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QUANTITATIVE APPROACH TO WEAK NONCOMPACTNESS IN THE POLYGON INTERPOLATION METHOD

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We study a quantitative approach to weak noncompactness of operators under the Cobos-Peetre polygon interpolation method for Banach N-tuples. In the case of operators acting between two J-spaces or two K-spaces obtained by this method we prove logarithmically convex-type inequalities for certain operator seminorm vanishing on the subspace of weakly compact operators. Geometrically speaking, in these estimates only some triangles inscribed in the polygon are involved. For operators acting from a J-space to a K-space we prove logarithmically convex-type estimates where all polygon vertices are included. In particular, the estimates obtained here give the new proofs of the results showing the relation between distribution of weakly compact operators among polygon vertices and weak compactness of operators under interpolation.

1. INTRODUCTION

A bounded linear operator $T\colon X \to Y$ acting between Banach spaces is said to be *weakly compact* if the image of the unit ball of X under T is a relatively weakly compact set in Y. This property is inherited by operators under the classical real and complex interpolation methods. More precisely, given Banach pairs (A_0, A_1) and (B_0, B_1) and interpolation spaces $A_{\theta,q}$, $B_{\theta,q}$ obtained by Lions and Peetre's **[20]** real method weak compactness of the restriction $T\colon A_0 \to B_0$ or $T\colon A_1 \to B_1$ implies that of $T\colon A_{\theta,q} \to B_{\theta,q}$ for all $\theta \in (0,1)$ and $q \in (1,\infty)$. An analogous implication holds for Calderón's **[3]** complex interpolation spaces $A_{[\theta]}$, $B_{[\theta]}$ with $\theta \in (0,1)$. For the real method we have even more: $T\colon A_{\theta,q} \to B_{\theta,q}$ with $\theta \in (0,1)$ and $q \in (1,\infty)$ is weakly compact if and only if so is $T\colon A_0 \cap A_1 \to B_0 + B_1$. The similar rule concerns some extensions of these methods to Banach N-tuples—the real methods of Yoshikawa **[27]** or Sparr **[25]** and (restricted to finite families) the complex method of Krein and Nikolova **[17]** or the so-called St. Louis **[11]** method (see **[2, 16, 21, 22, 23, 24]**).

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The behaviour of weak compactness for the Cobos-Peetre [**10**] polygon interpolation method for Banach N-tuples is different. The rule described above fails both for operators acting between two K-spaces and between two J-spaces obtained by this method. If we imagine that $T\colon A_j \to B_j$ is assigned to the jth vertex of a convex N-sided polygon Π, then T interpolated by J- or K-method for Π and $q \in (1,\infty)$ is weakly compact if so are $N-2$ restrictions $T\colon A_j \to B_j$ and the two left are located in some adjacent vertices (see [**4, 6**]). On the other hand, T acting from a J-space constructed for some $q \in (1,\infty)$ to the corresponding K-space is weakly compact if and only if so is the restriction of T acting from the intersection of all A_j to the sum of all B_j (see [**6**]).

In this paper using the seminorm introduced in [**19**], which measure deviation of bounded linear operators from weak compactness, we study behaviour of this property for operators under Cobos and Peetre's method. Following some norm estimates for operators proved in [**7, 8**] we establish logarithmically convex-type estimates for the norms of elements from J- and K-spaces with respect to the norms of their representations in some vector-valued l_q-spaces. In our main results we show that similar estimates hold for the seminorm. This is not the case for the seminorm based on the outer or inner measure related to the ideal of weakly compact operators (see [**5, 9**]). Weak noncompactness of an operator acting between two J-spaces or two K-spaces is estimated by weak noncompactness of restrictions $T\colon A_j \to B_j$ located at the vertices of selected triangles. In the case of an operator acting from a J-space to a K-space all restrictions are involved. In both cases logarithmic convexity holds within barycentric coordinates of interior points of Π (these points are the parameters of this method) with respect to polygon vertices. In particular, the results can be applied to weakly compact operators.

The space of all bounded linear operators between Banach spaces X and Y is denoted by $\mathcal{L}(X,Y)$. We write $\mathsf{B}(X)$ for the open unit ball of X. The convex hull of a set $A \subset X$ is denoted by $\operatorname{conv} A$.

2. The Cobos-Peetre polygon interpolation method

Let $\vec{A} = (A_1,\dots,A_N)$ be a *Banach N-tuple*, that is, a family of Banach spaces $A_1,\dots,A_N$ such that all of them are linearly and continuously embedded in a Hausdorff topological vector space E. The spaces

$$\Delta(\vec{A}) = A_1 \cap \cdots \cap A_N \quad \text{and} \quad \Sigma(\vec{A}) = A_1 + \cdots + A_N$$

are Banach spaces with norms

$$\|a\|_{\Delta(\vec{A})} = \max_{1\leqslant j\leqslant N} \|a\|_{A_j} \quad \text{and} \quad \|a\|_{\Sigma(\vec{A})} = \inf \sum_{j=1}^{N} \|a_j\|_{A_j},$$

where the infimum is taken over all decompositions $a_j \in A_j$, $a = \sum_{j=1}^{N} a_j$.

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We now recall the polygon interpolation method studied in [**10**]. Let $N \geqslant 3$ and $\Pi = P_1 \dots P_N$ be a convex polygon with vertices $P_j = (x_j, y_j)$, $j = 1, \dots, N$ in the affine plane $\mathbb{R}^2$. Given any positive numbers s and t we define the J-functional for every $a \in \Delta(\vec{A})$ by

$$J(s, t; a) = \max_{1 \leqslant j \leqslant N} \{ s^{x_j} t^{y_j} \|a\|_{A_j} \},$$

and the K-functional for every $a \in \Sigma(\vec{A})$ by

$$K(s, t; a) = \inf \left\{ \sum_{j=1}^{N} s^{x_j} t^{y_j} \|a_j\|_{A_j} : a_j \in A_j,\ a = \sum_{j=1}^{N} a_j \right\}.$$

These functionals equivalently renorm $\Delta(\vec{A})$ and $\Sigma(\vec{A})$, respectively.

In a discrete characterisation of the polygon interpolation method we put $s = 2^m$ and $t = 2^n$ for $z = (m, n) \in \mathbb{Z}^2$. For simplicity, let $J(2^z; a) = J(2^m, 2^n; a)$ and $K(2^z; a) = K(2^m, 2^n; a)$. We write $\langle \cdot\, , \cdot \rangle$ and $| \cdot |$ for the standard scalar product and norm in $\mathbb{R}^n$, respectively.

Set $q \in [1, \infty)$ and let $P \in \operatorname{Int} \Pi$ (the interior of Π). By the J-space $A_{P,q;J}$ we mean all $a \in \Sigma(\vec{A})$ for which there exists a family $u = \big(u(z)\big)_{z \in \mathbf{Z}^2} \subset \Delta(\vec{A})$ such that $a = \sum_{z \in \mathbf{Z}^2} u(z)$, convergence in $\Sigma(\vec{A})$, and

$$\|a\|_{A_{P,q;J}} = \inf \left(\sum_{z \in \mathbf{Z}^2} \left(2^{-\langle P, z \rangle} J\big(2^z; u(z)\big) \right)^q \right)^{1/q} < \infty,$$

the infimum being taken over all families u representing a just as before. By the K-space $A_{P,q;K}$ we mean all $a \in \Sigma(\vec{A})$ for which

$$\|a\|_{A_{P,q;K}} = \left(\sum_{z \in \mathbf{Z}^2} \left(2^{-\langle P, z \rangle} K(2^z; a) \right)^q \right)^{1/q} < \infty.$$

Under the above norms, $\Delta(\vec{A}) \subset A_{P,q;J} \subset A_{P,q;K} \subset \Sigma(\vec{A})$ with continuous inclusions. In the general case, $A_{P,q;J}$ and $A_{P,q;K}$ do not coincide. If Π is a triangle, then Cobos and Peetre's method is equivalent to Sparr's method for Banach 3-tuples. If Π is the unit square, then the polygon method coincides with Fernandez's [**13, 14**] method for 4-tuples. All these facts can be found in [**10**].

The definition given in [**10**] also covers $q = \infty$ but we shall not consider this case here. However, the norm estimates proved here for elements of J- or K-spaces can be easily extended to $q = \infty$. In this paper we deal with some discrete norms on $A_{P,q;J}$ and $A_{P,q;K}$ equivalent to $\|\cdot\|_{A_{P,q;J}}$ and $\|\cdot\|_{A_{P,q;K}}$, respectively. They will be introduced later.

For Banach N-tuples $\vec{A} = (A_1, \dots, A_N)$, $\vec{B} = (B_1, \dots, B_N)$ and a linear operator $T \colon \Sigma(\vec{A}) \to \Sigma(\vec{B})$ we write $T \colon \vec{A} \to \vec{B}$, if for $j = 1, \dots, N$ the restriction $T|A_j$ is a bounded operator into B_j.

Wherever we consider two families of interpolation spaces obtained by the polygon method for two Banach N-tuples $\vec{A}$ and $\vec{B}$ we assume that both families were obtained with respect to the same polygon Π.

From the interpolation viewpoint one of the basic facts concerning the polygon method is the following [**10**]: if $T\colon \vec{A} \to \vec{B}$, then $T\colon A_{P,q;J} \to B_{P,q;J}$, $T\colon A_{P,q;K} \to B_{P,q;K}$ and $T\colon A_{P,q;J} \to B_{P,q;K}$ are bounded for every $P \in \operatorname{Int}\Pi$.

3. Measures of weak noncompactness

We recall the measures of weak noncompactness introduced in [**19**]: γ for sets and, related to it, Γ for operators. Let (x_n) be a sequence in a Banach space X. We say that (y_n) is a sequence of successive convex combinations for (x_n) if there exist positive integers $0 = r_1 < r_2 < \cdots$ such that $y_n \in \operatorname{conv}\{x_i\}_{i=r_n+1}^{r_{n+1}}$ for each n. Vectors u_1, u_2 are said to be a pair of scc for (x_n) if $u_1 \in \operatorname{conv}\{x_i\}_{i=1}^{r}$ and $u_2 \in \operatorname{conv}\{x_i\}_{i=r+1}^{\infty}$ for some integer $r \geqslant 1$. The *convex separation* of (x_n) is defined by

$$\operatorname{csep}(x_n) = \inf\{\, \|u_1 - u_2\| : u_1, u_2 \text{ is a pair of scc for } (x_n)\,\}.$$

The *measure of weak noncompactness* γ is defined for every nonempty and bounded set $A \subset X$ by

$$\gamma(A) = \sup\{\operatorname{csep}(x_n)\colon (x_n) \subset \operatorname{conv} A\,\}.$$

The measure γ, based on James' criterion of weak compactness, has the following property: $\gamma(A) = 0$ if and only if A is a relatively weakly compact set in X. It is proved [**19**] that

$$\gamma(A) = \sup\{\, \lim_{m\to\infty} \lim_{n\to\infty} F_m(x_n) - \lim_{n\to\infty} \lim_{m\to\infty} F_m(x_n)\,\},$$

the supremum being taken over all $(x_n) \subset \operatorname{conv} A$ and $(F_m) \subset X^*$ with $\|F_m\| \leqslant 1$ and such that all the limits exist (here, X is taken over the real field), and

$$\gamma(A) = \sup \operatorname{dist}\bigl(x^{**}, \operatorname{conv}\{x_n\}\bigr),$$

the supremum being taken over all weak-star cluster points $x^{**} \in X^{**}$ of sequences $(x_n) \subset \operatorname{conv} A$ (here, x_n is identified with its canonical image in X^{**}). In general, γ is not equivalent to De Blasi's [**12**] measure of weak noncompactness.

Let X and Y be Banach spaces and let $\mathcal{W}(X,Y)$ be the subspace of $\mathcal{L}(X,Y)$ consisting of all weakly compact operators. The *measure of weak noncompactness of operators* is defined for every $T \in \mathcal{L}(X,Y)$ by

$$\Gamma(T) = \gamma\Bigl(T\bigl(\mathsf{B}(X)\bigr)\Bigr).$$

Clearly, $\Gamma(T) = 0$ if and only if $T \in \mathcal{W}(X,Y)$. The seminorm Γ is not equivalent to the so-called *weak essential norm* $\|T\|_{\mathcal{W}} = \operatorname{dist}\bigl(T, \mathcal{W}(X,Y)\bigr)$. A quantitative version of

Gantmacher's duality theorem holds for Γ (see [**18**]), which is not the case for $\|\cdot\|_W$ (see [**26**]). The measure Γ is equivalent to neither of the inner and outer measures (related to De Blasi's measure) for the ideal of weakly compact operators (see [**1, 15, 18**]). For more properties of γ and Γ we refer to [**18**] and [**19**].

Fix $q \in [1,\infty)$, $n \in \{1,2,\dots\}$ and $\zeta \in \mathbb{R}^n$. We denote by $l_q(\zeta, X)$ the Banach space of all families $x = \big(x(z)\big)_{z\in\mathbf{Z}^n} \subset X$ such that

$$\|x\|_{l_q(\zeta,X)} = \left(\sum_{z\in\mathbf{Z}^n}\left(2^{-\langle\zeta,z\rangle}\|x(z)\|_X\right)^q\right)^{1/q} < \infty.$$

The space $l_q(\zeta, X)$ is isometrically isomorphic to $l_q(0, X)$ with families indexed by natural numbers. Since γ is invariant under linear isometries, we can restate Theorem 3.6 from [**19**] for $l_q(\zeta, X)$. This theorem will be one of the key tools in our work.

THEOREM 3.1. *Let X and Y be Banach spaces, $T \in \mathcal{L}(X,Y)$ and $q \in (1,\infty)$. Suppose that $\widetilde{T} \in \mathcal{L}\big(l_q(\zeta,X), l_q(\zeta,Y)\big)$ is given by $\widetilde{T}x = \big(Tx(z)\big)$ for every $x = \big(x(z)\big) \in l_q(\zeta,X)$. Then $\Gamma(\widetilde{T}) = \Gamma(T)$.*

4. CASES $J \to J$ AND $K \to K$

Let us first set up some terminology. Let Ξ be a convex polygon with vertices $Q_1,\dots,Q_n$ in $\mathbb{R}^2$ and let Q belong to the interior or sides of Ξ. Any nonnegative $\theta_1,\dots,\theta_n$ such that $\sum_{j=1}^{n}\theta_j = 1$ and $Q = \sum_{j=1}^{n}\theta_j Q_j$ are called the *barycentric coordinates* of Q with respect to $Q_1,\dots,Q_n$. If moreover $\theta_j > 0$ for $j = 1,\dots,N$, then $\theta_1,\dots,\theta_n$ are said to be the *positive* barycentric coordinates of Q with respect to $Q_1,\dots,Q_n$. Let us recall two well-known facts. If Q is an interior point of Ξ, then there exist positive barycentric coordinates of Q with respect to $Q_1,\dots,Q_n$. If Ξ is a triangle, then the barycentric coordinates of Q with respect to Q_1, Q_2, Q_3 are unique.

To estimate Γ for operators acting between two J-spaces or two K-spaces we need certain estimates for the norms of elements from $A_{P,q;J}$ and $A_{P,q;K}$. Change of variables leads in this case to a discrete analogue of the function $D_{\alpha,\beta}$ considered in [**7**].

Let Π be a convex polygon with vertices $P_j = (x_j, y_j)$ for $j = 1,\dots,N$ and let $P = (\alpha,\beta)$ be an interior point of Π. For every $M = (M_1,\dots,M_N)$ with $M_j \geqslant 0$ for $j = 1,\dots,N$ define

$$d_P(M) = \inf_{z\in\mathbf{Z}^2}\max_{1\leqslant j\leqslant N}\left\{\,2^{\langle P_j - P, z\rangle}M_j\,\right\}.$$

Let $\mathcal{P}_P$ denote the set of all triples $\{j_1, j_2, j_3\}$ such that P belongs to the interior or sides of the triangle $\triangle P_{j_1}P_{j_2}P_{j_3}$. To cover in one formula the cases when P lies either in the interior of a triangle or on a diagonal of Π, we adopt here, as throughout, the convention that $0^0 = 1$.

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Lemma 4.1. *If $M = (M_1, \dots, M_N)$ and $M_j \geqslant 0$ for $j = 1, \dots, N$, then*

$$d_P(M) \leqslant c_P \max\{\, M_{j_1}^{\theta_1} M_{j_2}^{\theta_2} M_{j_3}^{\theta_3} \colon \{j_1, j_2, j_3\} \in \mathcal{P}_P \,\}, \tag{4.1}$$

where $\theta_1, \theta_2, \theta_3$ are the barycentric coordinates of P with respect to $P_{j_1}, P_{j_2}, P_{j_3}$ and

$$c_P = \max_{1 \leqslant j \leqslant N} \left\{\, 2^{\left(|x_j - \alpha| + |y_j - \beta|\right)/2} \,\right\}. \tag{4.2}$$

Proof: Fix $M = (M_1, \dots, M_N)$ and denote the right-hand side of (4.1) by δ. Let $P^+ = \{P_j \colon M_j > 0\}$ and $P^0 = \{P_j \colon M_j = 0\}$. Case $P^+ = \emptyset$ is obvious. Suppose that P^+ is nonempty and $\delta = 0$. Then each triangle containing P has a vertex from P^0 and if P lies on a diagonal of Π, one of the diagonal ends belongs to P^0. Therefore $P \notin \operatorname{conv} P^+$ and hence $0 = \inf\limits_{z \in \mathbb{Z}^2} \max\limits_{P_j \in P^+} \{2^{\langle P_j - P, z\rangle} M_j\} = d_P(M)$.

Suppose now that $\delta > 0$. By the proof of Theorem 1.9 in [7], there exist $r_0 \in \mathbb{R}^2$ and $\{j_1, j_2, j_3\} \in \mathcal{P}_P$ such that

$$\inf_{r \in \mathbb{R}^2} \max_{1 \leqslant j \leqslant N} \{\, 2^{\langle P_j - P, r\rangle} M_j \,\} = 2^{\langle P_{j_i} - P, r_0\rangle} M_{j_i} = M_{j_1}^{\theta_1} M_{j_2}^{\theta_2} M_{j_3}^{\theta_3}$$

for $i = 1, 2, 3$. Take $z_0 \in \mathbb{Z}^2$ and $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ with $|\rho_1|, |\rho_2| \leqslant 1/2$ such that $r_0 = z_0 - \rho$. Then $2^{\langle P_j - P, z_0\rangle} M_j \leqslant 2^{\langle P_j - P, \rho\rangle} M_{j_1}^{\theta_1} M_{j_2}^{\theta_2} M_{j_3}^{\theta_3}$ for $j = 1, \dots, N$. It follows that

$$d_P(M) \leqslant \max_{1 \leqslant j \leqslant N} \{\, 2^{\langle P_j - P, z_0\rangle} M_j \,\} \leqslant \max_{1 \leqslant j \leqslant N} \{\, 2^{\langle P_j - P, \rho\rangle} \,\} M_{j_1}^{\theta_1} M_{j_2}^{\theta_2} M_{j_3}^{\theta_3} \leqslant \delta,$$

which completes the proof. ∎

For our computations it will be more convenient to use some equivalent norms on $A_{P,q;J}$ and $A_{P,q;K}$. Put $X(j) = l_q(P - P_j, A_j)$ for $j = 1, \dots, N$. It is easy to check that $\|\cdot\|_{A_{P,q;J}}$ is equivalent to the norm given by

$$|||a|||_{A_{P,q;J}} = \inf \max_{1 \leqslant j \leqslant N} \|u\|_{X(j)}, \tag{4.3}$$

where the infimum is taken over all representations

$$\begin{cases} u = \big(u(z)\big)_{z \in \mathbb{Z}^2} \in X(1) \cap \dots \cap X(N) \\ a = \sum\limits_{z \in \mathbb{Z}^2} u(z) \text{ in } \Sigma(\vec{A}). \end{cases} \tag{4.4}$$

Similarly, $\|\cdot\|_{A_{P,q;K}}$ is equivalent to the norm given by

$$|||a|||_{A_{P,q;K}} = \inf \max_{1 \leqslant j \leqslant N} \|a_j\|_{X(j)}, \tag{4.5}$$

where the infimum is taken over all decompositions

$$\begin{cases} a_j = \big(a_j(z)\big)_{z \in \mathbb{Z}^2} \in X(j) \text{ for } j = 1, \dots, N \\ a = a_1(z) + \dots + a_N(z) \text{ for every } z \in \mathbb{Z}^2. \end{cases} \tag{4.6}$$

From now on, we assume that $A_{P,q;J}$ and $A_{P,q;K}$ are equipped with the norms described by (4.3) and (4.5), respectively.

PROPOSITION 4.2. *Let $\vec{A} = (A_1, \ldots, A_N)$ be a Banach N-tuple. Let $\Pi = P_1 \ldots P_N$ be a convex polygon and $P \in \operatorname{Int}\Pi$.*

(i) *If $a \in A_{P,q;J}$ and u is a representation of a as in (4.4), then*

$$|||a|||_{A_{P,q;J}} \leqslant c_P \max\left\{ \prod_{i=1}^{3} \|u\|_{X(j_i)}^{\theta_i} : \{j_1, j_2, j_3\} \in \mathcal{P}_P \right\}.$$

(ii) *If $a \in A_{P,q;K}$ and $a_1, \ldots, a_N$ is a decomposition of a as in (4.6), then*

$$|||a|||_{A_{P,q;K}} \leqslant c_P \max\left\{ \prod_{i=1}^{3} \|a_{j_i}\|_{X(j_i)}^{\theta_i} : \{j_1, j_2, j_3\} \in \mathcal{P}_P \right\}.$$

Here, c_P is given by (4.2), $X(j) = l_q(P - P_j, A_j)$ and $\theta_1, \theta_2, \theta_3$ are the barycentric coordinates of P with respect to $P_{j_1}, P_{j_2}, P_{j_3}$.

PROOF: If the elements $u = \big(u(z)\big)$ and $a_j = \big(a_j(z)\big)$ satisfy (4.4) and (4.6), respectively, then for every $w \in \mathbb{Z}^2$ do so $u^w = \big(u(z-w)\big)$ and $a_j^w = \big(a_j(z-w)\big)$. Clearly, $\|u^w\|_{X(j)} = 2^{\langle P_j - P, w\rangle} \|u\|_{X(j)}$ and $\|a_j^w\|_{X(j)} = 2^{\langle P_j - P, w\rangle} \|a_j\|_{X(j)}$. Combining these with Lemma 4.1, we obtain immediately our claims. ∎

In [**5**], some estimates of the type we consider in this paper were proved for the inner or outer measures related to operator ideals (in particular, to weakly compact operators) under the assumption that one of the N-tuples $\vec{A}$ or $\vec{B}$ reduces to a single Banach space. In the main result of this section no restrictions on N-tuples are required.

THEOREM 4.3. *Let $\vec{A} = (A_1, \ldots, A_N)$ and $\vec{B} = (B_1, \ldots, B_N)$ be Banach N-tuples and $q \in (1, \infty)$. Under the assumptions of Proposition 4.2, if $T\colon \vec{A} \to \vec{B}$, then*

$$\max\{\Gamma(T\colon A_{P,q;J} \to B_{P,q;J}), \Gamma(T\colon A_{P,q;K} \to B_{P,q;K})\}$$
$$\leqslant c_P \max\left\{ \prod_{i=1}^{3} \big(\Gamma(T\colon A_{j_i} \to B_{j_i})\big)^{\theta_i} : \{j_1, j_2, j_3\} \in \mathcal{P}_P \right\}.$$

PROOF: Write $X(j) = l_q(P - P_j, A_j)$ and $Y(j) = l_q(P - P_j, B_j)$ for $j = 1, \ldots, N$. Fix $\varepsilon > 0$.

CASE. $J \to J$. Let $(a_m) \subset \mathsf{B}(A_{P,q;J})$ and $b_m = Ta_m$. For each a_m there exists a representation $u_m = \big(u_m(z)\big)$ satisfying (4.4) and such that $u_m \in \mathsf{B}\big(X(j)\big)$ for $j = 1, \ldots, N$. Put $v_m = \big(Tu_m(z)\big)$. Of course, v_m is a representation of b_m as in (4.4) for $\vec{B}$.

Set $(v_m^0) = (v_m)$ and $(b_m^0) = (b_m)$. Consecutively for $j = 1, \ldots, N$, by [**19**, Theorem 2.1], we choose a sequence (v_m^j) of successive convex combinations for (v_m^{j-1}) such that

$$\|w_1^j - w_2^j\|_{Y(j)} \leqslant \operatorname{csep}(v_m^j)_{Y(j)} + \varepsilon$$

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for every pair w_1^j, w_2^j of successive convex combinations for (v_m^j). Here, $\mathrm{csep}(v_m^j)_{Y(j)}$ denotes the convex separation of (v_m^j) in the norm of $Y(j)$. Then $v_m^j = \sum_{i=r_m^j+1}^{r_{m+1}^j} t^j(i) v_i^{j-1}$ for some sequence of integers $0 = r_1^j < r_2^j < \cdots$ and nonnegative $t^j(r_m^j+1), \ldots, t^j(r_{m+1}^j)$ with sum 1 for $m = 1, 2, \ldots$ (the superscript j indicates the jth step). Put $b_m^j = \sum_{i=r_m^j+1}^{r_{m+1}^j} t^j(i) b_i^{j-1}$.

The relation successive convex combinations is transitive. Moreover, if (y_n) is a sequence of successive convex combinations for (x_n), then $\mathrm{csep}(x_n) \leqslant \mathrm{csep}(y_n)$. Thus we obtained the sequences (b_m^N) and (v_m^N) of successive convex combinations for (b_m) and (v_m), respectively, such that v_m^N is a representation of b_m^N as in (4.4) and

$$\left\| v_k^N - v_l^N \right\|_{Y(j)} \leqslant \mathrm{csep}(v_m^N)_{Y(j)} + \varepsilon$$

for $j = 1, \ldots, N$ and all k, l. Applying Proposition 4.2 (i) we obtain

$$\begin{aligned} \mathrm{csep}(b_m) &\leqslant \mathrm{csep}(b_m^N) \leqslant |||b_1^N - b_2^N|||_{B_{P,q;J}} \\ &\leqslant c_P \max\left\{ \prod_{i=1}^{3} \left\| v_1^N - v_2^N \right\|_{Y(j_i)}^{\theta_i} : \{j_1, j_2, j_3\} \in \mathcal{P}_P \right\} \\ &\leqslant c_P \max\left\{ \prod_{i=1}^{3} \left(\mathrm{csep}(v_m^N)_{Y(j_i)} + \varepsilon \right)^{\theta_i} : \{j_1, j_2, j_3\} \in \mathcal{P}_P \right\}. \end{aligned}$$

Define $\widetilde{T}_j \colon X(j) \to Y(j)$ for $j = 1, \ldots, N$ by $\widetilde{T}_j x = \big(Tx(z)\big)$ for every $x = \big(x(z)\big) \in X(j)$. Since $(v_m^N) \in \widetilde{T}_j\Big(\mathsf{B}\big(X(j)\big)\Big)$ for $j = 1, \ldots, N$, it follows that

$$\mathrm{csep}(b_m) \leqslant c_P \max\left\{ \prod_{i=1}^{3} \left(\Gamma(\widetilde{T}_{j_i}) + \varepsilon \right)^{\theta_i} : \{j_1, j_2, j_3\} \in \mathcal{P}_P \right\}.$$

By Theorem 3.1, $\Gamma(\widetilde{T}_j) = \Gamma(T \colon A_j \to B_j)$. Since the choice of ε and (a_m) was arbitrary, the proof of this case is complete.

Case. $K \to K$. Let now $(a_m) \subset \mathsf{B}(A_{P,q;K})$ and $b_m = Ta_m$. For each a_m there exists a decomposition $a_{j,m} = \big(a_{j,m}(z)\big) \in \mathsf{B}\big(X(j)\big)$, $j = 1, \ldots, N$ satisfying (4.6). Then $b_{j,m} = \big(Ta_{j,m}(z)\big)$, $j = 1, \ldots, N$ satisfy (4.6) for b_m and $\vec{B}$.

Similarly to Case $J \to J$ (see also [**19**, Theorem 3.8] for Banach pairs), we can find a sequence of integers $0 = p_1 < p_2 < \cdots$ and nonnegative $s(p_m+1), \ldots, s(p_{m+1})$ with sum 1 for $m = 1, 2, \ldots$ such that the sequences $(b'_{j,m})$ of successive convex combinations for $(b_{j,m})$ defined by $b'_{j,m} = \sum_{i=p_m+1}^{p_{m+1}} s(i) b_{j,i}$ for $j = 1, \ldots, N$ satisfy

$$\left\| b'_{j,k} - b'_{j,l} \right\|_{Y(j)} \leqslant \mathrm{csep}(b'_{j,m}) + \varepsilon$$

for all k, l. Putting $b'_m = \sum_{i=p_m+1}^{p_{m+1}} s(i)b_i$ we proceed as in the previous case applying now Proposition 4.2 (ii) and replacing b_m^N by b'_m and v_m^N with the norm of $Y(j)$ by $b'_{j,m}$. The assertion follows. ∎

Distributing weakly compact operators among the vertices of Π, we can now deduce a sufficient condition on weak compactness of $T\colon A_{P,q;J} \to B_{P,q;J}$ and $T\colon A_{P,q;K} \to B_{P,q;K}$ in the whole range of P. If $N \geqslant 4$, it is enough to consider those of P which lie on a diagonal of Π or, in other words, one of its barycentric coordinates with respect to some triangle is zero. To assure zero on the right-hand side of the inequality in Theorem 4.3, for each diagonal at least one of the operators located at the diagonal ends has to be weakly compact (according to our convention $0^0 = 1$, weak compactness at the third vertex is ignored). If Π is a triangle, weak compactness at any vertex is sufficient. In this way we get the same qualitative result as in [**6**] (see also [**4**]), where its optimality is examined as well.

COROLLARY 4.4. *If $T\colon A_j \to B_j$ is weakly compact for all indices $1 \leqslant j \leqslant N$ but two, say j_1 and j_2, such that the vertices P_{j_1} and P_{j_2} are adjacent, then $T\colon A_{P,q;J} \to B_{P,q;J}$ and $T\colon A_{P,q;K} \to B_{P,q;K}$ are weakly compact for all $P \in \operatorname{Int}\Pi$ and $q \in (1,\infty)$. In particular, if A_j is reflexive for all $1 \leqslant j \leqslant N$ but j_1 and j_2 as before, then so are $A_{P,q;J}$ and $A_{P,q;K}$.*

5. CASE $J \to K$

We first estimate K-norms of elements from a J-space by the norms of their representations in $l_q(P - P_j, A_j)$. Direct computations, similar to those in [**8**, Theorem 4.3] for operator norms, lead to estimates with disturbed barycentric coordinates of P in exponents. Moreover, the closer we are to some barycentric coordinates, the bigger is the constant of estimate. To avoid both problems, we shall use the fact observed in [**8**] that $A_{P,q;J}$ is embedded in Sparr's J- space and $A_{P,q;K}$ contains Sparr's K-space (both embeddings are continuous). Instead of Sparr's spaces equivalent Yoshikawa's [**27**] spaces will be used (for equivalence see [**25**, Remarks 4.5 and 4.6]).

We recall Yoshikawa's construction (our notation differs from the original one in [**27**]). Let $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$ and $\sum_{j=1}^{N} \theta_j = 1$ with all θ_j positive. Define the vectors $\xi_j = (\xi_2^j, \dots, \xi_N^j) \in \mathbb{R}^{N-1}$ by the following formula for $j = 1, \dots, N$ and $i = 2, \dots, N$:

$$\xi_i^j = \begin{cases} \theta_i - 1 & \text{if } j \geqslant 2 \text{ and } i = j \\ \theta_i & \text{elsewhere.} \end{cases} \tag{5.1}$$

Let $\vec{A} = (A_1, \dots, A_N)$ be a Banach N-tuple. Set $q \in [1,\infty)$ and let $V(j) = l_q(\xi_j, A_j)$ for $j = 1, \dots, N$. By the J-space $A_{\theta,q;J}$ we mean all $a \in \Sigma(\vec{A})$ which have a representation

u such that

$$\left\{\begin{array}{l} u = \big(u(w)\big)_{w\in\mathbb{Z}^{N-1}} \in V(1)\cap\cdots\cap V(N) \\ a = \sum\limits_{w\in\mathbb{Z}^{N-1}} u(w) \text{ in } \Sigma(\vec{A}). \end{array}\right. \tag{5.2}$$

The norm of $a \in A_{\theta,q;J}$ is given by

$$\|a\|_{A_{\theta,q;J}} = \inf \max_{1\leqslant j\leqslant N} \|u\|_{V(j)},$$

where the infimum is taken over all representations (5.2). By the K-space $A_{\theta,q;K}$ we mean all $a \in \Sigma(\vec{A})$ which have a decomposition $a_1, \dots, a_N$ such that

$$\left\{\begin{array}{l} a_j = \big(a_j(w)\big)_{w\in\mathbb{Z}^{N-1}} \in V(j) \text{ for } j = 1,\dots,N \\ a = a_1(w) + \cdots + a_N(w) \text{ for every } w \in \mathbb{Z}^{N-1}. \end{array}\right. \tag{5.3}$$

The norm of $a \in A_{\theta,q;J}$ is given by

$$\|a\|_{A_{\theta,q;K}} = \inf \max_{1\leqslant j\leqslant N} \|a_j\|_{V(j)},$$

where the infimum is taken over all decompositions (5.3).

The parameter $\theta = (\theta_1, \dots, \theta_N)$ will correspond to $P \in \operatorname{Int}\Pi$ with positive barycentric coordinates $\theta_1, \dots, \theta_N$. A particular role in our estimates will play the following constant:

$$c_\theta = 2^{\theta_2(1-\theta_2)+\cdots+\theta_N(1-\theta_N)}. \tag{5.4}$$

Note that $1 < c_\theta < 2$.

In the next two results we put some geometrical restriction on the polygon (concerning a triangle). Due to [**8**, Remark 4.1], any two polygons related by an affine isomorphism give equivalent interpolation spaces. Therefore our estimates remain valid in the general case up to a constant.

Proposition 5.1. *Let $\vec{A} = (A_1, \dots, A_N)$ be a Banach N-tuple. Let $\Pi = P_1 \dots P_N$ be a convex polygon with $P_1 = (0,0)$, $P_k = (1,0)$, $P_l = (0,1)$ for some $k, l \in \{2, \dots, N\}$ and let $\theta_1, \dots, \theta_N$ be some positive barycentric coordinates of $P \in \operatorname{Int}\Pi$ with respect to $P_1, \dots, P_N$. If $a \in A_{P,q;J}$ and u is a representation of a as in (4.4), then*

$$|||a|||_{A_{P,q;K}} \leqslant C_\theta \prod_{j=1}^{N} \|u\|_{X(j)}^{\theta_j},$$

where $X(j) = l_q(P - P_j, A_j)$ and C_θ depends only on $\theta_1, \dots, \theta_N$.

Proof:

STEP I. We first prove that for every $e \in A_{\theta,q;K}$

$$|||e|||_{A_{P,q;K}} \leqslant c_\theta \|e\|_{A_{\theta,q;K}}, \tag{5.5}$$

(compare to [8, Theorem 3.1]).

Define $f\colon \mathbb{Z}^2 \to \mathbb{Z}^{N-1}$ with $f(z) = (w_2, \dots, w_N)$, $z \in \mathbb{Z}^2$ satisfying

$$2^{w_i+\theta_i-1} \leqslant 2^{\langle P_i - P_1, z\rangle} < 2^{w_i+\theta_i}$$

for $i = 2, \dots, N$. Then

$$\begin{aligned} 2^{\langle P_1 - P, z\rangle} &= 2^{(1-\theta_1)\langle P_1, z\rangle} \prod_{i=2}^{N} (2^{\langle P_i, z\rangle})^{-\theta_i} \\ &\leqslant 2^{(1-\theta_1)\langle P_1, z\rangle} \prod_{i=2}^{N} \left(2^{w_i+\theta_i-1+\langle P_1, z\rangle}\right)^{-\theta_i} = \prod_{i=2}^{N} (2^{w_i+\theta_i-1})^{-\theta_i} = c_\theta 2^{-\langle \xi_1, f(z)\rangle}. \end{aligned}$$

Similarly, estimating from below we get

$$2^{\langle P_1 - P, z\rangle} > 2^{-\langle \xi_1, \xi_1\rangle} 2^{-\langle \xi_1, f(z)\rangle}.$$

If $2 \leqslant j \leqslant N$ and we put $I_j = \{2, \dots, N\} \backslash \{j\}$, then

$$\begin{aligned} 2^{\langle P_j - P, z\rangle} &= (2^{\langle P_j, z\rangle})^{1-\theta_j} 2^{-\theta_1\langle P_1, z\rangle} \prod_{i \in I_j} (2^{\langle P_i, z\rangle})^{-\theta_i} \\ &\leqslant \left(2^{w_j+\theta_j+\langle P_1, z\rangle}\right)^{1-\theta_j} 2^{-\theta_1\langle P_1, z\rangle} \prod_{i \in I_j} \left(2^{w_i+\theta_i-1+\langle P_1, z\rangle}\right)^{-\theta_i} \\ &= (2^{w_j+\theta_j})^{1-\theta_j} \prod_{i \in I_j} (2^{w_i+\theta_i-1})^{-\theta_i} = c_\theta 2^{-\langle \xi_j, f(z)\rangle} \end{aligned}$$

and

$$2^{\langle P_j - P, z\rangle} \geqslant 2^{-\langle \xi_j, \xi_j\rangle} 2^{-\langle \xi_j, f(z)\rangle}.$$

By the assumption on $\Delta P_1 P_k P_l$, if $y, z \in \mathbb{Z}^2$ and $y \neq z$, then

$$\max_{2 \leqslant i \leqslant N} \left|\langle P_i - P_1, y - z\rangle\right| \geqslant 1.$$

This shows that f is an injection. Fix $\varepsilon > 0$ and choose a representation e_j of e as in (5.3) and such that $\varepsilon + \|e\|_{A_{\theta,q;K}} \geqslant \max_{1 \leqslant j \leqslant N} \|e_j\|_{V(j)}$. It follows that

$$\begin{aligned} c_\theta\left(\varepsilon + \|e\|_{A_{\theta,q;K}}\right) &\geqslant c_\theta \max_{1 \leqslant j \leqslant N} \|e_j\|_{V(j)} \\ &\geqslant c_\theta \max_{1 \leqslant j \leqslant N} \left(\sum_{f(z) \in \mathbb{Z}^{N-1}} \left(2^{-\langle \xi_j, f(z)\rangle} \left\|e_j(f(z))\right\|_{A_j}\right)^q \right)^{1/q} \\ &\geqslant \max_{1 \leqslant j \leqslant N} \left(\sum_{z \in \mathbb{Z}^2} \left(2^{\langle P_j - P, z\rangle} \left\|e_j(f(z))\right\|_{A_j}\right)^q \right)^{1/q} \geqslant |||e|||_{A_{P,q;K}}. \end{aligned}$$

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Letting $\varepsilon \to 0$, we get (5.5).

STEP II. Let $d \in A_{\theta,q;J}$ and

$$\left\{ \begin{array}{l} Z_j = \left\{ w \in \mathbb{Z}^{N-1} : \langle \xi_j - \xi_k, w \rangle \geqslant 0,\ k = 1, \dots, N \right\} \\ C_\xi = \max\left\{ \sum\limits_{w \in Z_j} 2^{-\langle \xi_j, w \rangle} : j = 1, \dots, N \right\}. \end{array} \right. \tag{5.6}$$

Replacing in the proof of [27, Proposition 1.14] the 'continuous' norms by the 'discrete' ones considered here for Yoshikawa's spaces, we check that C_ξ is finite and

$$\|d\|_{A_{\theta,q;K}} \leqslant C_\xi \|d\|_{A_{\theta,q;J}} . \tag{5.7}$$

STEP III. In order to obtain the logarithmically convex-type inequality

$$\|d\|_{A_{\theta,q;J}} \leqslant c_\theta \prod_{j=1}^{N} \|y\|_{V(j)}^{\theta_j} \tag{5.8}$$

for every $d \in A_{\theta,q;J}$ and its representation $y = \big(y(w)\big)_{w \in \mathbf{Z}^{N-1}}$ satisfying (5.2), it is sufficient to change variables as follows. For $y \neq 0$ (case $y = 0$ is obvious), we put $y^v = \big(y(w + v)\big)_{w \in \mathbf{Z}^{N-1}}$, where $v = (v_2, \dots, v_N) \in \mathbb{Z}^{N-1}$ satisfies $2^{v_i + \theta_i - 1} \leqslant \|y\|_{V(i)} \|y\|_{V(1)}^{-1} < 2^{v_i + \theta_i}$ for $i = 2, \dots, N$. Proceeding analogously to the estimates for $2^{\langle P_j - P, z \rangle}$, we obtain

$$\max_{1 \leqslant j \leqslant N} \left\{ 2^{\langle \xi_j, v \rangle} \|y\|_{V(j)} \right\} \leqslant c_\theta \prod_{j=1}^{N} \|y\|_{V(j)}^{\theta_j}$$

Since $2^{\langle \xi_j, v \rangle} \|y\|_{V(j)} = \|y^v\|_{V(j)}$, (5.8) holds (compare to [27, Proposition 2.5]).

STEP IV. Let $u = \big(u(z)\big)_{z \in \mathbf{Z}^2}$ be a representation of $a \in A_{P,q;J}$ satisfying (4.4). Using the injection $f : \mathbb{Z}^2 \to \mathbb{Z}^{N-1}$ described in Step I, we define $\widehat{u} = \big(\widehat{u}(w)\big)_{w \in \mathbf{Z}^{N-1}}$ by

$$\widehat{u}(w) = \left\{ \begin{array}{ll} u(z) & \text{if } w = f(z) \\ 0 & \text{elsewhere.} \end{array} \right.$$

Then $a = \sum\limits_{w \in \mathbf{Z}^{N-1}} \widehat{u}(w)$ in $\Sigma(\vec{A})$ and for $j = 1, \dots, N$

$$\begin{aligned} \|\widehat{u}\|_{V(j)} &= \left(\sum_{f(z) \in \mathbf{Z}^{N-1}} \left(2^{-\langle \xi_j, f(z) \rangle} \left\| \widehat{u}(f(z)) \right\|_{A_j} \right)^q \right)^{1/q} \\ &\leqslant \left(\sum_{z \in \mathbf{Z}^2} \left(2^{\langle \xi_j, \xi_j \rangle} 2^{\langle P_j - P, z \rangle} \left\| u(z) \right\|_{A_j} \right)^q \right)^{1/q} = 2^{\langle \xi_j, \xi_j \rangle} \|u\|_{X(j)} . \end{aligned}$$

Thus $\widehat{u}$ is a representation of a as in (5.2). Applying (5.8) we get

$$\|a\|_{A_{\theta,q;J}} \leqslant c_\theta \prod_{j=1}^{N} \|\widehat{u}\|_{V(j)}^{\theta_j} \leqslant c_\theta \prod_{j=1}^{N} 2^{\theta_j \langle \xi_j, \xi_j \rangle} \|u\|_{X(j)}^{\theta_j} .$$

Since $\sum_{j=1}^{N} \theta_j \langle \xi_j, \xi_j \rangle = \sum_{j=2}^{N} \theta_j (1 - \theta_j)$, we have

$$\|a\|_{A_{\theta,q;J}} \leqslant c_\theta^2 \prod_{j=1}^{N} \|u\|_{X(j)}^{\theta_j} . \tag{5.9}$$

Finally, combining (5.5), (5.7) and (5.9), we get the assertion with

$$C_\theta = C_\xi c_\theta^3, \tag{5.10}$$

where C_ξ and c_θ given by (5.6) and (5.4), respectively, depend only on $\theta_1, \ldots, \theta_N$. ∎

In [6], it was proved that $T\colon A_{P,q;J} \to B_{P,q;K}$ with $P \in \operatorname{Int}\Pi$ and $q \in (1,\infty)$ is weakly compact if and only if so is $T\colon \Delta(\vec{A}) \to \Sigma(\vec{B})$. In particular, if $T\colon A_j \to B_j$ is weakly compact for at least one $j = 1, \ldots, N$, then so is $T\colon A_{P,q;J} \to B_{P,q;K}$ for all $P \in \operatorname{Int}\Pi$ and $q \in (1,\infty)$. This fact can be also derived as a corollary from our next result. Repeating arguments from the proof of Theorem 4.3, $J \to J$, and applying Proposition 5.1, we obtain the main theorem of this section.

Theorem 5.2. *Let $\vec{A} = (A_1, \ldots, A_N)$ and $\vec{B} = (B_1, \ldots, B_N)$ be Banach N-tuples and $q \in (1,\infty)$. Under the assumptions of Proposition 5.1, if $T\colon \vec{A} \to \vec{B}$, then*

$$\Gamma(T\colon A_{P,q;J} \to B_{P,q;K}) \leqslant C_\theta \prod_{j=1}^{N} (\Gamma(T\colon A_j \to B_j))^{\theta_j} ,$$

where C_θ given by (5.10) depends only on $\theta_1, \ldots, \theta_N$.

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