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ON THE TENSOR PRODUCTS OF JW-ALGEBRAS

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ABSTRACT. In this article we introduce and develop a theory of tensor products of JW-algebras. Since JW-algebras are so close to W^*-algebras, one can expect that the W^*-algebra tensor product theory will be actively involved. It is shown that if M and N are JW-algebras with centres Z_1 and Z_2 respectively, then $Z_1 \otimes Z_2$ is not the centre of the JW-tensor product $\mathrm{JW}(M\bar{\otimes}N)$ (see below for notation) of M and N, in general. Also, the type decomposition of $\mathrm{JW}(M\bar{\otimes}N)$ has been determined in terms of the type decomposition of the JW-algebras M and N which, essentially, rely on the relationship between the types of the JW-algebra and the types of its universal enveloping Von Neumann algebra.

1. **Introduction.** Throughout this paper we rely on the theory of tensor products of Von Neumann algebras. Our standard references will be [11, 13, 14, 15, 20]. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, then the minimal C^*-tensor product of $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \otimes_{\min} \mathcal{B}$. Given Von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ acting on Hilbert spaces H and K, respectively, *the Von Neumann tensor product of* $\mathcal{M}$ *and* $\mathcal{N}$ is the Von Neumann algebra generated by the algebraic tensor product $\mathcal{M} \otimes \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{B}(H \otimes K)$. It is known that if $\mathcal{M}$ and $\mathcal{N}$ are W^*-algebras, then there is a unique central projection Z in $(\mathcal{M} \otimes_{\min} \mathcal{N})^{**}$ such that $\mathcal{M} \otimes_{\min} \mathcal{N}$ is identified with a weak-$*$ dense C^*-subalgebra of $(\mathcal{M} \otimes_{\min} \mathcal{N})^{**}Z$. The W^*-algebra $(\mathcal{M} \otimes_{\min} \mathcal{N})^{**}Z$ is called *the* W^**-tensor product of* $\mathcal{M}$ *and* $\mathcal{N}$, and is denoted by $\mathcal{M}\bar{\otimes}\mathcal{N}$ [15, p. 66], [20, p. 221]. If $\pi_1\colon \mathcal{M} \longrightarrow \mathcal{B}(H)$ and $\pi_2\colon \mathcal{N} \longrightarrow \mathcal{B}(K)$ are faithful representations of W^*-algebras, then the W^*-tensor product of $\mathcal{M}$ and $\mathcal{N}$ is isomorphic to the Von Neumann tensor product of $\pi_1(\mathcal{M})$ and $\pi_2(\mathcal{N})$. Based on this fact, the Von Neumann tensor product of two Von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ will be, also, denoted by $\mathcal{M}\bar{\otimes}\mathcal{N}$ without specifying the Hilbert spaces on which $\mathcal{M}$ and $\mathcal{N}$ act.

If M is a JW-algebra, let $W^*(M)$ be the universal enveloping Von Neumann algebra of M, and let ϕ_M be the canonical involutary $*$-antiautomorphism of $W^*(M)$. Usually we will regard M to be a generating Jordan subalgebra of $W^*(M)$ so that ϕ_M fixes each point of M. The real Von Neumann algebra $\mathrm{RW}^*(M) = \{x \in W^*(M) : \phi_M(x) = x^*\}$ satisfies $\mathrm{RW}^*(M) \cap i\,\mathrm{RW}^*(M) = 0$, $W^*(M) = \mathrm{RW}^*(M) \oplus i\,\mathrm{RW}^*(M)$. If M is a JC-algebra, let $C^*(M)$ be the universal enveloping C^*-algebra of M. It is known that $C^*(M)^{**} \simeq W^*(M^{**})$. The reader is referred to [9, Chapter 7] for the properties of $C^*(M)$ and $W^*(M)$.

Let M be a JW-algebra. M is said to be *finite* if every family of orthogonal equivalent projections is finite. A projection e in M is said to be *finite* if eMe is a finite JW-algebra and to be *abelian* if eMe is abelian. Further, M is said to be of

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(a) *Type I* if $c_M(e) = 1$ for some abelian projection e of M, where $c_M(e)$ is the central support of e in M.
(b) *Type* I_{fin} if M is of Type I and finite.
(c) *Type* I_∞ if M is of Type I with no non-zero central finite projections,
(d) *Type* II_1 if M is finite without non-zero abelian projections,
(e) *Type* II_∞ if $c_M(e) = 1$ for some finite projection e of M, and M has no Type I or II_1 summand,
(f) *Type III* if M has no non-zero finite projections.

Every JW-algebra decomposes into a direct sum of some or all of the five types (b)–(f) [21, Theorem 13].

Given $n < \infty$ and orthogonal equivalent abelian projections $e_1, \ldots, e_n$ in M such that $\sum_{i=1}^n e_i = 1$, M is said to be of *Type* I_n; and M is of Type I_{fin} if and only if $M = \sum_{n\in S}^{\oplus} M_n$, where $S \subseteq \mathbb{N}$ (possibly infinite) and each M_n is a Type I_n JW-algebra.

In addition, the Type I_n JW-algebras ($n < \infty$) decompose into distinct types. In order to describe these we will say that:

(i) M is of Type $I_{n,\mathbb{F}}$, where $n < \infty$ and $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ if $M \simeq C\big(X, M_n(\mathbb{F})_{s.a}\big)$, for some compact hyperstonean space X.
(ii) M is of Type $I_{2,k}$ if every factor representation of M is onto the spin factor V_k. If $k < \infty$, this means that $M = C(X, V_k)$ for some compact hyperstonean space X; and when k is an infinite cardinal, it is equivalent to the existence of a weak-$*$ dense JC-subalgebra in M of the form $C(X, V_k)$, for some compact Hausdorff space X (Stacey [17]).

If M is of Type I_n, $3 \leq n < \infty$, then M is a direct sum of the three types described in (i). If M is of Type I_2 then there exists a subset $S \subseteq \mathbb{N}$ and a set S_∞ of infinite cardinals such that

$$M = \sum_{k\in S}{}^{\oplus} M_k \oplus \sum_{k\in S_\infty}{}^{\oplus} M_k,$$

where each M_k is of Type $I_{2,k}$. We will call $\sum_{k\in S_\infty}^{\oplus} M_k$ the Type $I_{2,\infty}$ part of M (see the proof of Theorem 2.7 of [4], references [9, 16, 17, 21]).

Let A and B be JC-algebras. We may suppose that A and B are canonically embedded in their respective universal enveloping C^*-algebras $C^*(A)$, $C^*(B)$. The completion $\mathrm{JC}(A \otimes_{\min} B)$ of the real Jordan algebra $J(A \otimes B)$ generated by $A \otimes B$ in $C^*(A) \otimes_{\min} C^*(B)$ is called *the* JC-*tensor product of A and B with respect to the minimal C^*-norm on* $C^*(A) \otimes C^*(B)$. For a detailed account of the theory of tensor products of JC-algebras, the reader is referred to [5].

2. **Tensor products of** JW-**algebras.**

Each JW-algebra sits inside a Von Neumann algebra so it is natural to seek means of exploiting the theory of tensor products of Von Neumann algebras in order to produce a theory of tensor products of JW-algebras. In this section, we introduce the definition of the tensor product of two JW-algebras. Theorem 2.9 is the main result of this section, and it provides a useful tool in the succeeding sections.

DEFINITION 2.1. Let $M \subseteq \mathcal{B}(H)_{s.a}$ and $N \subseteq \mathcal{B}(K)_{s.a}$ be JW-algebras, where H and K are complex Hilbert spaces. We denote by $J_{H,K}(M \otimes N)$ the Jordan subalgebra of $\mathcal{B}(H \otimes K)_{s.a}$ generated by the (real) algebraic tensor product $M \otimes N$ of M and N. The corresponding weak-operator closure will be denoted by $\mathrm{JW}_{H,K}(M \bar{\otimes} N)$. Then $\mathrm{JW}_{H,K}(M \bar{\otimes} N)$ is a JW-algebra, and is called *the* $\mathrm{JW}_{H,K}$*-tensor product of* M *and* N. If $[M]^-$, $[N]^-$ denote the Von Neumann subalgebras of $\mathcal{B}(H)$, $\mathcal{B}(K)$ generated by M, N respectively, then by definition, the Von Neumann algebra $[\mathrm{JW}_{H,K}(M \bar{\otimes} N)]^-$ generated by $\mathrm{JW}_{H,K}(M \bar{\otimes} N)$ in $\mathcal{B}(H \otimes K)$ is $[M]^- \bar{\otimes} [N]^-$.

DEFINITION 2.2. Given JW-algebras M and N, regarded as canonically embedded in $W^*(M)$ and $W^*(N)$, respectively, we define *the* JW*-tensor product of* M *and* N to be the JW-algebra generated by $M \otimes N$ in $W^*(M) \bar{\otimes} W^*(N)$. We denote it by $\mathrm{JW}(M \bar{\otimes} N)$.

Let M and N be JW-algebras. By [4, Theorem 2.7] we can realize $C^*(M)$ as the C^*-algebra of $W^*(M)$ generated by M. Consequently we have the natural inclusions

$$J(M \otimes N) \subset C^*(M) \otimes C^*(N) \subset W^*(M) \otimes W^*(N).$$

Thus, it is clear that $\mathrm{JW}(M \bar{\otimes} N)$ is the weak-$*$ closure of $J(M \otimes N)$ in $W^*(M) \bar{\otimes} W^*(N)$. Using [20, 4.4.22] we have further inclusions

$$J(M \otimes N) \subset \mathrm{JC}(M \otimes_{\min} N) \subset C^*(M) \otimes_{\min} C^*(N) \subset W^*(M) \otimes_{\min} W^*(N) \subset W^*(M) \bar{\otimes} W^*(N)$$

where $W^*(M) \otimes_{\min} W^*(N)$ is of course weak-$*$ dense in $W^*(M) \bar{\otimes} W^*(N)$. Thus we also deduce that $\mathrm{JC}(M \otimes_{\min} N)$ is weak-$*$ dense in $\mathrm{JW}(M \bar{\otimes} N)$.

LEMMA 2.3. *Let* $M \subseteq \mathcal{B}(H)_{s.a}$ *and* $N \subseteq \mathcal{B}(K)_{s.a}$ *be* JW*-algebras. Then there is a normal Jordan homomorphism of* $\mathrm{JW}(M \bar{\otimes} N)$ *onto* $\mathrm{JW}_{H,K}(M \bar{\otimes} N)$.

PROOF. Let $\iota_M: M \longrightarrow [M]^-$, and $\iota_N: N \longrightarrow [N]^-$ be the inclusion maps. Then consider the surjective normal $*$-homomorphisms $\hat{\iota}_M: W^*(M) \longrightarrow [M]^-$, $\hat{\iota}_N: W^*(N) \longrightarrow [N]$. This gives rise to the surjective normal $*$-homomorphism $\hat{\iota}_M \bar{\otimes} \hat{\iota}_N$ of $W^*(M) \bar{\otimes} W^*(N)$ onto $[M]^- \bar{\otimes} [N]^-$. Let π be the restriction of $\hat{\iota}_M \bar{\otimes} \hat{\iota}_N$ to $\mathrm{JW}(M \bar{\otimes} N)$. Thus $\pi\big(\mathrm{JW}(M \bar{\otimes} N)\big)$ is a JW-subalgebra of $([M]^- \bar{\otimes} [N]^-)_{s.a}$, by [9, 4.5.5 and 4.5.11]. Since M and N generate $\mathrm{JW}(M \bar{\otimes} N)$ and $\mathrm{JW}_{H,K}(H \otimes N)\pi$ is surjective, proving the lemma.

The next result is apparent from Lemma 2.3 and [20, 4.5.2].

LEMMA 2.4. *Let* M *and* N *be* JW*-algebras. If* $\pi_1: M \longrightarrow \mathcal{B}(H)_{s.a}$, *and* $\pi_2: N \longrightarrow \mathcal{B}(K)_{s.a}$ *are faithful normal representations of* M *and* N *which extend to faithful normal representations* $\hat{\pi}_1: W^*(M) \longrightarrow \mathcal{B}(H)$, $\hat{\pi}_2: W^*(N) \longrightarrow \mathcal{B}(K)$, *then*

$$\mathrm{JW}(M \bar{\otimes} N) \simeq \mathrm{JW}_{H,K}(M \bar{\otimes} N).$$

REMARK 2.5. Let $M \subseteq \mathcal{B}(H)_{s.a}$ be a universally reversible JW-algebra with no non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra, then $[M]^- \simeq W^*(M)$. Indeed, if $R(M) \cap iR(M) = 0$, then the conclusion follows by Lemma 2.3. If $R(M) \cap iR(M) \neq 0$, then $J = R(M)^{-} \cap iR(M)^{-} \neq 0$, by [19, Lemma 2.3], and is a non-zero weakly-closed two sided ideal of $[M]^{-}$ (see [18, Remark 2.2]). In addition,

$$J_{s.a} = \left(R(M)^{-} \cap iR(M)^{-}\right)_{s.a} \subset R(M)^{-}_{s.a} = M,$$

by [2, p. 279]. Hence, $J_{s.a}$ is a non-zero weakly-closed Jordan ideal of M. This contradiction proves the assertion.

COROLLARY 2.6. *Let $M \subset \mathcal{B}(H)_{s.a}$, and $N \subset \mathcal{B}(K)_{s.a}$ be universally reversible* JW-*algebras such that M and N contain no non-zero weakly-closed Jordan ideals isomorphic to the self-adjoint part of a Von Neumann algebra. Then*

$$\mathrm{JW}(M \bar{\otimes} N) \simeq \mathrm{JW}_{H,K}(M \bar{\otimes} N).$$

It is easy to see that if M is a JW-algebra, then M is reversible in $W^{*}(M)$ if and only if M is universally reversible. Thus, given a JC-algebra A, the separate weak $*$-continuity of multiplication together with the fact that $C^{*}(A)^{**} = W^{*}(A^{**})$ [9, 7.1.11] imply that A is universally reversible if and only if A^{**} is universally reversible.

PROPOSITION 2.7. *Let M and N be* JW-*algebras. Then* $\mathrm{JW}(M \bar{\otimes} N)$ *is universally reversible unless one of M, N has a non-zero abelian part and the other a non-zero Type $I_{2,k}$ part, for some cardinal number (possibly infinite) ≥ 4.*

PROOF. Suppose that the condition on M and N fails to occur. Then $\mathrm{JC}(M \otimes_{\min} N)$ is universally reversible, by [5, Corollary 1.5]. Hence $\mathrm{JW}(M \bar{\otimes} N)$ is universally reversible because it is the weak-closure of $\mathrm{JC}(M \otimes_{\min} N)$ in $W^{*}(M) \bar{\otimes} W^{*}(N)$.

On the other hand, suppose that M has a non-zero abelian part $M_a \simeq C_{\mathbb{R}}(X)$, say, where X is a compact hyperstonean space and N has a non-zero Type $I_{2,k}$ part M_k, for some cardinal number $k \geq 4$. Then using Stacey's result [17], and Grothendieck's result [20] we see that, since $M_a \bar{\otimes} M_k$ has a weakly dense subalgebra isomorphic to $C_{\mathbb{R}}(X) \otimes_{\lambda} \left(C(Y) \otimes_{\lambda} V_k\right) = C_{\mathbb{R}}(X \times Y) \otimes_{\lambda} V_k$, where $C_{\mathbb{R}}(Y) \simeq Z(M_k)$ and $\lambda =$ least cross norm. $\mathrm{JW}(M \bar{\otimes} N)$ has a non-zero Type $I_{2,k}$ part. Hence $\mathrm{JW}(M \bar{\otimes} N)$ is not universally reversible.

REMARK. Corollary 2.6 cannot be improved, and as examples of failure:

(a) Let $M = N = V_5\left(\simeq M_2(\mathbb{H})_{s.a}\right)$. By [3, p. 385]

$$\mathrm{JW}(M \bar{\otimes} N) \simeq \bigoplus_{1}^{4} M_{16}(\mathbb{R})_{s.a}.$$

However there is a natural embedding $\pi \colon V_5 \longrightarrow \mathcal{B}(H)$, where $H = \mathbb{C}^4$, in which case, $R\left(\pi(V_5)\right) \simeq M_2(\mathbb{H})$ and we see that

$$\mathrm{JW}_{H,H}(M \bar{\otimes} N) \simeq M_{16}(\mathbb{R})_{s.a},$$

and hence, $\mathrm{JW}(M \bar{\otimes} N) \not\simeq \mathrm{JW}_{H,H}(M \bar{\otimes} N)$.

(b) Put $M = N = M_n(\mathbb{C})_{s.a}$. We note that $M_n(\mathbb{C})_{s.a} \otimes M_n(\mathbb{C})_{s.a} = M_{n^2}(\mathbb{C})_{s.a}$. So from the natural identification $M_n(\mathbb{C}) = \mathcal{B}(H)$, where $H = \mathbb{C}^n$ we have

$$\mathrm{JW}_{H,H}(M\bar{\otimes}N) = M_{n^2}(\mathbb{C})_{s.a}.$$

On the other hand, we have

$$\begin{aligned}\mathrm{JW}(M\bar{\otimes}N) &= \Big(\mathrm{RW}^*(M)\bigotimes_{\mathbb{R}}\mathrm{RW}^*(N)\Big)_{s.a}\\ &= \Big(M_n(\mathbb{C})\bigotimes_{\mathbb{R}}M_n(\mathbb{C})\Big)_{s.a}\\ &= \big(M_{n^2}(\mathbb{C})_{s.a}\oplus M_{n^2}(\mathbb{C})\big)_{s.a}.\end{aligned}$$

That is $\mathrm{JW}(M\bar{\otimes}N) \not\simeq \mathrm{JW}_{H,H}(M\bar{\otimes}N)$.

LEMMA 2.8. *Let $M \subset \mathcal{B}(H)_{s.a}$ be a universally reversible* JW*-algebra. Then* $\mathrm{RW}^*(M) = R(M)^-$. *Further, if $R(M)\cap iR(M) = 0$, then $W^*(M) = [M]^-$.*

PROOF. The natural inclusion $M \longrightarrow \mathcal{B}(H)_{s.a}$ gives rise to the real weak $*$-continuous surjection $\pi\colon \mathrm{RW}^*(M) \longrightarrow R(M)^-$, where $\pi\circ\psi_M(x) = x$, for all $x \in M$, by the universal property. But note, $x \in \operatorname{Ker}\pi$ implies that $x^*x \in \mathrm{RW}^*(M)_{s.a} = \psi_M(M)$, so that with $\psi_M(y) = x^*x$ we have $y = \pi\circ\psi_M(y) = \pi(x^*x) = 0$. Hence $x = 0$, proving the first statement. The second statement is now immediate from the fact that $R(M)^-\cap iR(M)^- = 0$ [19, Lemma 2.3].

NOTATION. Let $\mathcal{R}$ and $\mathcal{S}$ be real subspaces of Von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, respectively. We denote by $\mathcal{R}\bar{\otimes}\mathcal{S}$ the real Von Neumann subalgebra of $\mathcal{M}\bar{\otimes}\mathcal{N}$ generated by the set $\{\sum_{j=1}^{n} r_j\otimes s_j, r_j \in \mathcal{R}, s_j \in \mathcal{S}, j = 1,\ldots,n\}$.

THEOREM 2.9. *Let M and N be* JW*-algebras, and ϕ_M, ϕ_N be the canonical $*$-antiautomorphisms of $W^*(M)$, $W^*(N)$, respectively. If* $\mathrm{JW}(M\bar{\otimes}N)$ *is universally reversible, then*

(i) $W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) = W^*(M)\bar{\otimes}W^*(N)$.

(ii) $\phi_M\bar{\otimes}\phi_N$ *is the canonical $*$-antiautomorphism of* $W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big)$, *in which case,* $\mathrm{JW}(M\bar{\otimes}N)$ *is exactly the self-adjoint fixed points of $\phi_M\bar{\otimes}\phi_N$.*

PROOF. Consider the $*$-antiautomorphism

$$\phi_M\bar{\otimes}\phi_N\colon W^*(M)\bar{\otimes}W^*(N) \longrightarrow W^*(M)\bar{\otimes}W^*(N),$$

and note that if $x \in W^*(M), y \in W^*(N)$, then

$$(\phi_M\bar{\otimes}\phi_N)^2(x\otimes y) = \phi_M^2(x)\otimes\phi_M^2(y) = x\otimes y.$$

Thus $(\phi_M\bar{\otimes}\phi_N)^2$ is the identity map on $W^*(M)\bar{\otimes}W^*(N)$ since it is normal. Now, consider the real $*$-subalgebras, $\mathrm{RW}^*(M) = \{x \in W^*(M)\ ;\ \phi_M(x) = x^*\}$, $\mathrm{RW}^*(N) = \{x \in W^*(N)\ ;\ \phi_N(x) = x^*\}$ and $\mathcal{R} = \{x \in W^*(M)\bar{\otimes}W^*(N)\ ;\ (\phi_M\bar{\otimes}\phi_N)(x) = x^*\}$. Note that $\mathcal{R}\cap i\mathcal{R} = 0$, so that $W^*(M)\bar{\otimes}W^*(N) = \mathcal{R}\oplus i\mathcal{R}$, by [9, 7.3.2].

Clearly, $\mathrm{RW}^*(M)\bar{\otimes}\,\mathrm{RW}^*(N)$ is the real Von Neumann subalgebra of $W^*(M)\bar{\otimes}W^*(N)$, generated by $M \otimes N$, and therefore by $\mathrm{JW}(M\bar{\otimes}N)$. Since $\mathrm{JW}(M\bar{\otimes}N)$ is universally reversible, we have

$$\mathrm{RW}^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) = \mathrm{RW}^*(M)\bar{\otimes}\,\mathrm{RW}^*(N),$$

by Lemma 2.8. If $x \in \mathrm{RW}^*(M)$, and $y \in \mathrm{RW}^*(N)$, then

$$(\phi_M\bar{\otimes}\phi_N)(x \otimes y) = \phi_M(x) \otimes \phi_N(y) = x^* \otimes y^* = (x \otimes y)^*.$$

Thus, $\mathrm{RW}^*(M) \otimes \mathrm{RW}^*(N) \subset \mathcal{R}$, and so, $\mathrm{RW}^*(M)\bar{\otimes}\,\mathrm{RW}^*(N) \subset \mathcal{R}$, since $\mathcal{R}$ is weak-$*$ closed, by [9, 7.3.2]. Then from the above and Lemma 2.8 it follows that

$$\mathrm{RW}^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) = \mathrm{RW}^*(M)\bar{\otimes}\,\mathrm{RW}^*(N) = \mathcal{R},$$

and that

$$W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) = W^*(M)\bar{\otimes}W^*(N).$$

Part (ii) is now immediate from (i), and the universal reversibility of $\mathrm{JW}(M\bar{\otimes}N)$.

Note that if M and N are JW-algebras with no Type I_2 part, then $\mathrm{JW}(M\bar{\otimes}N)$ is universally reversible and $W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) = W^*(M)\bar{\otimes}W^*(N)$.

PROPOSITION 2.10. *Let M be a universally reversible* JW-*algebra with no abelian part. Then the following are equivalent:*

(i) M contains a weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra.

(ii) There is a non-zero projection e in $Z\big(W^*(M)\big)$ such that $\phi_M(e) \neq e$.

(iii) $Z(M) \neq Z\big(W^*(M)\big)_{s.a}$.

PROOF. (i) $\Rightarrow$ (ii). Suppose that M contains a weakly-closed Jordan ideal J isomorphic to the self-adjoint part of a Von Neumann algebra. By [9, 4.3.6], there exists a central projection e in M such that $J = eM$. Then $W^*(eM) = eW^*(M)$ is a weakly-closed ideal of $W^*(M)$, and the restriction of ϕ_M to $W^*(eM)$ is the canonical $*$-antiautomorphism of $W^*(eM)$. Since M has no abelian part, neither does eM. Therefore, there is, by [9, 7.4.7], a projection f of $Z\big(W^*(eM)\big)$ and hence, of $Z\big(W^*(M)\big)$, such that $\phi_M(f) = e - f \neq f$.

(ii) $\Rightarrow$ (iii). Suppose that there is a projection e in $Z\big(W^*(M)\big)$ such that $\phi_M(e) \neq e$. Since M is universally reversible, it consists of all self-adjoint elements of $\mathrm{RW}^*(M)$, by [9, 7.3.3]. It follows that $e \notin M$, and hence $Z(M) \neq Z\big(W^*(M)\big)_{s.a}$.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Suppose that (iii) holds. Then $\phi_M(x) \neq x$, for some $x = x^* \notin Z(M)$. Hence there must be a projection z in $Z\big(W^*(M)\big)$ such that $\phi_M(z) \neq z$, proving (ii). By [9, 7.3.5], there are projections $e \in Z(M), f \in Z\big(W^*(M)\big)$ such that $e + f + \phi_M(f) = 1$ and $(1 - e)M \simeq fW^*(M)_{s.a}$. The condition (i) is immediate if $e \neq 1$. If $e = 1$, then ϕ_M leaves $Z\big(W^*(M)\big)$ pointwise invariant by [9, 4.2.15], a contradiction.

LEMMA 2.11. *Let M be a* JW-*algebra. If there exists a weakly closed ideal J of $W^*(M)$ such that $W^*(M) = J \oplus \phi_M(J)$, then $J_{s.a} \simeq$* $\mathrm{RW}^*(M)_{s.a}$ *$= M$ via $x \longrightarrow x \oplus \phi_M(x)$.*

PROOF. The proof is exactly the same as in Lemma 1.1. (ii) $\Rightarrow$ (i) of [6].

THEOREM 2.12. *Let $\mathcal{M}$ be a Von Neumann algebra with no abelian part, and let N be any* JW*-algebra. Then*

$$\mathrm{JW}(\mathcal{M}_{s.a}\bar{\otimes}N) \simeq \big(\mathcal{M}\bar{\otimes}W^*(N)\big)_{s.a}.$$

PROOF. Put $M = \mathcal{M}_{s.a}$, then $W^*(M) = \mathcal{M} \oplus \mathcal{M}^0$, where the inclusion map of M into $W^*(M)$ is given by $a \longrightarrow a \oplus a^0$, and the canonical $*$-antiautomorphism ϕ_M of $W^*(M)$ is defined by $\phi_M(a \oplus b^0) = b \oplus a^0$, by [9, 7.4.7]. Since M is a universally reversible JW-algebra, by [9, 7.4.6], which has no one-dimensional representations, $\mathrm{JW}(M\bar{\otimes}N)$ is universally reversible, by Proposition 2.7, and so

$$\begin{aligned} W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) &= W^*(M)\bar{\otimes}W^*(N), \\ &= (\mathcal{M} \oplus \mathcal{M}^0)\bar{\otimes}W^*(N), \\ &= \big(\mathcal{M}\bar{\otimes}W^*(N)\big) \oplus \big(\mathcal{M}^0\bar{\otimes}W^*(N)\big). \end{aligned}$$

Since $\phi_M(\mathcal{M}) = \mathcal{M}^0$, we have $(\phi_M\bar{\otimes}\phi_N)\big(\mathcal{M} \otimes W^*(N)\big) = \mathcal{M}^0 \otimes W^*(N)$. But $\phi_M\bar{\otimes}\phi_N$ is normal on $W^*(M)\bar{\otimes}W^*(N)$. Hence, it is normal on $\mathcal{M}\bar{\otimes}W^*(N)$, and so $(\phi_M\bar{\otimes}\phi_N)\big(\mathcal{M}\bar{\otimes}W^*(N)\big) = \mathcal{M}^0\bar{\otimes}W^*(N)$. That is

$$W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) = \big(\mathcal{M}\bar{\otimes}W^*(N)\big) \oplus (\phi_M\bar{\otimes}\phi_N)\big(\mathcal{M}\bar{\otimes}W^*(N)\big).$$

The described conclusion now follows from Lemma 2.11 because $\phi_M\bar{\otimes}\phi_N$ is the canonical $*$-antiautomorphism of $W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big)$.

REMARK 2.13. Let M_1, M_2, N be JW-algebras. Then

$$\mathrm{JW}\big((M_1 \oplus M_2)\bar{\otimes}N\big) = \mathrm{JW}(M_1\bar{\otimes}N) \oplus \mathrm{JW}(M_2\bar{\otimes}N).$$

This follows easily from the definitions because

$$\begin{aligned} W^*(M_1 \oplus M_2)\bar{\otimes}W^*(N) &= \big(W^*(M_1) \oplus W^*(M_2)\big)\bar{\otimes}W^*(N), \\ &= \big(W^*(M_1)\bar{\otimes}W^*(N)\big) \oplus \big(W^*(M_2)\bar{\otimes}W^*(N)\big). \end{aligned}$$

THEOREM 2.14. *Let $M \subseteq \mathcal{B}(H)_{s.a}$ and $N \subseteq \mathcal{B}(K)_{s.a}$ be universally reversible* JW-*algebras with no abelian part. Then the following are equivalent:*

(i) $\mathrm{JW}(M\bar{\otimes}N)$ *contains no non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra.*

(ii) Neither M nor N contains a non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra.

(iii) $\mathrm{JW}_{H,K}(M\bar{\otimes}N)$ *contains no non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra.*

If one (and hence all) of these conditions is satisfied, then

$$\mathrm{JW}(M\bar{\otimes}N) \simeq \mathrm{JW}_{H,K}(M\bar{\otimes}N).$$

PROOF. (i) $\Rightarrow$ (ii). Suppose that (i) holds, and assume that M contains a non-zero weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra. Then by Remark 2.13 and Theorem 2.12 we see that $\mathrm{JW}(M\bar{\otimes}N)$ contains a weakly-closed Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra. Hence (ii) follows by contradiction.

(ii) $\Rightarrow$ (i). If (ii) holds, then ϕ_M and ϕ_N fix the centres of $W^*(M)$ and $W^*(N)$, respectively, by Proposition 2.10. Since M and N are universally reversible, $W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) = W^*(M)\bar{\otimes}W^*(N)$, and so

$$\begin{aligned} Z\Big(W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big)\Big) &= Z\big(W^*(M)\bar{\otimes}W^*(N)\big) \\ &= Z\big(W^*(M)\big)\bar{\otimes}Z\big(W^*(N)\big), \end{aligned}$$

by [20, 4.5.11]. Since $\phi_M\bar{\otimes}\phi_N$ is the canonical $*$-antiautomorphism of $W^*(M)\bar{\otimes}W^*(N)$ and, by the above, fixes the centre of $W^*(M)\bar{\otimes}W^*(N)$, the conclusion follows from Proposition 2.10.

(i) $\Rightarrow$ (iii). By Lemma 2.3, $\mathrm{JW}_{H,K}(M\bar{\otimes}N)$ is isomorphic to a weakly-closed Jordan ideal of $\mathrm{JW}(M\bar{\otimes}N)$.

(iii) $\Rightarrow$ (ii). Let I be a weakly-closed Jordan ideal of M such that I is isomorphic to the self-adjoint part of a Von Neumann algebra. Then $\mathrm{JW}_{H,K}(I\bar{\otimes}N)$ is a weakly-closed Jordan ideal of $\mathrm{JW}_{H,K}(M\bar{\otimes}N)$, and is isomorphic to an ideal of $\mathrm{JW}(I\bar{\otimes}N)$, by Lemma 2.3. But the latter is isomorphic the self-adjoint part of a Von Neumann algebra, by Theorem 2.12.

The final statement follows from Corollary 2.6.

3. **On the centre of the tensor product of** JW-**algebras.** It is well-known (see 20, [4.5.11]) that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are Von Neumann algebras with centres Z_1 and Z_2, respectively, then the centre of the tensor product $\mathcal{M}_1\bar{\otimes}\mathcal{M}_2$ is $Z_1\bar{\otimes}Z_2$. It is natural to ask whether a similar result holds in the context of JW-algebras, and the JW-tensor product.

It transpires that the answer is "no" in general, but that the situation is manageable if we confine ourselves to universally reversible JW-algebras.

The following examples give some idea of the difficulties.

EXAMPLE 3.1. Let $M = N = \mathcal{B}(H)_{s.a}$, where H is a complex Hilbert space. Then

$$\mathrm{JW}(M\bar{\otimes}N) \simeq \big(\mathcal{B}(H)\bar{\otimes}\mathcal{B}(H)\big)_{s.a} \oplus \big(\mathcal{B}(H)\bar{\otimes}\mathcal{B}(H)\big)_{s.a} \simeq \mathcal{B}(H\otimes H)_{s.a} \oplus \mathcal{B}(H\otimes H)_{s.a}$$

so that

$$Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) = \mathbb{R}\oplus\mathbb{R} \neq \mathbb{R} = Z(M)\bar{\otimes}Z(N).$$

EXAMPLE 3.2. Let $M = N = V_{4n+1}$, for some $n < \infty$. Then using [6, Theorem 8] we have

$$\begin{aligned} Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) &= Z\big(\mathrm{JC}(V_{4n+1}\otimes V_{4n+1})\big) \\ &= \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R} \neq \mathbb{R} \\ &= Z(M)\bar{\otimes}Z(N). \end{aligned}$$

EXAMPLE 3.3. Let $M = N = V$ be an infinite dimensional spin factor. Then

$$\mathrm{JW}(M\bar{\otimes}N) = \big(\mathrm{RW}^*(V)\bar{\otimes}\,\mathrm{RW}^*(V)\big)_{s.a},$$

by Theorem 2.9, and Proposition 2.7. In this case, again $Z(M)\bar{\otimes}Z(N) = \mathbb{R}$. But $Z\big(\mathrm{JW}(M\bar{\otimes}N)\big)$ is infinite dimensional. This is because $Z\big(\mathrm{RW}^*(V)\big)$ is infinite dimensional. Indeed, $W^*(V) = W^*(V^{**}) = C^*(V)^{**}$ by [9, 6.1.7(iv) and 7.1.11] and $C^*(V)$ has an uncountable family of inequivalent irreducible representations [13, Section 6]. So, $C^*(V)^{**}$ certainly has an infinite orthogonal family of minimal central projections, and, since for any projection $e \in C^*(V)^{**}$ we have $e + \phi_V(e) \in \mathrm{RW}^*(V)$, it follows that $\mathrm{RW}^*(V)$ must have an infinite dimensional centre.

In view of the intractability, in this context, of the Type I_2 JW-algebras (described above) we will concentrate on the cases where both M and N are universally reversible. Even though it is not always (even then) the case that $Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) = Z(M)\bar{\otimes}Z(N)$, we can, nevertheless, describe $Z\big(\mathrm{JW}(M\bar{\otimes}N)\big)$ in terms of $Z(M)\bar{\otimes}Z(N)$ (Theorem 3.10).

PROPOSITION 3.4. *Let M and N be universally reversible* JW*-algebras such that each of the centres of $W^*(M)$ and $W^*(N)$ are pointwise fixed by their respective canonical $*$-antiautomorphism. Then*

$$Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) = Z(M)\bar{\otimes}Z(N).$$

PROOF. By [9, 7.3.3] $Z\big(W^*(M)\big)_{s.a} = Z(M)$, and $Z\big(W^*(N)\big)_{s.a} = Z(N)$. Now the fact that $W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) = W^*(M)\bar{\otimes}W^*(N)$, and $Z\big(W^*(M)\bar{\otimes}W^*(N)\big) = Z\big(W^*(M)\big)\bar{\otimes} Z\big(W^*(N)\big)$ together with Lemma 2 of [14] implies that

$$Z\Big(W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big)\Big)_{s.a} = Z(M)\bar{\otimes}Z(N).$$

But since $Z\big(W^*(M)\big)$, $Z\big(W^*(N)\big)$ are pointwise invariant under ϕ_M, ϕ_N respectively, and $\phi_M\bar{\otimes}\phi_N$ is normal on $W^*(M)\bar{\otimes}W^*(N)$, it follows that $Z\big(W^*(M)\big)\bar{\otimes}Z\big(W^*(N)\big)$ is pointwise $\phi_M\bar{\otimes}\phi_N$-invariant, from which it follows that

$$\begin{aligned} Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) &= Z\Big(W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big)\Big)_{s.a}, \\ &= Z(M)\bar{\otimes}Z(N), \end{aligned}$$

proving the proposition.

COROLLARY 3.5. *Let M and N be universally reversible* JW*-algebras such that neither contains a non-zero Jordan ideal isomorphic to the self-adjoint part of a Von Neumann algebra. Then $Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) = Z(M)\bar{\otimes}Z(N)$.*

PROOF. This is immediate from Proposition 3.4, and Proposition 2.10.

LEMMA 3.6. *Let $\mathcal{M}$ be a Von Neumann algebra without abelian part, and N a* JW-*algebra. Then*

$$Z\big(\mathrm{JW}(\mathcal{M}_{s.a}\bar{\otimes}N)\big) \simeq Z(\mathcal{M})_{s.a}\bar{\otimes}Z\big(W^*(N)\big)_{s.a}.$$

PROOF. Note first that for any Von Neumann algebra W, $Z(W_{s.a}) = Z(W)_{s.a}$. By Theorem 2.12, $\mathrm{JW}(\mathcal{M}_{s.a}\bar{\otimes}N) \simeq \big(\mathcal{M}\bar{\otimes}W^*(N)\big)_{s.a}$. So by [20, 4.5.11] and [14, Lemma 2] we have

$$\begin{aligned} Z\big(\mathrm{JW}(\mathcal{M}_{s.a}\bar{\otimes}N)\big) &\simeq Z\big(\mathcal{M}\bar{\otimes}W^*(N)\big)_{s.a}, \\ &= \Big(Z(\mathcal{M})\bar{\otimes}Z\big(W^*(N)\big)\Big)_{s.a}, \\ &= Z(\mathcal{M})_{s.a}\bar{\otimes}Z\big(W^*(N)\big)_{s.a}, \end{aligned}$$

and the proof is complete.

LEMMA 3.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be Von Neumann algebras without abelian part. Then*

$$Z\big(\mathrm{JW}(\mathcal{M}_{s.a}\bar{\otimes}\mathcal{N}_{s.a})\big) \simeq \big(Z(\mathcal{M})_{s.a}\bar{\otimes}Z(\mathcal{N})_{s.a}\big) \oplus \big(Z(\mathcal{M})_{s.a}\bar{\otimes}Z(\mathcal{N})_{s.a}\big).$$

PROOF. This is immediate since

$$\mathrm{JW}(\mathcal{M}_{s.a}\bar{\otimes}\mathcal{N}_{s.a}) \simeq (\mathcal{M}\bar{\otimes}\mathcal{N})_{s.a} \oplus (\mathcal{M}^0\bar{\otimes}\mathcal{N}^0)_{s.a}.$$

COROLLARY 3.8. *Let M be an abelian* JW-*algebra and N any universally reversible* JW-*algebra. Then*

$$Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) = M\bar{\otimes}Z(N).$$

PROOF. Observe that $\mathrm{JW}(M\bar{\otimes}N) = M\bar{\otimes}N$, and that $M = \mathcal{M}_{s.a}$ for some abelian Von Neumann algebra $\mathcal{M}$. By Lemma [9, 7.3.5] we can write $N = A \oplus W_{s.a}$, where the centre of $W^*(A)$ is pointwise fixed by ϕ_N and, W is a Von Neumann algebra. Then

$$Z\big(\mathrm{JW}(M\bar{\otimes}A)\big) = M\bar{\otimes}Z(A),$$

by Proposition 3.4. Also, using [20, 4.5.11],

$$Z\big(\mathrm{JW}(M\bar{\otimes}W_{s.a})\big) = Z\big(\mathcal{M}_{s.a}\bar{\otimes}Z(W)_{s.a}\big) = M\bar{\otimes}Z(W_{s.a}).$$

Hence,

$$Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) = M\bar{\otimes}Z(N).$$

REMARK. According to Proposition 2.10, a given universally reversible JW-algebra M has a decomposition

$$M = M_1 \oplus M_2 \oplus M_3,$$

where M_1 is the abelian part of M, the centre of $W^*(M_1 \oplus M_2)$ is pointwise fixed by its canonical $*$-antiautomorphism and, M_3 is isomorphic to the self-adjoint part of a Von Neumann algebra.

Retaining this notation, we prove:

THEOREM 3.10. *Let M and N be universally reversible* JW*-algebras. Then*

$$Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) \simeq \big(Z(M)\bar{\otimes}Z(N)\big) \oplus \big(Z(M_3)\bar{\otimes}Z(N_3)\big).$$

PROOF. By Remark 2.13, we have

$$\begin{aligned}\mathrm{JW}(M\bar{\otimes}N) = \mathrm{JW}\big((M_1 \oplus M_2)\bar{\otimes}(N_1 \oplus N_2)\big) \oplus \mathrm{JW}\big((M_1 \oplus M_2)\bar{\otimes}N_3\big)\\ \oplus \mathrm{JW}\big(M_3\bar{\otimes}(N_1 \oplus N_2)\big) \oplus \mathrm{JW}(M_3\bar{\otimes}N_3).\end{aligned}$$

Now,

$$Z\Big(\mathrm{JW}\big((M_1 \oplus M_2)\bar{\otimes}(N_1 \oplus N_2)\big)\Big) = Z(M_1 \oplus M_2)\bar{\otimes}Z(N_1 \oplus N_2),$$

by Proposition 3.4. By Lemma 3.6, we get

$$\begin{aligned}Z\Big(\mathrm{JW}\big((M_1 \oplus M_2)\bar{\otimes}N_3\big)\Big) &\simeq Z\big(W^*(M_1 \oplus M_2)\big)_{s.a}\bar{\otimes}Z(N_3),\\ &= Z(M_1 \oplus M_2)\bar{\otimes}Z(N_3).\end{aligned}$$

Similarly,

$$Z\Big(\mathrm{JW}\big(M_3\bar{\otimes}(N_1 \oplus N_2)\big)\Big) \simeq Z(M_3)\bar{\otimes}Z(N_1 \oplus N_2).$$

By Corollary 3.7, we get

$$Z\big(\mathrm{JW}(M_3\bar{\otimes}N_3)\big) \simeq \big(Z(M_3)\bar{\otimes}Z(N_3)\big) \oplus \big(Z(M_3)\bar{\otimes}Z(N_3)\big).$$

Collecting terms, we have

$$Z\big(\mathrm{JW}(M\bar{\otimes}N)\big) \simeq \big(Z(M)\bar{\otimes}Z(N)\big) \oplus \big(Z(M_3)\bar{\otimes}Z(N_3)\big),$$

as required.

4. **The type of the tensor products of** JW**-algebras.**

In this section we investigate the type of $\mathrm{JW}(M\bar{\otimes}N)$, where M and N are JW-algebras. We first study the type of $\mathrm{JW}(M\bar{\otimes}N)$ when M and N are of Type I. Then we complete the discussion of tensoring different types of JW-algebras. In one sense, the work can be considered as an application of Theorem 2.9, Theorem 8 of [1] and Table 11.1 of [11]. However, our main object is to establish the "multiplication table" (Theorem 4.9) which describes completely the type of the tensor product $\mathrm{JW}(M\bar{\otimes}N)$.

LEMMA 4.1. *Let M be a* JW*-algebra, and let N be a Type I_n* JW*-factor, $n < \infty$, and X any compact Hausdorff space. Suppose that M contains a weak-$*$ dense* JC*-subalgebra of the form $C(X, N)$. Then M is the same type as N.*

PROOF. If N is an infinite dimensional spin factor, then this follows from the result of Stacey [17]. So we may suppose that N is any finite dimensional factor. By [7, Corollary 5.1] for example, the fact that N is finite dimensional implies that $C(X, N)^{**} \simeq C(X)^{**} \otimes N \simeq C(Y) \otimes N$, for some compact hyperstonean space Y. But by hypothesis, there is a surjective normal homomorphism $C(X, N)^{**} \longrightarrow M$ and the described conclusion follows immediately from this.

LEMMA 4.2. *Let A, B be non-abelian finite dimensional* JW*-factors, and let X, Y be compact Hausdorff spaces. Then*

$$\mathrm{JC}\Big(C(X,A)\otimes_{\min} C(Y,B)\Big) = C(X\times Y)\otimes\Big(R^*(A)\otimes_{\mathbb{R}} R^*(B)\Big)_{s.a}.$$

PROOF. $\mathrm{JC}\big(C(X,A)\otimes_{\min} C(Y,B)\big)$ is universally reversible, by [4, Corollary 1.5]. Thus we see that, using [4, Theorem 2.2]

$$\begin{aligned} C_{\mathbb{R}}(X)\otimes_{\mathbb{R}} C_{\mathbb{R}}(Y)\otimes_{\mathbb{R}}\Big(R^*(A)\otimes_{\mathbb{R}} R^*(B)\Big)_{s.a} &= \Big(C_{\mathbb{R}}(X)\otimes_{\mathbb{R}} R^*(A)\otimes_{\mathbb{R}} C_{\mathbb{R}}(Y)\otimes_{\mathbb{R}} R^*(B)\Big)_{s.a}\\ &= \Big(C\big(X,R^*(A)\big)\otimes_{\mathbb{R}} C\big(Y,R^*(B)\big)\Big)_{s.a} \end{aligned}$$

is norm dense in $\mathrm{JC}\big(C(X,A)\otimes_{\min} C(Y,B)\big)$, and so,

$$\mathrm{JC}\Big(C(X,A)\otimes_{\min} C(Y,B)\Big) = C(X\times Y)\otimes\Big(R^*(A)\otimes_{\mathbb{R}} R^*(B)\Big)_{s.a},$$

as required.

4.3. From [12, Lemma 3.1.7] we have the following real algebra identifications:

$$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}=\mathbb{C}\oplus\mathbb{C},\qquad \mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}=M_2(\mathbb{C}),\qquad \mathbb{H}\otimes_{\mathbb{R}}\mathbb{H}=M_4(\mathbb{R}),$$

and consequently the following tensor product table for $M_n(\mathbb{F})\otimes M_m(\mathbb{F}')$

	$M_m(\mathbb{R})$	$M_m(\mathbb{C})$	$M_m(\mathbb{H})$
$M_n(\mathbb{R})$	$M_{nm}(\mathbb{R})$	$M_{nm}(\mathbb{C})$	$M_{nm}(\mathbb{H})$
$M_n(\mathbb{C})$	$M_{nm}(\mathbb{C})$	$M_{nm}(\mathbb{C})\oplus M_{nm}(\mathbb{C})$	$M_{2nm}(\mathbb{C})$
$M_n(\mathbb{H})$	$M_{nm}(\mathbb{H})$	$M_{2nm}(\mathbb{C})$	$M_{4nm}(\mathbb{R})$

Recall that the Type I_n factors, $n<\infty$, are $M_n(\mathbb{F})_{s.a}$, where $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and the spin factors [9, 5.3.8]. Also, recall that $R^*\big(M_n(\mathbb{F})\big)_{s.a}=M_n(\mathbb{F})$, whenever $n\geq 3$ and $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ [10, Corollary 1, p. 143]. Thus, using 4.1–4.3 together with the fact that $\mathrm{JC}(M\otimes_{\min}N)$ is weak $*$-dense in the JW-tensor product $\mathrm{JW}(M\bar{\otimes}N)$ we obtain the following, which is a complete description of $\mathrm{JW}(M\bar{\otimes}N)$, where M and N are JW-algebras of Type I_{fin} with no Type $I_{2,\infty}$ part.

THEOREM 4.4. *We have the following tensor product table for* $\mathrm{JW}(M\bar{\otimes}N)$
(i) $I_{n,\mathbb{F}}\bar{\otimes}I_{m,\mathbb{F}'}$ $(3\leq n,\ m<\infty,\ \mathbb{F},\mathbb{F}'=\mathbb{R},\mathbb{C},\mathbb{H})$.

	$I_{m,\mathbb{R}}$	$I_{m,\mathbb{C}}$	$I_{m,\mathbb{H}}$
$I_{n,\mathbb{R}}$	$I_{nm,\mathbb{R}}$	$I_{nm,\mathbb{C}}$	$I_{nm,\mathbb{H}}$
$I_{n,\mathbb{C}}$	$I_{nm,\mathbb{C}}$	$I_{nm,\mathbb{C}}$	$I_{2nm,\mathbb{C}}$
$I_{n,\mathbb{H}}$	$I_{nm,\mathbb{H}}$	$I_{2nm,\mathbb{C}}$	$I_{4nm,\mathbb{R}}$

(ii) $I_{n,\mathbb{F}}\bar{\otimes}I_{2,k}$ $(3 \leq n < \infty,\ \mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H};\ k = 2,\ldots,9)$

	$I_{2,2}$	$I_{2,3}$	$I_{2,4}$	$I_{2,5}$	$I_{2,6}$	$I_{2,7}$	$I_{2,8}$	$I_{2,9}$
$I_{n,\mathbb{R}}$	$I_{2n,\mathbb{R}}$	$I_{2n,\mathbb{C}}$	$I_{2n,\mathbb{H}}$	$I_{2n,\mathbb{H}}$	$I_{4n,\mathbb{H}}$	$I_{8n,\mathbb{C}}$	$I_{16n,\mathbb{R}}$	$I_{16n,\mathbb{R}}$
$I_{n,\mathbb{C}}$	$I_{2n,\mathbb{C}}$	$I_{2n,\mathbb{C}}$	$I_{4n,\mathbb{C}}$	$I_{4n,\mathbb{C}}$	$I_{8n,\mathbb{C}}$	$I_{8n,\mathbb{C}}$	$I_{16n,\mathbb{C}}$	$I_{16n,\mathbb{C}}$
$I_{n,\mathbb{H}}$	$I_{2n,\mathbb{H}}$	$I_{4n,\mathbb{C}}$	$I_{8n,\mathbb{R}}$	$I_{8n,\mathbb{R}}$	$I_{16n,\mathbb{R}}$	$I_{16n,\mathbb{C}}$	$I_{16n,\mathbb{H}}$	$I_{16n,\mathbb{H}}$

(iii) $I_{2,k}\bar{\otimes}I_{2,k'}$ $(k,k' = 2,\ldots,9)$

	$I_{2,2}$	$I_{2,3}$	$I_{2,4}$	$I_{2,5}$	$I_{2,6}$	$I_{2,7}$	$I_{2,8}$	$I_{2,9}$
$I_{2,2}$	$I_{2^2,\mathbb{R}}$	$I_{2^2,\mathbb{C}}$	$I_{2^2,\mathbb{H}}$	$I_{2^2,\mathbb{H}}$	$I_{2^3,\mathbb{H}}$	$I_{2^4,\mathbb{C}}$	$I_{2^5,\mathbb{R}}$	$I_{2^5,\mathbb{R}}$
$I_{2,3}$	$I_{2^2,\mathbb{C}}$	$I_{2^2,\mathbb{C}}$	$I_{2^3,\mathbb{C}}$	$I_{2^3,\mathbb{C}}$	$I_{2^4,\mathbb{C}}$	$I_{2^4,\mathbb{C}}$	$I_{2^5,\mathbb{C}}$	$I_{2^5,\mathbb{C}}$
$I_{2,4}$	$I_{2^2,\mathbb{H}}$	$I_{2^3,\mathbb{C}}$	$I_{2^4,\mathbb{R}}$	$I_{2^4,\mathbb{R}}$	$I_{2^5,\mathbb{R}}$	$I_{2^5,\mathbb{C}}$	$I_{2^5,\mathbb{H}}$	$I_{2^5,\mathbb{H}}$
$I_{2,5}$	$I_{2^2,\mathbb{H}}$	$I_{2^3,\mathbb{C}}$	$I_{2^4,\mathbb{R}}$	$I_{2^4,\mathbb{R}}$	$I_{2^5,\mathbb{R}}$	$I_{2^5,\mathbb{C}}$	$I_{2^5,\mathbb{H}}$	$I_{2^5,\mathbb{H}}$
$I_{2,6}$	$I_{2^3,\mathbb{H}}$	$I_{2^4,\mathbb{C}}$	$I_{2^5,\mathbb{R}}$	$I_{2^5,\mathbb{R}}$	$I_{2^6,\mathbb{R}}$	$I_{2^6,\mathbb{C}}$	$I_{2^6,\mathbb{H}}$	$I_{2^6,\mathbb{H}}$
$I_{2,7}$	$I_{2^4,\mathbb{C}}$	$I_{2^4,\mathbb{C}}$	$I_{2^5,\mathbb{C}}$	$I_{2^5,\mathbb{C}}$	$I_{2^6,\mathbb{C}}$	$I_{2^6,\mathbb{C}}$	$I_{2^7,\mathbb{C}}$	$I_{2^7,\mathbb{C}}$
$I_{2,8}$	$I_{2^5,\mathbb{R}}$	$I_{2^5,\mathbb{C}}$	$I_{2^5,\mathbb{H}}$	$I_{2^5,\mathbb{H}}$	$I_{2^6,\mathbb{H}}$	$I_{2^7,\mathbb{C}}$	$I_{2^8,\mathbb{R}}$	$I_{2^8,\mathbb{R}}$
$I_{2,9}$	$I_{2^5,\mathbb{R}}$	$I_{2^5,\mathbb{C}}$	$I_{2^5,\mathbb{H}}$	$I_{2^5,\mathbb{H}}$	$I_{2^6,\mathbb{H}}$	$I_{2^7,\mathbb{C}}$	$I_{2^8,\mathbb{R}}$	$I_{2^8,\mathbb{R}}$

COROLLARY 4.5. *If M and N are* JW*-algebras of Type I_{fin} without Type $I_{2,\infty}$ part then* $\mathrm{JW}(M\bar{\otimes}N)$ *is of the same type (without Type $I_{2,\infty}$ part).*

In order to complete the description of $\mathrm{JW}(M\bar{\otimes}N)$ for the remaining types of M and N we will need results which relate the type of M to that of $W^*(M)$.

LEMMA 4.6. *Let V be an infinite dimensional spin factor. Then $W^*(V)$ has no Type I_{fin} part.*

PROOF. On the contrary, suppose that $W^*(V)$ does have (non-zero) Type I_{fin} part. Then there is a central projection e of $W^*(V)$ such that $eW^*(V) \simeq C\big(X, M_n(\mathbb{C})\big)$ for some $n < \infty$ and some compact hyperstonean space X (see [9, 7.4.5]). Then $0 \neq eV \subset C\big(X, M_n(\mathbb{C})\big)$. Also $V \simeq eV$ because V is simple. Therefore, the postliminal C^*-algebra $C\big(X, M_n(\mathbb{C})\big)$ contains a copy of the antiliminal Clifford C^*-algebra $C^*(V)$. This is impossible, by [13, Proposition 6.2.9].

THEOREM 4.7. *Let M be a* JW*-algebra. Then*

(a) M is of Type I_{fin} without Type $I_{2,\infty}$ part if and only if $W^*(M)$ is of Type I_{fin}.

(b) if M is universally reversible then M is of Type I_∞ (resp. II_1, II_∞, III) if and only if $W^*(M)$ is of Type I_∞ (resp. II_1, II_∞, III).

(We note that M is universally reversible if it is of Type I_∞, II_1, II_∞ or III).

PROOF. (b) This is due to Ajupov [1, Theorem 8] and Størmer [19, Corollary 6.5].

(a) Suppose that $W^*(M)$ is of Type I_{fin}. Then M is certainly finite because any family of orthogonal equivalent projections of M are orthogonal and unitarily equivalent in $W^*(M)$ and so must be finite. Now (b) implies that M must be of Type I_{fin}. If M has a (non-zero) Type $I_{2,\infty}$ then it must contain a copy of an infinite dimensional spin factor V. But then $[V]^-$, the W^*-algebra generated in $W^*(M)$ by V is not of Type I_{fin}, by Lemma 4.6, because it is a quotient of $W^*(V)$. Since every W^*-subalgebra of a Type I_{fin} W^*-algebra is also of Type I_{fin}, this is a contradiction.

By (b), and in view of [4, Lemma 2.6], in order to prove the converse it is enough to suppose that $M = C(X, V_k)$, where $k < \infty$ and X is compact hyperstonean. But then (see Lemma 2.5 of [4], $W^*(M) = C\big(X, W^*(V_k)\big)$ which is of Type I_{fin}, and the proof is complete.

THEOREM 4.9. *The type of* $\mathrm{JW}(M\bar{\otimes}N)$ *for* JW-*algebras M and N is given in the following table*

	$I_{2,k<\infty}$ $I_{m\geq 3}$	I_∞	II_1	II_∞	III
$I_{2,k<\infty}$ $I_{n\geq 3}$	*see Theorem 4.4*	I_∞	II_1	II_∞	III
I_∞	I_∞	I_∞	II_∞	II_∞	III
II_1	II_1	II_∞	II_1	II_∞	III
II_∞	II_∞	II_∞	II_∞	II_∞	III
III	III	III	III	III	III

PROOF. Let M and N be any JW-algebra of any of the types occuring in the table. By Theorem 4.7 M and $W^*(M)$ (respectively, N and $W^*(N)$) are of the same type. In addition $\mathrm{JW}(M\bar{\otimes}N)$ and $W^*(M)\bar{\otimes}W^*(N)$ are of the same type. This is because, by Proposition 2.7 and Theorem 2.9, $\mathrm{JW}(M\bar{\otimes}N)$ is universally reversible with $W^*\big(\mathrm{JW}(M\bar{\otimes}N)\big) = W^*(M)\bar{\otimes}W^*(N)$, and so the claim follows from Theorem 4.7. The above table is now an immediate consequence of [15, Theorem 2.6.6], [20, Theorem 5.2.30].

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