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The Poincaré Inequality and Reverse Doubling Weights

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Abstract. We show that Poincaré inequalities with reverse doubling weights hold in a large class of irregular domains whenever the weights satisfy certain conditions. Examples of these domains are John domains.

1 Introduction

Let D be a domain in euclidean n-space $\mathbb{R}^n$, $n \geq 2$. Let $1 \leq p \leq q < \infty$. We say that D supports a (q, p)-Poincaré inequality with weights ν and μ, if there is a constant $c = c(q, p, \nu, \mu, D) < \infty$ such that

$$\inf_{a\in\mathbb{R}}\Big(\int_D |u(x) - a|^q \nu(x)\, dx\Big)^{\frac{1}{q}} \leq c\Big(\int_D |\nabla u(x)|^p \mu(x)\, dx\Big)^{\frac{1}{p}}, \tag{1.1}$$

where u is a Lipschitz function on D. If the inequality (1.1) holds for all Lipschitz functions u on D, then D is a (q, p)-Poincaré domain with weights ν and μ; we write $D \in \mathcal{P}\,(q, p, \nu, \mu)$. The constant c in (1.1) is called a *Poincaré constant.*

It is well known that for $\nu = \mu = 1$ bounded John domains are (q, p)-Poincaré domains for all $q \leq np/(n - p)$ when $p < n$, [1, Chapter 6]. Unbounded John domains satisfy the $\big(np/(n-p), p\big)$-Poincaré inequality with $\nu = \mu = 1$, [2, Corollary 4.6]. Examples of John domains are Lipschitz domains. But a John domain can have a rough boundary: a classical example is the Koch snowflake.

We prove that a bounded John domain D belongs to $\mathcal{P}(q, p, \nu, \mu)$ with $1 < p \leq q < \infty$ whenever ν and $\mu^{-\frac{1}{p-1}}$ are reverse doubling weights satisfying weak additional conditions; see Theorem 3.1. The result is extended to unbounded John domains in Corollary 4.2. We also show that the extra conditions on reverse doubling weights are not restrictive; see Section 4.

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2 Preliminaries

We assume that $1 < p \le q < \infty$. The abbreviation Q stands for the open cube $Q(x, r) = \{y \in \mathbb{R}^n : |x_i - y_i| < \frac{r}{2}, i = 1, \ldots, n\}$, where $x \in \mathbb{R}^n$ and $r > 0$. By a Whitney cube we mean a cube from a Whitney decomposition of a given domain, [8, VI 1]. If $t > 0$, then tQ denotes the cube Q dilated by a factor t.

We let $c(*, \ldots, *)$ denote a constant which depends only on the quantities appearing in the parentheses.

2.1 Reverse Doubling Weights

A *weight* (function) is a non-negative locally integrable function on $\mathbb{R}^n$.

A weight ν is a *doubling weight*, that is, ν satisfies a doubling condition, if there exists a constant $c < \infty$ such that

$$\int_{2Q} \nu(x)\, dx \le c \int_Q \nu(x)\, dx$$

for all cubes $Q \subset \mathbb{R}^n$.

A weight ν is a *reverse doubling weight* or satisfies a reverse doubling condition, if there exist constants $\delta \in (0, 1)$ and $\epsilon \in (0, 1)$ such that

$$\int_{\delta Q} \nu(x)\, dx \le \epsilon \int_Q \nu(x)\, dx$$

for all cubes $Q \subset \mathbb{R}^n$. We say that ν is a reverse doubling weight with a pair (δ, ϵ).

Doubling weights satisfy a reverse doubling condition. There are reverse doubling weights which are not doubling weights; see Example 5.2.

2.2 John Domains

We recall the definition of bounded John domains, [5]. A domain D is called an (α, β)-*John domain*, $0 < \alpha \le \beta < \infty$, if there is $x_0 \in D$ such that each $x \in D$ can be joined to x_0 by a curve $\vartheta \colon [0, l] \to D$ parametrized by its arc length with total length $l \le \beta$ and

$$\operatorname{dist}\big(\vartheta(t), \partial D\big) \ge \frac{\alpha}{l} t, \quad \text{for all } t \in [0, l].$$

The point x_0 is called a *John centre*. When a John centre is fixed, then α and β are fixed, and then by σD we mean the dilation of D by a factor $\sigma > 0$ with respect to the fixed John centre. Lipschitz domains are John domains, and the bounded (ϵ, δ)-domains of P. W. Jones are John domains, [3]. A classical example of an (ϵ, δ)-domain is the Koch snowflake. An example of a John domain which is not an (ϵ, δ)-domain is an (ϵ, δ)-domain from which an n-dimensional spire has been taken away; in the plane it is enough to take a slit away: $Q(0, 1) \setminus \{(x_1, 0) : 1/4 \le x_1 < 1/2\}$.

The above definition implies that D is bounded. The concept 'John domain' has been extended for unbounded domains, too, in [6] and [9]. We recall the definition.

Let E be a closed arc with endpoints a and b. The subarc between x and y is denoted by $E[x, y]$. For $x \in E \setminus \{a, b\}$ write

$$q(x) = \min\{\operatorname{dia}(E[a, x]), \operatorname{dia}(E[x, b])\}.$$

Let $\gamma \geq 1$. A domain D in $\mathbb{R}^n$ is a γ-John domain, if each pair of distinct points a and b in D can be joined by an arc E such that

$$\operatorname{cig} E(a, b) = \bigcup \big\{ B\big(x, q(x)/\gamma\big) \mid x \in E \setminus \{a, b\} \big\} \subset D.$$

The set $\operatorname{cig} E(a, b)$ is called a γ-cigar with core E joining a and b. Whenever D is bounded this gives exactly an (α, β)-John domain for some α and β. An unbounded John domain can be exhausted by bounded John domains according to the following result of J. Väisälä.

Theorem 2.1 [10, Theorem 4.6] *An η-John domain $D \subset \mathbb{R}^n$ can be written as the union of domains $D_1, D_2, \dots$ such that $\bar{D}_i$ is compact in D_{i+1} and D_i is an η_1-John domain with $\eta_1 = \eta_1(\eta, n)$.*

3 Main Result

We show that a bounded John domain is a Poincaré domain when the left hand side weight in (1.1) and the right hand side weight in (1.1) to the power $-1/(p-1)$ are reverse doubling weights and these weights satisfy certain conditions.

Theorem 3.1 *Let $1 < p \leq q < \infty$. Let ν and $\mu^{-\frac{1}{p-1}}$ be reverse doubling weights with respect to the pairs (δ_i, ϵ_i), $i = 1, 2$, respectively, such that*

$$\epsilon_1 < \delta_1^{(n-1)q/p} \quad \textit{and} \quad \epsilon_2 < \delta_2^{n-1}. \tag{3.1}$$

Then an (α, β)-John domain D belongs to $\mathcal{P}(q, p, \nu, \mu)$.

Proof We use the integral representation

$$|u(x) - u_A| \leq c(n)\Big(\frac{\beta}{\alpha}\Big)^{16n} \int_D |x - y|^{1-n} |\nabla u(y)|\, dy, \quad x \in D,$$

where $A = B^n\big(x_0, c(n)\alpha^{4n}/\beta^{5n}\big) \subset D$, from [4, Theorem 2.2 and Lemma 3.3], and Hölder's inequality with exponents p and $\frac{p}{p-1}$, to obtain

$$\begin{aligned} \int_D |u(x) - u_A|^q \nu(x)\, dx \\ \leq c(n, q)\Big(\frac{\beta}{\alpha}\Big)^{16nq} \int_D \Big(\int_D |x - y|^{1-n} |\nabla u(y)|^p \mu(y)\, dy \Big)^{\frac{q}{p}} \\ \times \Big(\int_D |x - y|^{1-n} \mu(x)^{-\frac{1}{p-1}}\, dy \Big)^{\frac{q(p-1)}{p}} \nu(x)\, dx. \end{aligned} \tag{3.2}$$

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Since D is a bounded John domain, there exist a cube Q and a constant $c(\alpha,\beta) > 0$ such that $D \subset Q$ and $c(\alpha,\beta)|D|^{\frac{1}{n}} = |Q|^{\frac{1}{n}}$. Here, $|*|$ is the Lebesgue measure of the sets in question. We fix $x \in Q$ and use the abbreviation $c(\alpha,\beta)|Q|^{\frac{1}{n}} = r$ and we exhaust $Q(x,r)$ with cubes $Q_i = k^{-i}Q(x,r)$, $i = 0,1,\dots$, where $k > 1$ is a fixed number. Hence, we obtain

$$\begin{aligned}\int_D |x-y|^{1-n}\mu(y)^{-\frac{1}{p-1}}\,dy &\le \int_{Q(x,r)} |x-y|^{1-n}\mu(y)^{-\frac{1}{p-1}}\,dy\\ &= \sum_{i=0}^{\infty}\int_{Q_i\setminus Q_{i+1}} |x-y|^{1-n}\mu(y)^{-\frac{1}{p-1}}\,dy\\ &\le \sum_{i=0}^{\infty}\int_{Q_i\setminus Q_{i+1}} k^{-(i+1)(1-n)}|Q|^{\frac{1-n}{n}}\mu(y)^{-\frac{1}{p-1}}\,dy\\ &\le \sum_{i=0}^{\infty}\int_{Q_i} k^{-(i+1)(1-n)}|Q|^{\frac{1-n}{n}}\mu(y)^{-\frac{1}{p-1}}\,dy.\end{aligned}$$

Whenever numbers $\delta_2 \in (0,1)$ and $\epsilon_2 \in (0,1)$ satisfy

$$\epsilon_2^{-\frac{\log k}{\log \delta_2}} k^{n-1} < 1, \quad \text{that is,} \quad \epsilon_2 < \delta_2^{n-1},$$

then the reverse doubling property of $\mu^{-\frac{1}{p-1}}$, with these $\delta_2 \in (0,1)$ and $\epsilon_2 \in (0,1)$ yields,

$$\begin{aligned}\int_Q |x-y|^{1-n}\mu(y)^{-\frac{1}{p-1}}\,dy &\le |Q|^{\frac{1-n}{n}}\sum_{i=0}^{\infty}\epsilon_2^{-\frac{\log k}{\log\delta_2}i}k^{-(i+1)(1-n)}\int_{Q(x,r)}\mu(y)^{-\frac{1}{p-1}}\,dy\\ &\le c(\epsilon_2,\delta_2)|Q|^{\frac{1-n}{n}}\int_{Q(x,r)}\mu(y)^{-\frac{1}{p-1}}\,dy\\ &\le c(\epsilon_2,\delta_2)|Q|^{\frac{1-n}{n}}\int_{3Q}\mu(y)^{-\frac{1}{p-1}}\,dy.\end{aligned}$$

Inequality (3.2) and the generalized Minkowski's inequality [8, p. 271] yield

$$\begin{aligned}(3.3)\qquad &\int_D |u(x)-u_A|^q\nu(x)\,dx\\ &\le c(\epsilon_2,\delta_2,n,q)\Big(\frac{\beta}{\alpha}\Big)^{16nq}\Big(\int_{3Q}\mu(y)^{-\frac{1}{p-1}}\,dy\Big)^{\frac{q(p-1)}{p}}\\ &\quad\times\int_D\Big(\int_D |x-y|^{1-n}|\nabla u(y)|^p\mu(y)\nu(x)^{\frac{p}{q}}\,dy\Big)^{\frac{q}{p}}\,dx\end{aligned}$$

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$$\le c(\epsilon_2,\delta_2,n,q)\Big(\frac{\beta}{\alpha}\Big)^{16nq}\Big(\int_{3Q}\mu(y)^{-\frac{1}{p-1}}\,dy\Big)^{\frac{q(p-1)}{p}}$$
$$\times\Big(\int_D\Big(\int_D(|x-y|^{1-n}|\nabla u(y)|^p\mu(y)\nu(x)^{\frac{p}{q}})^{\frac{q}{p}}\,dx\Big)^{\frac{p}{q}}\,dy\Big)^{\frac{q}{p}}$$
$$\le c(\epsilon_2,\delta_2,n,q)\Big(\frac{\beta}{\alpha}\Big)^{16nq}\Big(\int_{3Q}\mu(y)^{-\frac{1}{p-1}}\,dy\Big)^{\frac{q(p-1)}{p}}$$
$$\times\Big(\int_D\Big(\int_D(|x-y|^{1-n}\nu(x)^{\frac{p}{q}})^{\frac{q}{p}}\,dx\Big)^{\frac{p}{q}}|\nabla u(y)|^p\mu(y)\,dy\Big)^{\frac{q}{p}}.$$

Since ν satisfies a reverse doubling condition, similar calculations as above imply

$$\int_Q(|x-y|^{1-n})^{\frac{q}{p}}\nu(x)\,dx\le c(\epsilon_1,\delta_1,q/p)|Q|^{\frac{(1-n)q}{pn}}\int_{3Q}\nu(x)\,dx, \tag{3.4}$$

whenever $\epsilon_1<\delta_1^{(n-1)q/p}$. Inequalities (3.3) and (3.4) yield

$$\int_D|u(x)-u_A|^q\nu(x)\,dx\le c|3Q|^{(\frac{1}{n}-1)q}\Big(\int_{3Q}\nu(x)\,dx\Big)\Big(\int_{3Q}\mu(x)^{-\frac{1}{p-1}}\,dx\Big)^{\frac{(p-1)q}{p}}$$
$$\times\Big(\int_D|\nabla u(x)|^p\mu(x)\,dx\Big)^{\frac{q}{p}},$$

where $c=c(\epsilon_1,\epsilon_2,\delta_1,\delta_2,n,p,q)(\beta/\alpha)^{16nq}$ whenever $\epsilon_1<\delta_1^{(n-1)q/p}$ and $\epsilon_2<\delta_2^{n-1}$. Since ν and $\mu^{-\frac{1}{p-1}}$ are locally integrable,

$$|3Q|^{(\frac{1}{n}-1)q}\Big(\int_{3Q}\nu(x)\,dx\Big)\Big(\int_{3Q}\mu(x)^{-\frac{1}{p-1}}\,dx\Big)^{\frac{(p-1)q}{p}}\le c(D,\nu,\mu,p,q)<\infty. \tag{3.5}$$

Thus the assertion follows. ■

Remark 3.2 Since a cube in $\mathbb{R}^n$ is a John domain, our main theorem is valid for cubes also. Previously, for cubes the following result was proved by E. Sawyer and R. Wheeden, [7, Theorem 5]. If ν is a reverse doubling weight and μ is a weight, then a cube $Q_0\subset\mathbb{R}^n$ is a (q,p)-Poincaré domain with $1<p<q<\infty$, whenever there exists a constant $c<\infty$ such that the inequality

$$|Q|^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}}\Big(\fint_Q\nu(x)\,dx\Big)^{\frac{1}{q}}\Big(\fint_Q\mu(x)^{-\frac{1}{p-1}}\,dx\Big)^{\frac{p-1}{p}}\le c \tag{3.6}$$

holds for all cubes $Q\subset 8Q_0$. Note that our theorem does not require that the condition (3.6) should be valid for all cubes, but only to $3Q$ where Q is a cube to which D is included; see (3.5).

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4 The Poincaré Inequality with Reverse Doubling Weights in Unbounded Domains

If an unbounded domain can be exhausted by (q, p, ν, μ)-Poincaré domains with a fixed Poincaré constant, then this unbounded domain is also Poincaré domain under certain conditions.

Theorem 4.1 *Let D in $\mathbb{R}^n$ be an unbounded domain such that $D = \bigcup_{i=1}^{\infty} D_i$, where $D_i \in \mathcal{P}(q, p, \nu, \mu)$ with Poincaré constants $c(D_i) \le c_0$ for some constant c_0 and $D_i \subset \bar{D}_i \subset D_{i+1}$, $i = 1, 2, \dots$, and $\int_{D_1} \nu(x)\, dx > 0$. Let there be cubes Q_i such that $D_i \subset Q_i$ and*

$$|3Q_i|^{(\frac{1}{n}-1)q} \Big(\int_{3Q_i} \nu(x)\, dx \Big) \Big(\int_{3Q_i} \mu(x)^{-\frac{1}{p-1}}\, dx \Big)^{\frac{(p-1)q}{p}} \le c_1.$$

Then also $D \in \mathcal{P}(q, p, \nu, \mu)$.

Proof We proceed as in [2, Theorem 4.1] where the case $\nu = \mu = 1$ is considered. Let u be a Lipschitz function in D. We may assume that $\int_D u(x)\nu(x)\, dx < \infty$. Set

$$u_i = \frac{1}{\int_{D_i} \nu(x)\, dx} \int_{D_i} u(x)\nu(x)\, dx, \quad i = 1, 2, \dots .$$

We show that there is a convergent subsequence (u_{i_j}) of (u_i) and a number $b \in \mathbb{R}$ such that $\lim_{j\to\infty} u_{i_j} = b$ and

$$\Big(\int_D |u(x) - b|^q \nu(x)\, dx \Big)^{\frac{1}{q}} \le c \Big(\int_D |\nabla u(x)|^p \mu(x)\, dx \Big)^{\frac{1}{p}}.$$

We have to find an upper bound for $(|u_i|)$ which does not depend on i. Since

$$\begin{aligned} |u_i| &= \Big(\int_{D_1} \nu(x)\, dx \Big)^{-1} \int_{D_1} |u_i| \nu(x)\, dx \\ &\le \Big(\int_{D_1} \nu(x) \Big)^{-1} \Big(\int_{D_1} |u_i - u(x)| \nu(x)\, dx + \int_{D_1} |u(x)| \nu(x)\, dx \Big) \end{aligned}$$

and $\int_{D_1} |u(x)|\nu(x)\, dx < \infty$, we have to prove that also

$$\int_{D_1} |u_i - u(x)| \nu(x)\, dx < \infty.$$

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Since $D_1 \subset D_i \subset D$ and D_i is a Poincaré domain,

$$\begin{aligned}\int_{D_1} |u_i - u(x)|\nu(x)\,dx &\le \Big(\int_{D_1} \nu(x)\,dx\Big)^{1-1/q}\Big(\int_{D_1} |u_i - u(x)|^q \nu(x)\,dx\Big)^{1/q}\\ &\le \Big(\int_{D_1} \nu(x)\,dx\Big)^{1-1/q}\Big(\int_{D_i} |u_i - u(x)|^q \nu(x)\,dx\Big)^{1/q}\\ &\le c_0\Big(\int_{D_1} \nu(x)\,dx\Big)^{1-1/q}\Big(\int_{D_i} |\nabla u(x)|^p \mu(x)\,dx\Big)^{1/p}\\ &\le c_0\Big(\int_{D_1} \nu(x)\,dx\Big)^{1-1/q}\Big(\int_{D} |\nabla u(x)|^p \mu(x)\,dx\Big)^{1/p} < \infty.\end{aligned}$$

Hence (u_i) is a bounded sequence and there is a convergent subsequence (u_{i_j}) of (u_i) and a number $b \in \mathbb{R}$ such that $\lim_{j\to\infty} u_{i_j} = b$. Since $\int_D u(x)\nu(x)\,dx < \infty$, in fact the number $b = 0$. We rewrite (u_j) for the subsequence (u_{i_j}). Since

$$\lim_{j\to\infty} \chi_{D_j}(x)|u(x) - u_j|^q = \chi_D(x)|u(x) - b|^q,$$

Fatou's lemma and the fact that D_j is a Poincaré domain with a constant c_0 imply

$$\begin{aligned}\int_D |u(x) - b|^q \nu(x)\,dx &= \int_D \lim_{j\to\infty} \chi_{D_j}(x)|u(x) - u_j|^q \nu(x)\,dx\\ &= \liminf_{j\to\infty}\Big(c_0 \int_{D_j} |\nabla u(x)|^p \mu(x)\,dx\Big)^{q/p}\\ &\le \Big(c_0 \int_D |\nabla u(x)|^p \mu(x)\,dx\Big)^{q/p}.\end{aligned}$$

Hence also D is a Poincaré domain. ■

Corollary 4.2 *Let $1 < p \le q < \infty$. Let ν and $\mu^{-\frac{1}{p-1}}$ be reverse doubling weights with a pair (δ_i, ϵ_i), $i = 1, 2$, respectively, such that (3.1) holds. Then an unbounded John domain D is $\mathbb{R}^n$ is a (q, p, ν, μ)-Poincaré domain if for John domains $D_1, D_2, \dots$ in D's exhaustion there are cubes Q_i such that $D_i \subset Q_i$ and*

$$|3Q_i|^{(\frac{1}{n}-1)q}\Big(\int_{3Q_i} \nu(x)\,dx\Big)\Big(\int_{3Q_i} \mu(x)^{-\frac{1}{p-1}}\,dx\Big)^{\frac{(p-1)q}{p}} \le c_1.$$

Proof Theorems 2.1, 3.1, and 4.1. ■

5 Examples

We show that the conditions $\epsilon_1 < \delta_1^{(n-1)q/p}$ and $\epsilon_2 < \delta_2^{n-1}$ on reverse doubling weight constants (ϵ_i, δ_i), $i = 1, 2$, in (3.1) are not restrictive.

Example 5.1 Let v be a weight such that $0 < m \le v(x) \le M < \infty$ for all $x \in \mathbb{R}^n$. We show that the conditions in (3.1) are not restrictive. Let $\delta \in (0,1)$. We have

$$\int_{\delta Q} v(x)\,dx \le M|\delta Q| = M\delta^n|Q| = Mm^{-1}\delta^n m|Q| \le Mm^{-1}\delta^n \int_Q v(x)\,dx$$

for any bounded cube $Q \subset \mathbb{R}^n$. If we write $\epsilon = Mm^{-1}\delta^n$, then $\epsilon < \delta^{n-1}$ whenever $\delta < mM^{-1}$. Further, if $q < np/(n-1)$, then $\epsilon < \delta^{(n-1)q/p}$ whenever $\delta < (\frac{m}{M})^{p/np-q(n-1)}$.

We show that there are nontrivial unbounded reverse doubling weights such that the conditions hold. These weights are not doubling in the classical sense:

Example 5.2 We consider the case $n = p = 2$. Let

$$\mu(x) = e^{-(x_1+x_2)} \quad \text{and} \quad \nu(x) = e^{x_1+x_2}.$$

Let $\delta \in (0,1)$. Then ν and $\mu^{\frac{1}{p-1}}$ satisfy a reverse doubling condition with δ and $\epsilon = \delta^2$ and this ϵ is the smallest possible $\epsilon \in (0,1)$.

In the n-case, $n \ge 2$, and $p > 1$, we define

$$\mu(x) = e^{-(p-1)(x_1+\cdots+x_n)}$$

and set

$$\nu(x) = e^{x_1+\cdots+x_n}$$

for all $x \in \mathbb{R}^n$. For each $\delta \in (0,1)$ we can choose $\epsilon = \delta^n$ and the condition $\epsilon < \delta^{n-1}$ is valid as well as $\epsilon < \delta^{(n-1)q/p}$ whenever $q < np/(n-1)$.

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