < \lambda} \deg(I - (L + \lambda JP)^{-1}(N + \lambda JP), \Omega, 0),$$

where $\deg(I - (L + \lambda JP)^{-1}(N + \lambda JP), \Omega, 0)$ is the generalized degree for A-proper maps if $I - (L + \lambda JP)^{-1}(N + \lambda JP)$ is A-proper with respect to $\Gamma_0$, otherwise $\deg(I - (L + \lambda JP)^{-1}(N + \lambda JP), \Omega, 0) = \emptyset$.

**Theorem 2.6.** The generalized degree defined by (2.3) has the following properties.

1. If $\Omega_1$ and $\Omega_2$ are disjoint open subsets of $\Omega$ such that $0 \notin (L - N)(D(L) \cap \overline{\Omega \setminus (\Omega_1 \cup \Omega_2)})$, then

$$\deg_{\Gamma_0}(L - N, \Omega, 0) \subseteq \deg_{\Gamma_0}(L - N, \Omega_1) + \deg_{\Gamma_0}(L - N, \Omega_2, 0).$$

2. If $H(t, x) : [0, 1] \times \overline{\Omega} \rightarrow Y$ satisfies $0 \notin \cup_{t \in [0, 1]}(I - H(t, \cdot))(D(L) \cap \partial\Omega)$, and $\{I - (L + \lambda JP)^{-1}(H(t, \cdot) + \lambda JP)\}_{t \in [0, 1]}$ is a homotopy of A-proper maps with respect to $\Gamma_0$ for all $\lambda > 0$, then $\deg_{\Gamma_0}(L - H(t, \cdot), \Omega, 0)$ does not depend on $t \in [0, 1]$.

3. If $\deg_{\Gamma_0}(L - N, \Omega, 0) \neq \{0\}$, then $0 \notin (L - N)(D(L) \cap \Omega)$.

4. If $\Omega$ is a symmetric neighbourhood of 0, and $N : \overline{\Omega} \rightarrow Y$ is an odd mapping such that $I - (L + \lambda JP)^{-1}(N + \lambda JP)$ is A-proper with respect to $\Gamma_0$ for some $\lambda > 0$, and $0 \notin (L - N)(\partial\Omega \cap D(L))$, then $\deg_{\Gamma_0}(L - N, \Omega, 0)$ does not contain even numbers.

**Proof.** The proof is standard. We prove (2) and omit the others. Since $0 \notin \cup_{t \in [0, 1]}(I - H(t, \cdot))(D(L) \cap \partial\Omega)$, it follows that $0 \notin \cup_{t \in [0, 1]}(I - (L + \lambda JP)^{-1}(H(t, \cdot) + \lambda JP))(\partial\Omega)$ for all $\lambda > 0$. By Theorem 2.1 of [12], we know that $\deg(I - (L + \lambda JP)^{-1}(H(t, \cdot) + \lambda JP), \Omega, 0)$ does not depend on $t \in [0, 1]$ for each $\lambda > 0$. Thus (2) follows from (2.3). □

**Theorem 2.7.** Suppose that $(L + \lambda JP)^{-1} : Y \rightarrow X$ is a continuous compact mapping for each $\lambda > 0$, and $0 \notin \overline{\Omega} \subset X$ is an open bounded subset, $N : \overline{\Omega} \rightarrow Y$ is a continuous bounded mapping such that $Lx \neq Nx, x \notin \eta Px$ for all $x \in \partial\Omega \cap D(L), \eta > 0$, where $P, Q$ are projections as in section 1. Then $\deg(L - N, \Omega, 0) = [1]$.

**Proof.** Let $\Gamma_0 = (X_n, P_n)$ be a projectionally complete scheme for $X$. Since $(L + \lambda JP)^{-1} : Y \rightarrow X$ is continuous and compact for each $\lambda > 0$, it follows that $\{I - (L + \lambda JP)^{-1}t(N + \lambda JP)\}_{t \in [0, 1]}$ is a homotopy of A-proper maps with respect to $\Gamma_0$. We claim that $x \neq (L + \lambda JP)^{-1}t(N + \lambda JP)x$ for all $(t, x) \in [0, 1] \times (\partial\Omega \cap D(L))$, $\lambda > 0$. If this is not true, then there exist $\lambda_0 > 0$, $(t_0, x_0) \in [0, 1] \times \partial\Omega$ such that $x_0 = (L + \lambda_0 JP)^{-1}t_0(Nx_0 + \lambda_0 JPx_0)$. Thus we have $x_0 \in D(L)$, and $Lx_0 + \lambda_0 JPx_0 = t_0(Nx_0 + \lambda_0 JPx_0)$.

Obviously, $t_0 \neq 1$, therefore $(1 - t_0)\lambda_0 JPx_0 = t_0Q Nx_0$, which is a contradiction to one of our assumptions. Consequently, the A-proper degree $\deg(I - (L + \lambda JP)^{-1}(N + \lambda \eta$x_0)$ does not depend on $t \in [0, 1]$ for each $\lambda > 0$. Thus (2) follows from (2.3). □

Theorem 2.7. Suppose that $(L + \lambda JP)^{-1} : Y \rightarrow X$ is a continuous compact mapping for each $\lambda > 0$, and $0 \notin \overline{\Omega} \subset X$ is an open bounded subset, $N : \overline{\Omega} \rightarrow Y$ is a continuous bounded mapping such that $Lx \neq Nx, x \notin \eta Px$ for all $x \in \partial\Omega \cap D(L), \eta > 0$, where $P, Q$ are projections as in section 1. Then $\deg(L - N, \Omega, 0) = [1]$.

**Proof.** Let $\Gamma_0 = (X_n, P_n)$ be a projectionally complete scheme for $X$. Since $(L + \lambda JP)^{-1} : Y \rightarrow X$ is continuous and compact for each $\lambda > 0$, it follows that $\{I - (L + \lambda JP)^{-1}t(N + \lambda JP)\}_{t \in [0, 1]}$ is a homotopy of A-proper maps with respect to $\Gamma_0$. We claim that $x \neq (L + \lambda JP)^{-1}t(N + \lambda JP)x$ for all $(t, x) \in [0, 1] \times (\partial\Omega \cap D(L))$, $\lambda > 0$. If this is not true, then there exist $\lambda_0 > 0$, $(t_0, x_0) \in [0, 1] \times \partial\Omega$ such that $x_0 = (L + \lambda_0 JP)^{-1}t_0(Nx_0 + \lambda_0 JPx_0)$. Thus we have $x_0 \in D(L)$, and $Lx_0 + \lambda_0 JPx_0 = t_0(Nx_0 + \lambda_0 JPx_0)$.

Obviously, $t_0 \neq 1$, therefore $(1 - t_0)\lambda_0 JPx_0 = t_0Q Nx_0$, which is a contradiction to one of our assumptions. Consequently, the A-proper degree $\deg(I - (L + \lambda JP)^{-1}(N + \lambda \eta$x_0)$ does not depend on $t \in [0, 1]$ for each $\lambda > 0$. Thus (2) follows from (2.3). □
\[
\lambda J P, \Omega, 0) = \deg(I, \Omega, 0) = \{1\}. \quad \Box
\]

\textbf{Corollary 2.8.} Suppose that \( H \) is a separable Hilbert space, and \((L + \lambda J P)^{-1} : H \to X \) is a continuous compact mapping for each \( \lambda > 0 \), and \( 0 \in \Omega \subset X \) is an open bounded subset, and \( N : \Omega \to H \) is a continuous bounded mapping such that \( Lx \neq Nx \) for all \( x \in \partial \Omega \cap D(L) \), \( QNx \neq 0 \) for \( x \in \partial \Omega \cap D(L) \cap \text{Ker}(P) \), \((QN_x, JP_x) < 0\) for all \( x \in \partial \Omega \cap D(L) \cap (\text{Ker}(P))^c \), where \( P, Q \) are projections as in section 1. Then \( \deg(L - N, \Omega, 0) = \{1\} \).

\textit{Proof.} From our assumptions, we have \( QNx \neq \eta JPx \) for all \( x \in \partial \Omega \cap D(L), \eta > 0 \). Thus the conclusion follows from Theorem 2.7. \( \Box \)

\section{3. An Example.} Consider the following wave equation

\[
\begin{align*}
&u_{tt}(t, x) - u_{xx}(t, x) - h(u(t, x)) = f(t, x), \quad t \in (0, 2\pi), \quad x \in (0, \pi), \\
&u(t, 0) = u(t, \pi) = 0, \quad t \in (0, 2\pi), \\
&u(0, x) = u(2\pi, x), \quad x \in (0, \pi),
\end{align*}
\]

(E 3.1)

where \( h : R \to R \) is a continuous function satisfying

\[
|h(u)| \leq \delta |u| + \gamma, \quad (3.1)
\]

and \( f(\cdot) \in L^2((0, 2\pi) \times (0, \pi)) \), where \( \delta > 0, \gamma > 0 \) are constants.

We say \( u \in L^2((0, 2\pi) \times (0, \pi)) \) is a weak solution of (E 3.1) if

\[
(u, v_{tt} - v_{xx}) - (h(u(t, x)), v) = (f(t, x), v)
\]

for all \( v \in C^2([0, 2\pi] \times [0, \pi]) \) with \( v(t, 0) = v(t, \pi) = 0 \) for \( t \in [0, 2\pi] \), and \( v(2\pi, x) = v(0, x) \) for \( x \in [0, \pi] \).

Let \( L : D(L) \subset L^2((0, 2\pi) \times (0, \pi)) \to L^2((0, 2\pi) \times (0, \pi)) \) be the wave operator \( Lu = u_{tt} - u_{xx} \). Then it is well known that \( L \) is self-adjoint, densely defined, closed, and \( \text{Ker}(L) \) is finite dimensional with \( \text{Ker}(L)^c = \text{Im}(L) \). Thus \( L \) is a Fredholm mapping of zero index type. Let \( P : L^2((0, 2\pi) \times (0, \pi)) \to \text{Ker}(L) \) be the projection, then \((L + \lambda P)^{-1} : L^2((0, 2\pi) \times (0, \pi)) \to D(L) \) is compact for all \( \lambda > 0 \).

Let \( N : L^2((0, 2\pi) \times (0, \pi)) \to L^2((0, 2\pi) \times (0, \pi)) \) be defined by \( Nu(t, x) = h(u(t, x)) + f(t, x) \) for \( u(t, x) \in L^2((0, 2\pi) \times (0, \pi)) \). By (3.1), \( N \) is a bounded continuous mapping. For each \( \eta > 0 \), consider the following equation

\[
\begin{align*}
&u_{tt}(t, x) - u_{xx}(t, x) + nu(t, x) - h(u(t, x)) = f(t, x), \quad t \in (0, 2\pi), \quad x \in (0, \pi), \\
&u(t, 0) = u(t, \pi) = 0, \quad t \in (0, 2\pi), \\
&u(0, x) = u(2\pi, x), \quad x \in (0, \pi),
\end{align*}
\]

(E 3.2)

where \( h, f \) are as in (E 3.1). Let \( u_\eta \) be the weak solution of (E 3.2) if it exists, and we set \( S = \{u_\eta : \eta > 0\} \). Now we have the following alternative result.

\textbf{Theorem 3.1.} \( S \) is unbounded in \( L^2((0, 2\pi) \times (0, \pi)) \) or (E 3.1) has a weak solution.
Proof. We may assume that $S$ is bounded in $L^2((0, 2\pi) \times (0, \pi))$. So there exists $r_0 > 0$ such that

$$
\|u_0\|_{L^2} < r_0, \text{ for all } u_0 \in S. \quad (3.2)
$$

Let $\Omega = \{u(t, x) \in L^2((0, 2\pi) \times (0, \pi)) : \|u\|_{L^2} < r_0\}$. By (3.2), we know $PNu \neq \eta Pu$ for all $u \in C^2([0, 2\pi] \times [0, \pi]) \cap \partial \Omega$, and $\eta > 0$. We may assume that $Lu \neq Nu$ for all $u \in C^2([0, 2\pi] \times [0, \pi]) \cap \partial \Omega$.

By Theorem 2.7, we have $\text{deg}(L - N, \Omega, 0) = \{1\}$, thus (E 3.1) has a weak solution.

REFERENCES