Glasgow Math. J. **59** (2017) 493–502.
doi:10.1017/S001708951600029X.

THE ŁOJASIEWICZ EXPONENT FOR WEIGHTED HOMOGENEOUS POLYNOMIAL WITH ISOLATED SINGULARITY

OULD M. ABDERRAHMANE

Déparement de Mathématiques, Université des Sciences, de Technologie et de Médecine BP. 880, Nouakchott, Mauritanie
e-mail: ymoine@univ-nkc.mr

(Received 18 May 2015; revised 23 September 2015; accepted 2 January 2016; first published online 10 June 2016)

Abstract. The purpose of this paper is to give an explicit formula of the Łojasiewicz exponent of an isolated weighted homogeneous singularity in terms of its weights.

2010 *Mathematics Subject Classification.* Primary 14B05, 32S05.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at 0.The Łojasiewicz exponent $L(f)$ of f is by definition

$$L(f) = \inf\{\lambda > 0 \ : \mid \mathrm{grad} f \mid \geq \mathrm{const.} \mid z \mid^{\lambda} \ \text{ near zero} \},$$

It is well known (see [**11**]) that the Łojasiewicz exponent can be calculated by means of analytic paths

$$L(f) = \sup\left\{ \frac{\mathrm{ord}(\mathrm{grad} f(\varphi(t)))}{\mathrm{ord}(\varphi(t))} \ : \ 0 \neq \varphi(t) \in \mathbb{C}\{t\}^n, \ \varphi(0) = 0 \right\}, \tag{1}$$

where $\mathrm{ord}(\phi) := \inf_i\{\mathrm{ord}(\phi_i)\}$ for $\phi \in \mathbb{C}\{t\}^n$. By definition, we put $\mathrm{ord}(0) = +\infty$.

Łojasiewicz exponents have important applications in singularity theory, for instance, Teissier [**22**] showed that C^0-sufficiency degree of f (i.e., the minimal integer r such that f is topologically equivalent to $f + g$ for all g with $\mathrm{ord}(g) \geq r + 1$) is equal to $[L(f)] + 1$, where $[L(f)]$ denote integral part of $L(f)$. Despite deep research of experts in singularity theory, it is not proved yet that Łojasiewicz exponent $L(f)$ is a topological invariant of f (in contrast to the Milnor number). An interesting mathematical problem is to give formulas for $L(f)$ in terms of another invariants of f or an algorithm to compute it. In the two-dimensional case, there are many explicit formulas for $L(f)$ in various terms (see [**5, 6, 10**]). Estimations of the Łojasiewicz exponent in the general case can be found in [**1, 7, 14, 19**].

The aim of this paper is to compute the Łojasiewicz exponent for the classes of weighted homogeneous isolated singularities in terms of the weights. In particular, we generalize a formula for $L(f)$ of Krasiński, Oleksik and Płoski [**9**] for weighted homogeneous surface singularity. Here, we give an alternative proof of the result of Tan, Yau and Zuo [**21**]. We were motivated by their papers. However, our considerations are based on other ideas. More precisely, we use the notion of weighted homogenous filtration introduced by Paunescu in [**18**], the geometric characterization of μ-constancy in [**13, 22**] and the result of Varchenko [**23**], which described the μ-constant stratum of weighted homogeneous singularities in terms of the mixed Hodge structures.

Moreover, we show that the Łojasiewicz exponent is invariant for all μ-constant deformation of weighted homogeneous singularity, which gives an affirmative partial answer to Teissier's conjecture [**22**].

NOTATION. To simplify the notation, we will adopt the following conventions: for a function $F(z, t)$ we denote by ∂F the gradient of F and by $\partial_z F$ the gradient of F with respect to variables z.

Let $\varphi,\ \psi : (\mathbb{C}^n, 0) \to \mathbb{R}$ be two function germs. We say that $\varphi(x) \lesssim \psi(x)$ if there exists a positive constant $C > 0$ and an open neighbourhood U of the origin in $\mathbb{C}^n$ such that $\varphi(x) \leq C\,\psi(x)$, for all $x \in U$. We write $\varphi(x) \sim \psi(x)$ if $\varphi(x) \lesssim \psi(x)$ and $\psi(x) \lesssim \varphi(x)$. Finally, $|\varphi(x)| \ll |\psi(x)|$ (when x tends to x_0) means $\lim_{x\to x_0} \frac{\varphi(x)}{\psi(x)} = 0$.

1. Weighted homogeneous filtration. Let $\mathbb{N}$ be the set of non-negative integers and $\mathcal{O}_n$ denote the ring of analytic function germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The Milnor number of a germ f, denoted by $\mu(f)$, is algebraically defined as the dim $\mathcal{O}_n/J(f)$, where $J(f)$ is the Jacobian ideal in $\mathcal{O}_n$ generated by the partial derivatives $\{\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\}$. Let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be the deformation of f given by $F(z, t) = f(z) + \sum c_\nu(t) z^\nu$, where $c_\nu : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ are germs of holomorphic functions. We use the notation $F_t(z) = F(z, t)$ when t is fixed.

From now, we shall fix a system of positive integers $w = (w_1, \dots, w_n) \in (\mathbb{N} - \{0\})^n$, the weights of variables z_i, $w(z_i) = w_i$, $1 \leq i \leq n$, and a positive integer $d \geq 2w_i$ for $i = 1, \dots, n$, then a polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$ is called weighted homogeneous of degree d with respect the weight $w = (w_1, \dots, w_n)$ (or type $(d; w)$) if f may be written as a sum of monomials $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ with

$$\alpha_1 w_1 + \cdots + \alpha_n w_n = d. \tag{2}$$

Comparing these weights with the $w' = (w'_1 \dots, w'_n)$ defined in [**9, 21**], from (2), we get $w'(z_i) = \frac{d}{w_i}$ for $i = 1, \dots, n$, so it follows that $w'_i \geq 2$ if and only if $d \geq 2w_i$. Also, we have

$$\max_{i=1}^{n} \left(w'_i - 1\right) = \max_{i=1}^{n} \left(\frac{d}{w_i} - 1\right).$$

There is another (weaker) definition of a weighted homogeneous polynomial. A polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$ is called a weak weighted homogeneous polynomial, if there exist n integers positive (weights) $w = (w_1, \dots, w_n)$ such that f may be written as a sum of monomials $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ with

$$\alpha_1 w_1 + \cdots + \alpha_n w_n = d.$$

We droop only the hypothesis $d \geq 2w_i$ for $i = 1, \dots, n$.

We may introduce (see [**18**]) the function $\rho(z) = (|z_1|^{\frac{2}{w_1}} + \cdots + |z_n|^{\frac{2}{w_n}})^{\frac{1}{2}}$. We also consider the spheres associated to this ρ

$$S_r = \{z \in \mathbb{C}^n \ :\ \rho(z) = r\}, \qquad r > 0.$$

https://doi.org/10.1017/S001708951600029X Published online by Cambridge University Press

Here $\cdot$ means the weighted action, with respect to the $\mathbb{C}^*$ action defined below

$$t \cdot z = (t^{w_1} z_1, \ldots, t^{w_n} z_n).$$

DEFINITION 1. Using ρ, we define a singular Riemannian metric on $\mathbb{C}^n$ by the following bilinear form

$$\left\langle \rho^{w_i} \frac{\partial}{\partial x_i}, \rho^{w_j} \frac{\partial}{\partial x_j} \right\rangle = \delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

We will denote by grad_w and $\| \, \|_w$, the corresponding gradient and norm associated with this Riemannian metric (for more details about these see [**18**]).

Let $f \in \mathcal{O}_n$. We denote the Taylor expansion of f at the origin by $\sum c_\nu z^\nu$. Setting $H_j(z) = \sum c_\nu z^\nu$ where the sum is taken over n with $< w, \nu >= w_1\nu_1 + \cdots + w_n\nu_n = j$, we can write the weak weighted Taylor expansion f

$$f(z) = H_d(z) + H_{d+1}(z) + \cdots \ ; H_d \neq 0.$$

We call d the weak weighted degree of f and H_d the weak weighted initial form of f about the weight. Furthermore, for any $f \in \mathcal{O}_n$ we get

$$\|\mathrm{grad}_w f(z)\|_w \lesssim \rho^{d_w(f)}(z), \tag{3}$$

where $d_w(f)$ denotes the degree of f with respect to w. Indeed, as all non-zero z, we find $\frac{1}{\rho(z)} \cdot z \in S_1$, moreover, we have $\frac{\partial H_j}{\partial z_i}$ is zero or a weak weighted homogeneous polynomial of degree $d - w_j$, then we obtain

$$\|\mathrm{grad}_w H_j \left(\frac{1}{\rho(z)} \cdot z \right) \|_w = \frac{\|\mathrm{grad}_w H_j(z)\|_w}{\rho(z)^j} \lesssim 1.$$

Therefore,

$$\| \ \mathrm{grad}_w f(z) \ \|_w \lesssim \sum_{j \geq d_w(f)} \| \ \mathrm{grad}_w H_j(z) \ \|_w \lesssim \rho^{d_w(f)}(z).$$

PROPOSITION 2. *Let $f \in \mathcal{O}_n$ be a weak weighted homogeneous isolated singularity of type $(d; w)$ at $0 \in \mathbb{C}^n$. Then*

$$\|grad_w f(z)\|_w \gtrsim \rho(z)^d. \tag{4}$$

Proof. Since f has only isolated singularity at the origin, then for small values of r we have

$$\|\mathrm{grad}_w f(z)\|_w = \left(\sum_{i=1}^{n} \left| \rho^{w_i}(z) \frac{\partial f}{\partial z_i}(z) \right|^2 \right)^{\frac{1}{2}} \gtrsim 1, \ \forall z \in S_r. \tag{5}$$

https://doi.org/10.1017/S001708951600029X Published online by Cambridge University Press

On the other hand, $\frac{\partial f}{\partial z_i}$ is weak weighted homogeneous of degree $d - w_i$ for $i = 1, \ldots, n$ and also, $\frac{r}{\rho(z)} \cdot z \in S_r$ for all non-zero z. Thus, by (5) we obtain

$$\|\mathrm{grad}_w f\left(\frac{r}{\rho(z)} \cdot z\right)\|_w = r^d \frac{\|\mathrm{grad}_w f(z)\|_w}{\rho(z)^d} \gtrsim 1.$$

This completes the proof of the proposition. □

2. The maximal and minimal coordinates. The class of weak weighted homogeneous polynomials is broader than the class of weighted homogeneous polynomials. In order to extend the main result announced by Tan, Yau and Zuo in **[21]** to this class, we introduce the maximal and the minimal coordinate.

Definition 3. Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a weak weighted homogenous of type $(d; w_1, \ldots, w_n)$, we set

$$\begin{aligned} M(w) &= \{i \in \{1, \ldots, n\} \mid d < 2w_i\}, \\ I_{\max_1} &= \max\{i \in M(w)\}, \ I_{\max_2} = \max\left\{i \in M(w) - \{I_{\max_1}\}\right\}, \ldots \\ I_{\max_k} &= \max\left\{i \in M(w) - \{I_{\max_1}, \ldots, I_{\max_{k-1}}\}\right\}, \end{aligned}$$

where k is the cardinal of $M(w)$. We have $M(w) = \{I_{\max_1}, \ldots, I_{\max_k}\}$. We set

$$I_{\min_1} = \begin{cases} I_{\max_1} \text{ if } z_i z_{I_{\max_1}} \text{ don't appear in } f \ \forall i = 1, \ldots n, \\ \min\left\{i \in \{1, \ldots, n\} \mid z_i z_{I_{\max_1}} \text{ appear in } f\right\}. \end{cases},$$

$$I_{\min_2} = \begin{cases} I_{\max_2} \text{ if } z_i z_{I_{\max_2}} \text{ don't appear in } f \ \forall i = 1, \ldots n, \\ \min\left\{i \in \{1, \ldots, n\} - \{I_{\min_1}\} \mid z_i z_{I_{\max_2}} \text{ appear in } f\right\}. \end{cases},$$

$\ldots\ldots$

$$I_{\min_k} = \begin{cases} I_{\max_k} \text{ if } z_i z_{I_{\max_k}} \text{ don't appear in } f \ \forall i = 1, \ldots n, \\ \min\left\{i \in \{1, \ldots, n\} - \{I_{\min_1}, \ldots, I_{\min_{k-1}}\} \mid z_i z_{I_{\max_2}} \text{ appear in } f\right\}. \end{cases}.$$

We put $I(f) = \left\{I_{\min_1}, \ldots, I_{\min_k}\right\}$, we define the maximal coordinates of the variables, the z_i, for $i \in M(w)$, i.e., the coordinates of weights $w(z_i) = w_i > \frac{d}{2}$, also we called the minimal coordinates of the variables, the z_i, for $i \in I(f)$. Finally, we set $M(f) = M(w) \cup I(f)$, $\ell(f)$ the cardinal of $M(f)$ and $w_{M(f)} = (w_1, \ldots, \widehat{w_k}, \ldots, w_n)$, where the hat means omission of all w_k such that $\mathsf{k} \in M(f)$.

3. The results. The main result of this paper is the following:

Theorem 4. *Let $f \in \mathcal{O}_n$ be a weak weighted homogeneous polynomial of type $(d; w_1, \ldots, w_n)$ defining an isolated singularity at the origin. Then*

$$L(f) = \begin{cases} \max_{i \notin M(f)}(\frac{d}{w_i} - 1) & \text{if } \ell(f) < n \\ 1 & \text{if } \ell(f) = n. \end{cases}$$

Note that if $d \geq 2w_i$ for all $i = 1, \ldots, n$, then $M(f) = \emptyset$ and we recover the following Theorem 5.

https://doi.org/10.1017/S001708951600029X Published online by Cambridge University Press

THEOREM 5. *Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be weighted homogeneous polynomial of type $(d; w)$, with $d \geq 2w_i$ for $i = 1, \ldots, n$ defining an isolated singularity at the origin $0 \in \mathbb{C}^n$. Then*

$$L(f) = \max_{i=1}^{n}\left(\frac{d}{w_i} - 1\right).$$

COROLLARY 6. *Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weak weighted homogeneous polynomial of type $(d; w)$, defining an isolated singularity at the origin in $\mathbb{C}^n$. For any deformation $F_t(z) = f(z) + \sum c_\nu(t) z^\nu$ for which $\mu(F_t) = \mu(f)$ is called μ-constant, then $L(F_t)$ is also constant.*

COROLLARY 7. *Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function If the weak weighted initial forms H_d of f define an isolated singularity at the origin, then*

$$L(f) = \begin{cases} \max_{i \notin M(H_d)}(\frac{d}{w_i} - 1) & \text{if } \ell(H_d) < n \\ 1 & \text{if } \ell(H_d) = n. \end{cases}$$

4. Proofs of the Theorem 5. Before starting the proofs, we will recall some important results on the geometric characterization of μ-constancy.

THEOREM 8 *Greuel [**8**], Lê-Saito [**13**], Teissier [**22**]. Let $F: (\mathbb{C}^n \times \mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be the deformation of a holomorphic $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity. The following statements are equivalent*:

(1) F is a μ-constant deformation of f.
(2) $\frac{\partial F}{\partial t_j} \in \overline{J(F_t)}$, where $\overline{J(F_t)}$ denotes the integral closure of the Jacobian ideal of F_t generated by the partial derivatives of F with respect to the variables $z_1, \ldots, z_n$.
(3) The deformation $F(z, t) = F_t(z)$ is a Thom map, that is,

$$\sum_{j=1}^{m} \left|\frac{\partial F}{\partial t_j}\right| \ll \|\partial F\| \text{ as } (z, t) \to (0, 0).$$

(4) The polar curve of F with respect to $\{t = 0\}$ does not split, that is,

$$\{(z, t) \in \mathbb{C}^n \times \mathbb{C}^m \mid \partial_z F(z, t) = 0\} = \{0\} \times \mathbb{C}^m \text{ near } (0, 0).$$

4.1. Proof of Theorem 5. First, by the Proposition 2 we get

$$\|\text{grad}_w f(z)\|_w^2 = \sum_{i=1}^{n} \left|\rho^{w_i}(z)\frac{\partial f}{\partial z_i}(z)\right|^2 \gtrsim \rho(z)^{2d}.$$

Therefore,

$$\rho(z)^{\min\{w_i\}} \|\text{grad} f(z)\| \gtrsim \rho(z)^d.$$

Hence,

$$\|\text{grad} f(z)\| \gtrsim \left(\sum_{i=1}^{n} |z_i|^{\frac{1}{w_i}}\right)^{d-\min\{w_i\}} \gtrsim |z|^{\frac{d-\min w_i}{\min w_i}} = |z|^{\max_{i=1}^{n}(\frac{d}{w_i}-1)},$$

https://doi.org/10.1017/S001708951600029X Published online by Cambridge University Press

it follows that $L(f) \leq \max_{i=1}^{n}(\frac{d}{w_i} - 1)$.

In order to show the opposite inequality we need the following lemma.

Lemma 9. *Let $f \in \mathcal{O}_n$ be a weighted homogeneous isolated singularity of type $(d; w)$. Suppose that $w_k = \min_{i=1}^{n} w_i$ and $V_{z_k}(f) \nsubseteq \{z_k = 0\}$, where*

$$V_{z_k}(f) = \left\{ z \in \mathbb{C}^n \ : \ \frac{\partial f}{\partial z_1}(z) = \cdots = \frac{\partial f}{\partial z_{k-1}}(z) = \frac{\partial f}{\partial z_{k+1}}(z) = \cdots = \frac{\partial f}{\partial z_n}(z) = 0 \right\}.$$

Then $L(f) = \frac{d}{w_k} - 1 = \max_{i=1}^{n}(\frac{d}{w_i} - 1)$.

Proof. See **[9]**, Proposition 2. □

We now want to prove the opposite inequality. Modulo a permutation coordinate of $\mathbb{C}^n$, we may assume that $w_1 \leq w_2 \leq \cdots \leq w_n$. Since f be a weighted homogeneous of degree d with isolated singularity. It is easy to check that the monomial $z_1^{q_1}$ or $z_1^{q_1} z_i$ appear in the expansion of f. There are three cases to be considered.

Case 1. In this case, we suppose $z_1 z_i$ appear in the expansion of f, since f defining an isolated singularity at the origin $0 \in \mathbb{C}^n$, there exist the terms $z_n^{q_n}$ or $z_n^{q_n} z_j$ with non-zero coefficients in f.

We first consider the case whereby $z_n^{q_n} z_j$ appear in f, from the hypotheses $d = w_1 + w_i \geq 2w_n \geq \cdots \geq 2w_1$, then we may write

$$\begin{aligned} d &= q_n w_n + w_j \geq (q_n - 1)w_n + w_i + w_j \geq w_i + w_j \geq w_i + w_1 \\ &= d \geq w_i + w_i \geq w_1 + w_i = d. \end{aligned}$$

Therefore, $q_n = 1$ and $w_1 = w_2 = \cdots = w_n$.

We will next consider the case whereby $z_n^{q_n}$ appear in f, since $\partial f(0) = 0$, we have $q_n \geq 2$, it follows that

$$d = q_n w_n \geq (q_n - 1)w_n + w_i \geq w_1 + w_i = d,$$

hence $q_n = 2$ and $w_1 = w_2 = \cdots = w_n$.

In the homogenous case $w_1 = w_2 = \cdots = w_n$, for any non-zero $a \in \mathbb{C}^n$, along the curve $\varphi(t) = t \cdot a = (t^{w_1} a_1, \ldots, t^{w_n} a_n)$, we obtain $\partial f(\varphi(t)) = t^{d-w_1} \partial f(a)$, it follows from (1) that

$$L(f) \geq \frac{\mathrm{ord}(\partial f(\varphi(t))}{\mathrm{ord}(\varphi(t))} = \frac{d}{w_1} - 1 = \max_{i=1}^{n}(\frac{d}{w_i} - 1).$$

This ends the proof of Theorem 5 in the first case.

Case 2. In this case, we suppose $z_1^{q_1}$ appear in the expansion of f and $z_1 z_i$ doesn't appear for $i = 2, \ldots, n$. Take an analytic path $\varphi(t) = (t, 0, \ldots, 0)$, then from (1) we get

$$L(f) \geq \frac{\mathrm{ord}(\partial f(\varphi(t))}{\mathrm{ord}(\varphi(t))} = q_1 - 1 = \frac{d}{w_1} - 1 = \max_{i=1}^{n}\left(\frac{d}{w_i} - 1\right).$$

This ends the proof of Theorem 5 in the second case.

https://doi.org/10.1017/S001708951600029X Published online by Cambridge University Press

Case 3. In this case, we suppose that $z_1^{q_1} z_i$ appear in the expansion of f with $q_1 \geq 2$. By lemma 9, it is enough to prove that $V_{z_1}(f) \not\subseteq \{z_1 = 0\}$. Indeed, suppose that $V_{z_1}(f) \subset \{z_1 = 0\}$. Then, we let the deformation $F(z, t) = f(z) + t z_1^{q_1}$ of f. Since,

$$V_t(F) = \{(z, t) \in \mathbb{C}^n \times \mathbb{C} \mid \partial_z F(z, t) = 0\} \subset V_{z_1}(F_t) = V_{z_1}(f) \subset \{z_1 = 0\},$$

this means that

$$\partial_z F(z, t) = 0 \text{ if and only if } \partial f(z) = 0.$$

Since f defining an isolated singularity, and hence, by (4) in Theorem 8 we get that F_t is μ-constant. According to the result of Varchenko's theorem [**23**], the monomial $z_1^{q_1}$ verifies $d_w(z_1^{q_1}) = q_1 w_1 \geq d$. But $d = q_1 w_1 + w_i > q_1 w_1 \geq d$, which is a contradiction. This completes the proof of Theorem 5.

5. Proofs of the Theorem 4, Corollary 6 and Corollary 7

5.1.Proof of Theorem 4

Proof. Without loss of generality, we suppose that $w_1 \leq w_2 \leq \cdots \leq w_n$. By the proof of the Theorem 5, only the first case in the opposite inequality can be considered. This remains us to consider the case where $z_1 z_i$ appears in the expansion of f.

Let $d < 2w_{n-k+1} \leq \cdots \leq 2w_n$, and so $M(w) = \{n, \ldots, n-k+1\} = \{I_{\max_1}, \ldots, I_{\max_k}\}$. Since f defining an isolated singularity, it is easy to check that the monomial z_n^q or $z_n^q z_j$ appear in expansion of f, then $q w_n = d$ or $q w_n + w_j = d$, but $\partial f(0) = 0$ and $d < 2w_n$, so that $z_n z_j$ appear in f. Moreover, for any monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ of f with $\alpha_n \neq 0$, we have

$$2w_n > d = \sum_{j<n} \alpha_j w_j + \alpha_n w_n \geq \left(\sum_{j<n} \alpha_j \right) w_1 + w_n \geq w_1 + w_i = d.$$

Then, $\sum_{j<n} \alpha_j = 1$, $\alpha_n = 1$ and $w_i = w_n$.

Therefore we may write

$$f(z) = a_{I_{\min_1}} z_{I_{\min_1}} z_n + \sum_{j \neq I_{\min_1}} a_j z_j z_n + f(z_1, \ldots, z_{n-1}, 0), \quad a_{I_{\min_1}} \neq 0,$$

for $a_j \neq 0$, we have $d = w_n + w_{I_{\min_1}} = w_n + w_j = w_1 + w_i$, so we obtain $w_j = w_{I_{\min_1}} = w_1$. After permutation of coordinates with same weights it can be written as

$$f(z) = z_1 z_n + \sum_{j>1} a_j z_j z_n + f(z_1, \ldots, z_{n-1}, 0).$$

Then we may assume, by a change of coordinates $\xi_1 = z_1 + \sum_{j>1} a_j z_j$, that $f(z) = z_1 z_n + f(z_1, \ldots z_{n-1}, 0) = z_1(z_n + g(z)) + f(0, z_2 \ldots, z_{n-1}, 0)$ also by a change of coordinates $\xi_n = z_n + g(z)$, we can assume that

$$f(z) = z_1 z_n + f(0, z_2, \ldots, z_{n-1}, 0).$$

We set $h(z_1, \dots, z_{n-2}) = f(0, z_1, \dots, z_{n-2}, 0)$, obviously implies $L(f) = L(h)$. For $M(f) \neq \{1, \dots, n\}$, it follows by elimination of the maximal and minimal coordinates that $L(f) = L(h)$, where $h \in \mathcal{O}_{n-\ell(f)}$ be weighted homogenous of type $(d; w_{M(f)})$. Therefore by Theorem 5, we get

$$L(f) = L(h) = \max_{i \notin M(f)} (\frac{d}{w_i} - 1).$$

For $M(f) = \{1, \dots, n\}$, then we can suppose, by the splitting lemma, that $f(z) = z_1^2 + \dots + z_n^2$, thus $L(f) = 1$. The Theorem 4 is proved. □

5.2. Proof of Corollary 6. Let $f_t(z) = f(z) + \sum_\nu c_\nu(t) z^\nu$ be a deformation μ-constant of a weighted homogeneous polynomial f of degree d with isolated singularity. Since $c_\nu(0) = 0$, we can write

$$f_t(z) = f(z) + t g_t(z).$$

By a result of Varchenko's theorem [**23**], the deformation g_t verifies $d_w(g_t) \geq d$ for all t. This together with (3) and (4) gives

$$\begin{aligned} \|\mathrm{grad}_w f_t(z)\|_w &\geq \|\mathrm{grad}_w f(z)\|_w - |t| \|\mathrm{grad}_w g_t(z)\|_w \\ &\gtrsim \rho^d(z), \qquad \text{as } |t| \ll 1. \end{aligned}$$

Moreover, by a similar argument to the proof of the first inequality in Theorem 5 we find the following:

$$L(f_t) \leq \max_{i=1}^{n} \left(\frac{d}{w_i} - 1 \right).$$

Also, using the processes of elimination of the maximal and minimal coordinates, we can get

$$L(f_t) \leq \begin{cases} \max_{i \notin M(f)} (\frac{d}{w_i} - 1) & \text{if } \ell(f) < n \\ 1 & \text{if } \ell(f) = n. \end{cases}$$

By the semi-continuity of the Łojasiewicz exponent in holomorphic μ-constant families of isolated singularities [**20, 22**], we find that

$$L(f) = \begin{cases} \max_{i \notin M(f)} (\frac{d}{w_i} - 1) & \text{if } \ell(f) < n \\ 1 & \text{if } \ell(f) = n. \end{cases} \leq L(f_t).$$

Then the result follows.

5.3. Proof of Corollary 7. Let $d = d_w(f)$, it says that f can be written in the form

$$f(z) = H_d(z) + H_{d+1}(z) + \cdots ; H_d \neq 0.$$

Take the following family of singularities

$$f_t(z) := H_d(z) + t H_{d+1}(z) + t^2 H_{d+2}(z) + \cdots , t \in \mathbb{C}.$$

It follows from the isolated singularity of H_d and the theorem in ([**2**] Section 12.2) that f_t is a μ-constant deformation of H_d, and so by the above corollary we find

$$L(f) = L(f_t) = L(H_d) = \begin{cases} \max_{i \notin M(H_d)}(\frac{d}{w_i} - 1) & \text{if } \ell(H_d) < n \\ 1 & \text{if } \ell(H_d) = n. \end{cases}$$

This completes the proof of the corollary.

REMARK 10. For three variables $n = 3$, Krasiński, Oleksik and Płoski proof in [**9**] that

$$L(f) = \min\left(\max_{i=1}^{n}\left(\frac{d}{w_i} - 1\right),\ \mu(f)\right).$$

But this is not valid for n greater than 3, indeed, let

$$f(z_1, z_2, z_3, z_4) = z_1z_4 + z_1^{10} + z_2^5 + z_3^5,$$

is weak weighted homogeneous of type (10; 1, 2, 2, 9). Since $\mu(f) = \prod_i^n(\frac{d}{w_i} - 1)$ by the Milnor–Orlik formula [**16**], then $\mu(f) = 16$. Moreover, it easy to cheek that $L(f) = 4$ and $\max_{i=1}^n(\frac{d}{w_i} - 1) = 10$, hence $L(f) < \min(\max_{i=1}^n(\frac{d}{w_i} - 1),\ \mu(f))$.

EXAMPLE 11. Let

$$f(z) = z_1z_6 + z_1^{12} + z_2z_5 + z_3^4 + z_4^3 + z_2^6,$$

f is weak weighted homogenous of type (12; 1, 2, 3, 4, 10, 11) with isolated singularity, since $M(w) = \{5, 6\}$ and $M(f) = \{1, 2, 5, 6\} \subsetneq \{1, \dots, 6\}$, then by Theorem 4 we get

$$L(f) = \max_{i \notin M(f)}\left(\frac{d}{w_i} - 1\right) = 3.$$

EXAMPLE 12. Let

$$f(z) = z_1z_6 + z_2z_5 + z_3z_4,$$

f is weak weighted homogenous of type (12; 1, 2, 3, 9, 10, 11) defining an isolated singularity, since $M(w) = \{4, 5, 6\}$ and $M(f) = \{1, \dots, 6\}$, then by Theorem 4 we get $L(f) = 1$. Also f can be seen as weighted homogenous of type (2; 1, 1, 1, 1, 1, 1), and hence by Theorem 5, $L(f) = 1$.

NOTE. A. Parusiński called my attention to S. Brzostowski's result [**3**], which has independently proved the Theorem 5 of this paper. But his proof is different from ours.

REFERENCES

1. O. M. Abderrahmane, On the Łojasiewicz exponent and Newton polyhedron, *Kodai Math. J.* **28** (2005), 106–110.

2. V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps,* Vol. 1, Mono- graphs Math., vol. 82 (Birkhäuser, Boston, 1985).

3. S. Brzostowski, *The Łojasiewicz Exponent of Semiquasihomogeneous Singularities,* http://arxiv.org/abs/1405.5179v1, May 2014.

4. S. Brzostowski, T. Krasiński and G. Oleksik, A conjecture on the Lojasiewicz exponent, *J. Singularities* **6** (2012), 124–130.

5. J. Chadzyński and T. Krasiński, The Łojasiewicz exponent of an analytic mapping of two complex variables at an isolated zero, in *Singularities* (PWN, Warszawa, 1988), 139–146.

6. A. Lenarcik, On the Łojasiewicz exponent of the gradient of a holomorphic function, in *Singularities SymposiumÎ Łojasiewicz 70*, Banach Center Publications, vol. 44 (PWN, Warszawa, 1998), 149–166.

7. T. Fukui, Łojasiewicz type inequalities and Newton diagrams, *Proc. Amer. Math. Soc.* **112** (1991), 1169–1183.

8. G.-M. Greuel, Constant Milnor number implies constant multiplicity for quasihomogeneous singularities, *Manuscritpta Math.* **56** (1986), 159–166.

9. T. Krasiński, G. Oleksik and A. Płoski, The Łojasiewicz exponent of an isolated weighted homogeneous surface singularity, *Proc. Amer. Math. Soc.* **137** (2009), 3387–3397.

10. T. C. Kuo and Y. C. Lu, On analytic function germs of two complex variables, *Topology* **16** (1977), 299Î–310.

11. M. Lejeune-Jalabert and B. Teissier, *Clôture integrale des idéaux et équisingularite, Séminaire Lejeune-Teissier Centre de Mathématiques, École Polytechnique* (Université Scientifique et Medicale de Grenoble, 1974).

12. D. T. Lê and C. P. Ramanujam, Invariance of MilnorÈs number implies the invariance of topological type, *Amer. J. Math.* **98** (1976), 67–78.

13. D.T. Lê and K. Saito, La constence du nombre de Milnor donne des bonnes stratifications, *Compt. Rendus Acad. Sci. Paris, série A* **272** (1973), 793–795.

14. B. Lichtin, Estimation of Łojasiewicz exponents and Newton polygons, *Invent. Math.* **64** (1981), 417–429.

15. J. Milnor, *Singular points of complex hypersurfaces* (Princeton University Press, Princeton, NJ, 1968).

16. J. Milnor and P. Orlik, Isolated singularities defined by weighted homogeneous polynomials, *Topology* **9** (1970), 385–393.

17. D. B. O'Shea, Topologically trivial deformations of isolated quasihomogeneous singularities are equimultiple, *Proc. A.M.S.* **101**(2) (1987), 260–262.

18. L. Paunescu, A weighted version of the Kuiper-Kuo-Bochnak-Łojasiewicz theorem, *J. Algebr. Geom.* **2** (1993), 69–79.

19. A. Płoski, Sur l'exposant d'une application analytique. I, *Bull. Polish Acad. Sci. Math.* **32** (1984), 669–673.

20. A. Ploski, Semicontinuity of the Lojasiewicz exponent, *Univ. Iagel. Acta Math.* **48** (2010), 103–110.

21. S. Tan, S. S.-T. Yau and H. Zuo, Łojasiewicz inequality for weighted homogeneous polynomial with isolated singularity, *Proc. Amer. Math. Soc.* **138** (2010), 3975–3984.

22. B. Teissier, Variétés polaires, *Invent. Math.* **40** (1977), 267–292

23. A. N. Varchenko, A lower bound for the codimension of the stratum μ-constant in term of the mixed Hodge structure, *Vest. Mosk. Univ. Mat.* **37** (1982), 29–31