Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1129–1131

PRESENTATIONS OF THE GROUPS SL(2, m) AND PSL(2, m)

J. G. SUNDAY

1. In this paper, we refine the presentations of Behr and Mennicke [**1**] for SL(2, m) and PSL(2, m) where m is odd. The group SL(2, m) is first shown to be presented by the following system of generators and relations:

$$S^m = T^2 = (ST)^3 = (S^{\frac{1}{2}(m+1)}TS^4T)^2, T^4 = 1. \tag{1.1}$$

The group PSL(2, m) appears as the factor group

$$S^m = T^2 = (ST)^3 = (S^{\frac{1}{2}(m+1)}TS^4T)^2 = 1. \tag{1.2}$$

This simplification then permits us to use the results of Schur [**3**] to establish three-relation presentations for these groups. SL(2, m) is ultimately presented by

$$S^m = T^2 = (ST)^3 = (S^{\frac{1}{2}(m+1)}TS^4T)^2, \tag{1.3}$$

and PSL(2, m) is presented by

$$S^m = 1,\ T^2 = (ST)^3,\ (S^{\frac{1}{2}(m+1)}TS^4T)^2 = 1. \tag{1.4}$$

These results do not depend on the restriction of m to odd primes p which Zassenhaus [**4**] imposed. In addition, they simplify the Zassenhaus presentation

$$S^p = (ST)^3,\ T^2 = 1,\ (S^{\frac{1}{2}(p^2+1)}TS^2T)^3 = 1, \tag{1.5}$$

of PSL(2, p), at the same time removing the exceptional case $p \equiv 17 \pmod{28}$ for which he must use the presentation

$$S^p = (ST)^3,\ T^2 = 1,\ (S^{\frac{1}{2}(p+1)}TS^2T)^3 = 1, \tag{1.6}$$

and the exceptional case $p \equiv 3 \pmod{28}$ for which neither of his presentations suffices to define PSL(2, p).

2. Gunning [**2**, pp. 8–10] gives a description of the group SL(2, m) which consists of 2×2 matrices of determinant 1 whose entries belong to the ring of integers modulo m. In terms of the prime factorization $m = \prod p^c$, the order of this group is

$$m^3 \prod (1 - 1/p^2).$$

The presentation

$$A^m = 1,\ (AB)^3 = B^2,\ B^4 = 1,\ (A^{\frac{1}{2}(m+1)}BA^2B)^3 = 1 \tag{2.1}$$

Received October 20, 1971 and in revised form, January 10, 1972.

for SL(2, m) was discovered by Behr and Mennicke [**1**, p. 1433] when m is odd. Let Z denote the central element B^2, and define $S = AZ$ and $T = BZ$. An equivalent presentation is obviously

$$(2.2) \qquad S^m = T^2 = (ST)^3 = Z, Z^2 = 1, (S^{\frac{1}{2}(m+1)}TS^2T)^3 = Z^{\frac{1}{2}(m+1)}.$$

Note that the elements

$$S = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

fulfill the relations (2.2). Coxeter noticed that they also satisfy

$$(S^{\frac{1}{2}(m+1)}TS^4T)^2 = Z.$$

Therefore, to show that (1.1) defines the same group SL(2, m), it is enough to show that (1.1) implies $(S^{\frac{1}{2}(m+1)}TS^2T)^3 = Z^{\frac{1}{2}(m+1)}$, where Z is the central element T^2. Letting $q = \frac{1}{2}(m + 1)$, it follows from (1.1) that

$$\begin{aligned}(S^qTS^2T)S(S^qTS^2T)^{-1} &= S^{q-1}(STSTZ)(TSTST)S^{-2}TS^{-q} \\ &= S^{q-1}T^{-1}S^{-1}S^{-1}ZS^{-2}TS^{-q} \\ &= S^{q-1}(S^qTS^4T)^{-1} \\ &= ZTS^4T^{-1}.\end{aligned}$$

Taking qth powers, noting that $S^{2m} = Z^2 = 1$, we find

$$(S^qTS^2T)S^q(S^qTS^2T)^{-1} = Z^qTS^2T^{-1} = Z^{q-1}TS^2T.$$

Finally,

$$Z^q = (S^qTS^4T)^2Z^{q-1} = S^qTS^2T(Z^{q-1}TS^2T)S^qTS^2TTS^2T = (S^qTS^2T)^3,$$

as required.

Now, let G be one of the groups defined by either (1.3) or (1.4). In the commutator quotient group of G, which is obtained by adding the relation $ST = TS$ to whichever of (1.3) or (1.4) defines G, the element $T = S^{-3}$ is the identity. Hence, in G, every element of the subgroup $\langle T^2 \rangle$ belongs to the commutator subgroup. Furthermore, $\langle T^2 \rangle$ is normal and $G/\langle T^2 \rangle$, presented by (1.2), is isomorphic to PSL(2, m). Since the commutator quotient group of PSL(2, m) is either C_1 or C_3 and the multiplicator is a 2-group, Schur's theory [**3**, p. 96] implies that G is either SL(2, m) or PSL(2, m). Since the group defined by (1.3) has SL(2, m) in the form (1.1) as a factor group it must in fact be SL(2, m). Finally since $(S^{\frac{1}{2}(m+1)}TS^4T)^2 = 1$ in (1.4), the group defined by (1.4) must be PSL(2, m).

References

1. H. Behr and J. Mennicke, *A Presentation of the groups* PSL(2, p), Can. J. Math. *20* (1968), 1432–1438.

2. R. C. Gunning, *Lectures on modular forms* (Princeton University Press, Princeton, 1962).
3. J. Schur, *Untersuchungen uber die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. *132* (1907), 85–137.
4. H. J. Zassenhaus, *A Presentation of the groups* PSL(2, p) *with three defining relations*, Can. J. Math. *21* (1969), 310–311.

University of Toronto,
Toronto, Ontario

https://doi.org/10.4153/CJM-1972-118-x Published online by Cambridge University Press