Canad. Math. Bull. Vol. **38** (2), 1995 pp. 215–217

ON THE PERIODIC RADICAL OF A RING

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ABSTRACT. Let R be a ring and $P(R)$ the sum of all periodic ideals of R. We prove that $P(R)$ is the intersection of all prime ideals P_α such that $\frac{R}{P_\alpha}$ contains no nontrivial periodic ideals. We also prove that $P(R) = 0$ if and only if R is a subdirect product of prime rings R_α with $P(R_\alpha) = 0$.

A ring R is called *periodic* if for each $x \in R$, there exist distinct positive integers m and n such that $x^m = x^n$. In [1] Bell and Klein showed that any ring R has a maximal periodic ideal $P(R)$, defined as the sum of all periodic ideals; they proved that $\frac{R}{P(R)}$ has no nontrivial periodic ideals. It is our purpose to develop further the theory of $P(R)$.

The symbols Z and Z^+ will denote the ring of integers and the set of positive integers. For $a \in R$, the symbol (a) will denote the principal ideal generated by a.

We begin with

LEMMA 1 [2]. *Suppose that for each x in the ring R, there exist a positive integer $n = n(x)$ and a polynomial $p(X) = p_x(X) \in Z[X]$ such that $x^n = x^{n+1}p(x)$. Then R is periodic.*

THEOREM 1. *$P(R) = \bigcap_\alpha P_\alpha$, where the intersection is taken over the set of prime ideals P_α such that $\frac{R}{P_\alpha}$ contains no nontrivial periodic ideals (If there are no prime ideals P_α such that $\frac{R}{P_\alpha}$ contains no nontrivial periodic ideals, we say that the intersection is R).*

PROOF. If $P(R) = R$ the result is obviously correct. Suppose then that $P(R) \neq R$.

Let P_α be a prime ideal of R such that $\frac{R}{P_\alpha}$ contains no nontrivial periodic ideals. If $P(R) \not\subseteq P_\alpha$, then $\frac{P_\alpha+P(R)}{P_\alpha}$ is a nontrivial periodic ideal of $\frac{R}{P_\alpha}$, which is a contradiction. Thus $P(R) \subseteq P_\alpha$, and hence $P(R) \subseteq \bigcap_\alpha P_\alpha$.

On the other hand, for any $a \in R - P(R)$, (a) is not periodic. Thus, by Lemma 1 there exists b in (a) such that $b^n - b^{n+1}p(b) \neq 0$ for all $n \in Z^+$ and all $p(X) \in Z[X]$. Let $H = \{b^n - b^{n+1}p(b) \mid n \in Z^+, p(X) \in Z[X]\}$, and let $\mathcal{A}$ be the set of all ideals P in R with $P \cap H = \emptyset$. Then $\mathcal{A} \neq \emptyset$ since $0 \in \mathcal{A}$, so there exists a maximal element P_β in $\mathcal{A}$ by Zorn's lemma.

We claim that P_β is a prime ideal of R. Let A, B be ideals of R such that $A \not\subseteq P_\beta$ and $B \not\subseteq P_\beta$. Since $A + P_\beta \neq P_\beta$, $B + P_\beta \neq P_\beta$, both $A + P_\beta$ and $B + P_\beta$ meet H, say $b^m - b^{m+1}f(b) \in A + P_\beta$, $b^n - b^{n+1}g(b) \in B + P_\beta$ for some $m, n \in Z^+, f(X), g(X) \in Z[X]$. Then $b^{m+n} - b^{m+n+1}h(b) = \big(b^m - b^{m+1}f(b)\big)\big(b^n - b^{n+1}g(b)\big) \in (A+P_\beta)(B+P_\beta) \subseteq AB+P_\beta$, where $h(X) = f(X) + g(X) - Xf(X)g(X)$. But $b^{m+n} - b^{m+n+1}h(b) \notin P_\beta$, hence $AB \not\subseteq P_\beta$. Then P_β is prime.

Received by the editors September 22, 1993; revised April 20, 1994.
AMS subject classification: 16N60, 16N80.

https://doi.org/10.4153/CMB-1995-030-7 Published online by Cambridge University Press

Next we prove that $\frac{R}{P_\beta}$ contains no nontrivial periodic ideals. Let $I \supset P_\beta$ be an ideal of R and $\frac{I}{P_\beta}$ a nontrivial periodic ideal of $\frac{R}{P_\beta}$. Then, by the maximality of P_β there exist an integer $m \in Z^+$ and a polynomial $f(X) \in Z[X]$ such that $b^m - b^{m+1}f(b) \in I$, so there exist distinct positive integers s and t with $s < t$ such that $\Big(\big(b^m - b^{m+1}f(b)\big) + P_\beta\Big)^s = \Big(\big(b^m - b^{m+1}f(b)\big) + P_\beta\Big)^t$, and therefore $\big(b^m - b^{m+1}f(b)\big)^s - \big(b^m - b^{m+1}f(b)\big)^t \in P_\beta$, which contradicts the choice of P_β since $\big(b^m - b^{m+1}f(b)\big)^s - \big(b^m - b^{m+1}f(b)\big)^t$ can be written in the form $b^{ms} - b^{ms+1}l(b)$, $l(X) \in Z[X]$. Then $\frac{R}{P_\beta}$ contains no nontrivial periodic ideals.

Note that $a \notin P_\beta$. Then $a \notin \bigcap_\beta P_\beta$, and hence $a \notin \bigcap_\alpha P_\alpha$. Then $\bigcap_\alpha P_\alpha \subseteq P(R)$. This completes the proof of Theorem 1.

LEMMA 2. *Let a ring R be a subdirect product of rings R_α with $P(R_\alpha) = 0$. Then $P(R) = 0$.*

PROOF. Let π_α be the natural homomorphism of R onto R_α. If $P(R) \neq 0$, then $\pi_\alpha\big(P(R)\big) \neq 0$ for some α, so $0 \neq \pi_\alpha\big(P(R)\big) \subseteq P\big(\pi_\alpha(R)\big)$, which is a contradiction. Hence $P(R) = 0$.

An immediate consequence of Theorem 1 and Lemma 2 is the following

THEOREM 2. *Let R be any ring. Then $P(R) = 0$ if and only if R is a subdirect product of prime rings R_α with $P(R_\alpha) = 0$.*

An alternative way of stating the results of Bell and Klein is that periodicity is a radical property in the sense of Kurosh and Amitsur, and that $P(R)$ is the corresponding radical which we may call the periodic radical of R. Of course, $P(R)$ contains the nil radical $N(R)$; hence it is appropriate to inquire what properties $P(R)$ and $N(R)$ share.

Divinsky [3] defines a class M of rings to be a *special class* if it satisfies the following three properties:

(1) Every ring in the class M is a prime ring.
(2) Every nonzero ideal of a ring in M is also a ring in M.
(3) If A is a ring of M which is an ideal of some ring K, and if A^* is the two-sided annihilator of A in K, then $\frac{K}{A^*}$ is in M.

With each special class M there is associated a radical property, and the corresponding radical of a ring R turns out to be the intersection of all ideals P_α of R such that $\frac{R}{P_\alpha}$ is in M [3, Lemma 80]. Radicals which arise in this way are called *special radicals*, and the nil radical is known to be a special radical. A trivial modification of the proof of Theorem 65 of [3] shows that the class of prime rings with no nonzero periodic ideals is a special class; and this result, combined with Theorem 1, yields

THEOREM 3. *The periodic radical is a special radical.*

ACKNOWLEDGEMENT. The author wishes to express his indebtedness and gratitude to the referee for his helpful suggestions and valuable comments.

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