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ON LINEAR ALMOST PERIODIC SYSTEMS WITH BOUNDED SOLUTIONS

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It is proved that in every neighbourhood of a system of linear differential equations with almost periodic skew-adjoint matrix with frequency module $\mathcal{F}$ there exists a system with frequency module contained in the rational hull of $\mathcal{F}$ possessing all almost periodic solutions.

1. INTRODUCTION

Let us consider the system of linear differential equations

$$\frac{dx}{dt} = A(t)x, \tag{1}$$

where $x \in \mathbb{C}^n$, $A(t)$ is an n-dimensional skew-adjoint matrix, $A(t) + A^*(t) = 0$, $t \in \mathbb{R}$, and we also suppose that $A(t)$ is a continuous function and it is Bohr almost periodic with frequency module $\mathcal{F}$. Let $X_A(t)$ be the fundamental matrix for system (1), $X_A(0) = I$, where I is the identity matrix.

The function $X_A(t)$ need not be almost periodic in t. The aim of this paper is to prove that in any neighbourhood of the matrix-function $A(t)$ (in the uniform topology on the real axis) there exists a skew-adjoint matrix-function $C(t)$ such that $C(t)$ and $X_C(t)$ are almost periodic with frequencies belonging to the rational hull of $\mathcal{F}$.

We note that in [**3, 4**] this statement was proved for systems with almost periodic matrix $A(t)$ which has a frequency basis of dimension two or three. If the matrix $A(t)$ is periodic in t the statement is trivial, taking into account Floquet's theorem.

2. MAIN RESULT

Let us denote by $U(n)$ the set of all unitary matrices of dimension n and by $SU(n)$ the set of unitary matrices of dimension n with determinant equal to 1. For $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ we define the norm $\|x\| = \left(\sum_{j=1}^{n} x_j \overline{x}_j\right)^{1/2}$, where $\overline{x}$ is the complex conjugate of x. The corresponding norm $\|A\|$ for the n-dimensional matrix

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A is defined as follows: $\|A\| = \sup\{\|Ax\| : x \in \mathbb{C}^n,\ \|x\| = 1\}$. Thus $\|Ax\| = 1$ for $A \in U(n)$.

The frequency module $\mathcal{F}$ of the almost periodic function $A(t)$ is defined to be the $\mathbb{Z}$-module of the real numbers, generated by the λ such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{2\pi i t\lambda} A(t)\,dt \neq 0. \tag{2}$$

$\mathcal{F}_{rat}$ is the rational hull of $\mathcal{F}$, that is, $\mathcal{F}_{rat} = \{\lambda/m : \lambda \in \mathcal{F},\ m \in \mathbb{Z}\}$.

Theorem 1. *In every neighbourhood of the system of linear differential equations (1) with almost periodic skew-adjoint matrix with frequency module $\mathcal{F}$ there exists a system with frequency module contained in the rational hull of $\mathcal{F}$ possessing all almost periodic solutions with frequencies belonging to $\mathcal{F}_{rat}$.*

The proof of theorem is preceded by two lemmas.

Lemma 1. *For $\varepsilon > 0$ and for positive integers n and d there exists a number $N(\varepsilon, n, d)$ such that for every compact metric space X of dimension at most d and for every homotopic trivial $u : X \to U(n)$ there exists a sequence $u = u_0, u_1, \dots,$ $u_{N(\varepsilon,n,d)} = I$ of continuous maps from X to $U(n)$ with*

$$\sup_{\varphi\in X} \|u_k(\varphi) - u_{k+1}(\varphi)\| \leqslant \varepsilon \quad \textit{for all} \quad k.$$

Proof: Let us consider a continuous homotopic trivial $a(\varphi) : X \to U(n)$. The function $a(\varphi)$ can be rewritten in the form $a(\varphi) = (\det a(\varphi))^{1/n} a_1(\varphi)$, where $a_1(\varphi) : X \to SU(n)$ is a homotopic trivial map. $D(\varphi) = (\det a(\varphi))^{1/n}$ is a homotopic trivial map from X to $U(1) = \{x \in \mathbb{C} : |x| = 1\}$. Hence there is $\theta_0 \in [0, 2\pi)$ such that the point $e^{i\theta_0}$ does not have an inverse image under the map $D(\varphi)$ and the function $D(\varphi)$ has the representation $D(\varphi) = e^{i\alpha(\varphi)}$, where $\alpha(\varphi) : X \to \mathbb{R}$ is a continuous function and $\theta_0 < \alpha(\varphi) < \theta_0 + 2\pi$.

The function $h(\varphi, t) = \exp(it\alpha(\varphi)) : X \times [0,1] \to U(1)$ is a homotopy from $D(\varphi)$ to the identity. Let us consider a sequence of functions

$$v_k(\varphi) = \exp\left(i\frac{k}{N_1}\alpha(\varphi)\right), \quad 0 \leqslant k \leqslant N_1$$

with natural N_1. We get the estimate

$$\begin{aligned} |v_{k+1} - v_k| &= \left|\exp\left(i\frac{k+1}{N_1}\alpha(\varphi)\right) - \exp\left(i\frac{k}{N_1}\alpha(\varphi)\right)\right| = \left|\exp\frac{i\alpha(\varphi)}{N_1} - 1\right| \\ &= \left|\cos\frac{i\alpha(\varphi)}{N_1} + i\sin\frac{i\alpha(\varphi)}{N_1} - 1\right| = 2\sin\frac{i\alpha(\varphi)}{2N_1} \leqslant \frac{2(\theta_0 + 2\pi)}{2N_1} \leqslant \frac{4\pi}{N_1}. \end{aligned}$$

We choose N_1 so that $4\pi/N_1 < \varepsilon/2$.

By Lemma 3.1 [5] for $\varepsilon > 0$ and for positive integers $n \geqslant 2$ and d there exists $N_2(\varepsilon, n, d)$ such that for every compact metric space X of dimension at most d and every homotopic trivial $u : X \to SU_n$, there exists a sequence $u = u_0, u_1, \dots, u_{N_2(\varepsilon,n,d)} = I$ of continuous maps from X to SU_n with

$$\sup_{\varphi \in X} \|u_k(\varphi) - u_{k+1}(\varphi)\| \leqslant \varepsilon/2 \quad \text{for all} \quad k.$$

By taking $N(\varepsilon, n, d) = \max\{N_1, N_2(\varepsilon, n, d)\}$ we complete the proof. □

REMARK 1. Analysis of [**5**, proof of Lemma 3.1] and [**1**, Lemmas 1.3 and 4.3] shows that Lemma 1 remains valid with $U(n)$ replaced by a compact Riemannian manifold Y with finite fundamental group $\pi_1(Y)$. Therefore Lemma 1 is valid for homotopic trivial maps from a compact metric space X of dimension at most d to the group $SO(n)$ of n-dimensional orthogonal matrices with determinant 1 for $n \geqslant 3$ (because $\pi_1(SO(n)) = \mathbb{Z}_2$ if $n \geqslant 3$ [**2**]).

LEMMA 2. *Suppose that the continuous function $A(\varphi) : T_m \to U(m)$ satisfies*

$$\sup_{\varphi \in T_m} \|A(\varphi) - I\| \leqslant \varepsilon \leqslant \frac{1}{2}. \tag{3}$$

Then there exists a continuous logarithm of the function $A(\varphi)$ defined on the torus T_m such that

$$\sup_{\varphi \in T_m} \|\ln A(\varphi)\| \leqslant \frac{4\sqrt{2}\varepsilon}{1 - 2\varepsilon}. \tag{4}$$

PROOF: We use the formula

$$\ln A(\varphi) = \frac{1}{2\pi i} \int_{\partial\Omega} (\lambda I - A(\varphi))^{-1} \ln \lambda \, d\lambda, \tag{5}$$

where the simply connected domain Ω in the complex plane contains the closure of the set of eigenvalues of $A(\varphi)$, $\varphi \in T_m$ and it does not contain zero [**6**].

By assumption (3) the eigenvalues of the matrix $A(\varphi)$, $\varphi \in T_m$ are contained inside the circle of radius ε with centre at the point $(1, 0)$ of the complex plane. The function $\ln A(\varphi)$ is continuous on the torus T_m.

Let $\partial\Omega$ in (5) be the circle of radius 2ε with centre at the point $(1, 0)$ of the complex plane,

$$\partial\Omega = \{\lambda : \lambda = 1 + 2\varepsilon e^{i\varphi}, \ \varphi \in [0, 2\pi]\}.$$

Hence for $\lambda \in \partial\Omega$ we get

$$\left\|(\lambda I - A(\varphi))^{-1}\right\| \leqslant \frac{1}{\varepsilon} \tag{6}$$

Let us estimate $|\ln \lambda|$ for $\lambda \in \partial\Omega$. These λ have the form

$$\lambda = 1 + 2\varepsilon e^{i\varphi} = \rho e^{i\theta},$$

where

$$\rho = \sqrt{1 + 4\varepsilon \cos\varphi + 4\varepsilon^2}, \;\; \theta = \arctan \frac{2\varepsilon \sin\varphi}{1 + 2\varepsilon \cos\varphi}.$$

Hence we have

$$\ln \lambda = \ln \rho + i\theta + 2\pi i k, \;\; k \in \mathbb{Z}.$$

Let $k = 0$, then

$$|\ln \lambda| \leqslant \sqrt{(\ln \rho)^2 + \theta^2}.$$

Further, $\ln \rho$ satisfies the inequalities

$$-\ln\left(1 + \frac{2\varepsilon}{1 - 2\varepsilon}\right) = \frac{1}{2}\ln\left(1 - 4\varepsilon + 4\varepsilon^2\right) \leqslant \ln \rho \leqslant \frac{1}{2}\ln\left(1 + 4\varepsilon + 4\varepsilon^2\right) = \ln(1 + 2\varepsilon),$$

hence

$$|\ln \rho| \leqslant \left|\ln\left(1 + \frac{2\varepsilon}{1 - 2\varepsilon}\right)\right| \leqslant \frac{2\varepsilon}{1 - 2\varepsilon}.$$

Similarly we get the estimate

$$|\theta| \leqslant \left|\arctan \frac{2\varepsilon}{1 - 2\varepsilon}\right| \leqslant \frac{2\varepsilon}{1 - 2\varepsilon}.$$

Therefore we conclude that

$$|\ln \lambda| \leqslant \frac{2\sqrt{2}\varepsilon}{1 - 2\varepsilon}. \tag{7}$$

Using (6) and (7) we get the estimate for the right-hand side of (5):

$$\sup_{\varphi \in T_m} \|\ln A(\varphi)\| \leqslant \frac{1}{2\pi}\frac{1}{\varepsilon}\frac{2\sqrt{2}\varepsilon}{1 - 2\varepsilon} 2\varepsilon 2\pi = \frac{4\sqrt{2}\varepsilon}{1 - 2\varepsilon},$$

which completes the proof of the lemma. ∎

REMARK 2. If $A(\varphi) \in SO(n)$ then it can be shown that the matrix $\ln A(\varphi)$ in (5) is real and satisfies the estimates (4).

PROOF OF THEOREM: Let $\varepsilon > 0$ be arbitrary. For the almost periodic function $A(t)$ with frequency module $\mathcal{F}$ there exists a quasiperiodic function $A_1(t)$ with frequencies belonging to $\mathcal{F}$ such that $\sup_{t\in\mathbb{R}} \|A(t) - A_1(t)\| < \varepsilon$. Hence we can consider system (1) with quasiperiodic matrix $A(t)$.

We can rewrite $A(t)$ in the form $A(t) = B(\omega_1 t, \dots, \omega_{m+1} t)$, where the continuous skew-adjoint matrix $B(\varphi_1, \dots, \varphi_{m+1})$ is periodic with period 1 in each coordinate φ_i, $i = 1, \dots, m+1$, and m is some positive integer. The real constants $\omega_1, \dots, \omega_{m+1}$ are rationally independent. Without loss of generality we may assume that $\omega_{m+1} = 1$. The $(m+1)$-dimensional torus $T_{m+1} = \mathbb{R}^{m+1}/\mathbb{Z}^{m+1}$ is the hull of the quasiperiodic function $A(t)$.

Consider now the collection of systems

$$\frac{dx}{dt} = B(\varphi \cdot t)x, \tag{8}$$

where $\varphi \cdot t = \omega t + \varphi$ is the irrational twist flow on the torus T_{m+1}, $\varphi = (\varphi_1, \dots, \varphi_{m+1}) \in T_{m+1}$. Then $A(t) = B(\varphi_0 \cdot t)$, where $\varphi_0 = (0, \dots, 0)$. Let $\Phi(\varphi, t)$ be the fundamental matrix for the system (8), $\Phi(\varphi, 0) = I$. It forms a cocycle

$$\Phi(\varphi, t_1 + t_2) = \Phi(\varphi \cdot t_1, t_2)\Phi(\varphi, t_1). \tag{9}$$

We consider the torus T_{m+1} as a product $T_{m+1} = T_m \times T_1$ of the m-dimensional torus T_m and of the circle T_1. Then $\varphi = (\psi, \xi)$, $\psi \in T_m$, $\xi \in T_1$ and $\Phi(\varphi, t) = \Phi(\psi, \xi, t)$.

For fixed t the function $\Phi(\psi, 0, t)$ forms a mapping $T_m \to U(n)$ which is homotopic to the identity in the space $U(n)$.

We consider the fundamental matrix $\Phi(\psi, 0, N)$, where the number

$$N = N\left(\frac{\varepsilon}{16}, m, n\right)$$

is the same as in Lemma 1. By Lemma 1 there exists a sequence of maps $M(\psi, k) : T_m \to U(n)$ such that $M(\psi, 0) = I$, $M(\psi, N) = \Phi^*(\psi, 0, N)$ and

$$\sup_{\psi \in T_m} \|M(\psi, k) - M(\psi, k+1)\| \leqslant \frac{\varepsilon}{16} \quad \text{for} \quad k = 0, \dots, N-1.$$

Obviously,

$$\sup_{\psi \in T_m} \|M(\psi, k+1)M^*(\psi, k) - I\| \leqslant \varepsilon/16, \qquad k = 0, \dots, N-1. \tag{10}$$

By Lemma 2 for $\varepsilon \in (0, \varepsilon_0]$ with sufficiently small $\varepsilon_0 > 0$, there exists a logarithm $\ln(M(\psi, k+1)M^*(\psi, k))$ continuous on T_m with

$$\sup_{\psi \in T_m} \|\ln(M(\psi, k+1)M^*(\psi, k))\| \leqslant \frac{\varepsilon}{2}. \tag{11}$$

Let $\alpha(t) : [0,1] \to [0,1]$ be a differentiable monotone increasing function satisfying the conditions $\alpha(0) = \alpha'(0) = \alpha'(1) = 0$, $\alpha(1) = 1$, $\alpha'(t) < 2$. We construct the function

$$N(\psi, t) = \exp[\alpha(t-k)\ln(M(\psi, k+1)M^*(\psi, k))]M(\psi, k)$$

for $t \in [k, k+1)$, $k = 0, \dots, N-1$.

$$\begin{aligned}\frac{\partial N(\psi,t)}{\partial t} &= \exp[\alpha(t-k)\ln(M(\psi,k+1)M^*(\psi,k))] \\ &\quad \times \ln(M(\psi,k+1)M^*(\psi,k))\alpha'(t-k)M(\psi,k), \\ \frac{\partial N(\psi,t)}{\partial t}N^*(\psi,t) &= \alpha'(t-k)\ln(M(\psi,k+1)M^*(\psi,k))\end{aligned}$$

for $t \in [k, k+1]$. By construction, the function $N(\psi, t)$ is continuously differentiable with respect to t, for $t \in [0, N]$. Using (11) we obtain for $t \in [0, N]$

$$\sup_{\psi \in T_m} \left\| \frac{\partial N(\psi,t)}{\partial t} N^*(\psi,t) \right\| \leqslant \varepsilon.$$

Let us consider the function $\Psi(\psi, t) = \Phi(\psi, 0, t)N(\psi, t)$. It satisfies the conditions $\Psi(\psi, 0) = \Psi(\psi, N) = I$, $\psi \in T_m$. We extend the function $\Psi(\psi, t)$ to the intervals $t < 0$ and $t > N$ by the formula

$$\Psi(\psi, t + kN) = \Psi(\psi \cdot kN, t), \quad k \in \mathbb{Z}, \tag{12}$$

where $\psi = (\psi_1, \dots, \psi_m)$, $\psi \cdot t = (\omega_1 t + \psi_1, \dots, \omega_m t + \psi_m)$. The function $\Psi(\psi, t)$ is uniformly continuous on the set $T_m \times [0, N]$. Therefore for $\varepsilon > 0$ there exists $\delta > 0$ such that if $\rho(\psi_1, \psi_2) < \delta$ then $\|\Psi(\psi_1, t) - \Psi(\psi_2, t)\| < \varepsilon$. Here $\rho(.,.)$ stands for the metric on the torus T_m.

For the irrational twist flow $\psi \cdot t$ on the torus T_m there exists a relatively dense set of integers q_δ such that $\rho(\psi, \psi \cdot Nq_\delta) < \delta$ for all $\psi \in T_m$. Then

$$\|\Psi(\psi, t) - \Psi(\psi, t + Nq_\delta)\| = \|\Psi(\psi, t) - \Psi(\psi \cdot Nq_\delta, t)\| < \varepsilon$$

for $t \in [0, N]$, $\psi \in T_m$. Therefore for the function $\Psi(\psi, t)$ there exists a relatively dense set of ε-almost periods, the function $\Psi(\psi, t)$ is almost periodic in t, and for fixed ψ it satisfies the system

$$\frac{d\Psi(\psi,t)}{dt} = B_1(\psi \cdot t, t)\Psi(\psi, t), \tag{13}$$

where

$$B_1(\psi \cdot t, t) = \frac{\partial \Psi(\psi,t)}{\partial t} \Psi^*(\psi,t), \ \psi \in T_m, \ t \in \mathbb{R}.$$

The function $B_1(\psi, t)$ is periodic in t with a period N and

$$B_1(\psi,t) = B_1(\psi, t+N), \ \psi \in T_m, \ t \in \mathbb{R}. \tag{14}$$

In order to verify equality (14) let us consider

$$\begin{aligned} B_1((\psi_1 \cdot t)\cdot N, t+N) &= \frac{\partial \Psi(\psi_1, t+N)}{\partial t} \Psi^*(\psi_1, t+N) \\ &= \frac{\partial \Psi(\psi_1 \cdot N, t)}{\partial t} \Psi^*(\psi_1 \cdot N, t) = B_1((\psi_1 \cdot N)\cdot t, t). \end{aligned}$$

Thus we obtain (14) if $\psi_1 \cdot (N+t) = \psi$. Hence the matrix $B_1(\psi,t)$ is periodic in each coordinate and $B_1(\psi \cdot t, t)$ is quasiperiodic.

For $t = Nq + t_1, \ 0 \leqslant t_1 < N, \ \varphi = (\psi, 0)$ we get the following estimate:

$$\begin{aligned} &\|B_1(\psi \cdot t, t) - B(\varphi \cdot t)\| \\ &\quad = \left\| \frac{\partial \Psi(\psi,t)}{\partial t} \Psi^*(\psi,t) - \frac{\partial \Phi(\psi,0,t)}{\partial t} \Phi^*(\psi,0,t) \right\| \\ &\quad = \left\| \frac{\partial \Psi(\psi \cdot Nq, t_1)}{\partial t} \Psi^*(\psi \cdot Nq, t_1) - \frac{\partial \Phi(\psi \cdot Nq, 0, t_1)}{\partial t} \Phi^*(\psi \cdot Nq, 0, t_1) \right\| \\ &\quad = \left\| \Phi(\psi \cdot Nq, 0, t_1) \frac{\partial N(\psi \cdot Nq, t_1)}{\partial t} N^*(\psi \cdot Nq, t_1) \Phi^*(\psi \cdot Nq, 0, t_1) \right\| \\ &\quad \leqslant \left\| \frac{\partial N(\psi \cdot Nq, t_1)}{\partial t} N^*(\psi \cdot Nq, t_1) \right\| \leqslant \varepsilon. \end{aligned}$$

The numbers $\omega_1, \dots, \omega_m, 1$ form a basis of $\mathcal{F}$ and the quasiperiodic function $B_1(\psi \cdot t, t)$ has the following expansion in the Fourier series:

$$B_1(\psi \cdot t, t) = \sum_{k,l} a_{kl} e^{2\pi i (k_1\omega_1 + \dots + k_m\omega_m + l/N)t}.$$

Therefore the frequencies of the function $B_1(\psi \cdot t, t)$ belong to $\mathcal{F}_{rat}$. A direct computation by (2) shows that the frequencies of the almost periodic function $\Psi(\psi, t)$ belong to $\mathcal{F}_{rat}$.

Thus we have constructed in the ε-neighbourhood of the almost periodic skew-adjoint function $A(t)$ an almost periodic skew-adjoint function $A_1(t) = B_1(\psi_0 \cdot t)$, $\psi_0 = (0, \dots, 0) \in T_m$ with frequencies belonging to $\mathcal{F}_{rat}$ and such that the fundamental matrix $X_{A_1}(t)$ is almost periodic. The proof of the theorem is now complete. ∎

REMARK 3. Theorem 1 remains valid for systems (1) considered in a real space $x \in \mathbb{R}^n$. In this case the matrix $A(t)$ is skew-symmetric and the fundamental matrix $X_A(t)$ is orthogonal for all $t \in \mathbb{R}$. The proof is practically identical to that for the complex case with regard to Remarks 1 and 2. We note that the case $n = 1$ is trivial and the result for the case $n = 2$ was proved in [4].

In [7] it is proved that those systems with k-dimensional frequency basis of the almost periodic function $A(t)$ whose solutions are not almost periodic form a subset of the second category (an intersection of a countable set of everywhere dense subsets) in the space of all systems (1) with k-dimensional frequency basis of $A(t)$. Therefore by Theorem 1 we obtain the following

COROLLARY 1. *Systems with k-dimensional frequency basis of $A(t)$ and with an almost periodic fundamental matrix form an everywhere dense set of the first category in the space of all systems (1) with k-dimensional frequency basis of the skew-adjoint matrix $A(t)$.*

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