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FUNCTORIAL RADICALS AND NON-ABELIAN TORSION, II

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The object of this paper is to complete and continue some matters in [**1**].

In [**1**], Section 2, the torsion and torsion-free functors, whose operation on the category of abelian groups are well known, were extended to the category of all groups as follows. For a group A, put $t_0(A)=0$ and $t_1(A)=$ the subgroup of A generated by the torsion elements of A. Inductively define $t_{n+1}(A)/t_n(A)=t_1(A/t_n(A))$, for every positive integer n. Then $T(A)=\bigcup_n t_n(A)$ is the smallest subgroup H of A such that A/H is torsion-free, [**1**], Th. 2.2. A group A satisfying $T(A)=A$ was called a *pre-torsion group*. In [**1**], 2.12 an example was constructed of a group A satisfying $t_1(A)\neq t_2(A)=A$. The question was posed whether for every positive integer n there exist groups A, satisfying $t_{n-1}(A)\neq t_n(A)=A$. Here we give an affirmative answer. In fact, such groups will be constructed, as well as pre-torsion groups A with $t_k(A)\neq A$ for every positive integer k, see Section 1.

In [**1**], Section 4, results concerning radicals and pre-radicals on the category (Λ,Σ)-mod of modules over a near-ring Λ distributively generated by a monoid Σ, were briefly presented. In Section 2 of this paper, proofs are supplied, as promised in [**1**], and some more results are given.

1.

We denote by $A*B$ the free product of groups A, B. For groups $A<B$ we denote by $[A]$ the normal closure of A in B.

Lemma 1.1. *For groups A, B and for $n\in\mathbb{N}$, $t_n(A*B)=[t_n(A)*t_n(B)]$.*

Proof. The only periodic elements in $A*B$ are conjugates of periodic elements in A or in B. Hence $t_1(A*B)$ is indeed the normal closure of $t_1(A)*t_1(B)$ in $A*B$. In proceeding from n to $n+1$, it suffices to show that

$$t_1(A*B/[t_n(A)*t_n(B)])=[t_{n+1}(A)*t_{n+1}(B)]/[t_n(A)*t_n(B)].$$

Now

$$A*B/[t_n(A)*t_n(B)]\cong A/t_n(A)*B/t_n(B)$$

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(see [3], p. 194), hence

$$t_1(A*B/[t_n(A)*t_n(B)])\cong[t_1(A/t_n(A))*t_1(B/t_n(B))]=[t_{n+1}(A)/t_n(A)*t_{n+1}(B)/t_n(B)];$$

here the normal closure is taken in $A/t_n(A)*B/t_n(B)$. But

$$t_{n+1}(A)/t_n(A)*t_{n+1}(B)/t_n(B)\cong t_{n+1}(A)*t_{n+1}(B)/[t_n(A)*t_n(B)],$$

hence the normal closure of this group is isomorphic to

$$[t_{n+1}(A)*t_{n+1}(B)]/[t_n(A)*t_n(B)]$$

and our claim is established for $n+1$.

Definition 1.2. A group A will be called *n-torsion-generated* (n a positive integer) if

$$t_n(A)=A\neq t_{n-1}(A) \tag{1}$$

In [1] a 1-torsion-generated group was said to be torsion-generated.

1.3. We construct inductively groups A_n which are n-torsion-generated. Clearly, any non-trivial torsion-generated group (for instance any non-trivial periodic group) may serve as A_1. Suppose A_n has been constructed which is n-torsion-generated. Take two copies of A_n, say B_n^1, B_n^2 and put $H_n=B_n^1*B_n^2$. It follows (by 1.1 and by [1], 2.16) that H_n is n-torsion-generated. By assumption there exist $b_i\in B_n^i\backslash t_{n-1}(B_n^i)$, $i=1,2$. Then $(b_1b_2)^m\notin t_{n-1}(H_n)$ for every positive integer m. Add a free generator to H_n, namely consider $H_n*\langle v_{n+1}\rangle$ and define A_{n+1} to be the quotient group of H_n modulo $v_{n+1}^2=b_1b_2$. Clearly $t_{n+1}(A_{n+1})=A_{n+1}$ but $t_n(A_{n+1})\neq A_{n+1}$ since $v_{n+1}\notin t_n(A_{n+1})$.

Observe that the construction may begin with any non-trivial group which is generated by its periodic elements. For example take $A_1=\langle x; x^2=1\rangle$ a group of order 2. Then, by the construction $H_2=\langle x,y;x^2=y^2=1\rangle$ and $A_2=\langle x,y,v;x^2=y^2=1,$ $v^2=xy\rangle$. This is precisely the group A of [1], 2.12, namely $\langle x_1,x_2;x_1^2=1,(x_1x_2^2)^2=1\rangle$, via the mapping $x_1\mapsto x$, $x_2\mapsto v$ (so $x_1x_2^2\mapsto y$).

Observation. A_2 is 2-solvable since $A_2/[v]$ is clearly a group of order 2. Hence the fact $T(A)=t_1(A)$, which is true for nilpotent groups ([1], 2.11), is not true for solvable groups.

1.4. The construction in 1.3 exhibits a multitude of n-torsion-generated groups (which may be constructed to be finitely presented). It may be generalised in the following sense. Consider a set of groups $\mathscr{S}$, $|\mathscr{S}|\geqq 2$, with $t_n(H)=H$ for all $H\in\mathscr{S}$ and with $t_{n-1}(H_0)\neq H_0$ for at least one H_0 in $\mathscr{S}$. Take a free product $G=\left(\underset{H\in\mathscr{S}}{*}H\right)*F$ with F free and consider any $1\neq f\in F$, $1\neq h_j\in H_j\in\mathscr{S}$ (for $j=1,\dots,r$), $h_0\in H_0\backslash t_{n-1}(H_0)$, $k\geqq 2$. Then G modulo $f^k=h_0h_1\dots h_r$ is $(n+1)$-torsion-generated.

1.5. Every n-torsion-generated group is evidently pre-torsion, $T(A)=A$. We construct a group A_ω with

$$T(A_\omega)=A_\omega \neq t_n(A) \qquad \text{for all} \quad n\in\mathbb{N}. \tag{2}$$

In 1.3 consider $A_n=B_n^1 \rightarrowtail H_n \twoheadrightarrow A_{n+1}$. This is clearly an embedding, so we take the limit $A_\omega=\bigcup_{n\in\mathbb{N}} A_n$. Then, by [**1**] 2.16, A_ω satisfies (2). (Clearly, A_ω may be constructed to be countably presented.) The following is established.

Theorem 1.6. *Every n-torsion-generated group may be embedded into a $(n+1)$-torsion-generated group. Every n-torsion-generated group may be embedded into a pre-torsion group which is not k-torsion-generated for every $k\in\mathbb{N}$.*

2.

We consider the collection Rad of *radicals* on (Λ, Σ)-mod, namely functors on (Λ, Σ)-mod which are normal subfunctors of the identity and satisfy $R(X/R(X))=0$ for all X. We assume the condition (a) of [**1**], hence the word "normal" may be omitted. Each radical R determines the class $\mathscr{B}_R$ of radical objects, which are the (Λ,Σ)-modules X satisfying $R(X)=X$, and the class $\mathscr{C}_R$ of semisimple objects, i.e., X such that $R(X)=0$, see [**1**].

For radicals R, S the composed functor RS is a radical, as shown in the next proposition. Is there any relationship between the classes of semisimple objects $\mathscr{C}_R$, $\mathscr{C}_S$ and $\mathscr{C}_{RS}$? Employing a common construction in varieties [**3**], we define $\mathscr{C}_R\circ\mathscr{C}_S=$ the collection of (Λ,Σ)-modules X such that there is a normal submodule X' in X with $X'\in\mathscr{C}_R$ and $X/X'\in\mathscr{C}_S$. It turns out that $\mathscr{C}_{RS}$ is precisely $\mathscr{C}_R\circ\mathscr{C}_S$. Moreover, the class $\mathbb{C}=\{\mathscr{C}_R \mid R\in \text{Rad}\}$ with the operation just defined turns into a monoid which is an epimorphic image of Rad, as shown by

Proposition 2.1. (i) Rad *is a monoid with respect to composition of functors*

(ii) $\mathbb{C}$ *is a monoid with respect to the operation* $\circ$ *defined above;*

(iii) *The map* $R\mapsto\mathscr{C}_R$ *is an epimorphism of* Rad *onto* $\mathbb{C}$.

Proof. For $R, S\in\text{Rad}$, RS is clearly a normal subfunctor of the identity. Under the natural epimorphism $\phi: A/RS(A)\to A/S(A)$, $\phi(S(A/RS(A)))\subset S(A/S(A))=0$, and so $S(A/RS(A))\subset \ker\phi=S(A)/RS(A)$. Therefore $RS(A/RS(A))\subset R(S(A)/RS(A))=0$. Now $A\in\mathscr{C}_{RS}$ if and only if $RS(A)=0$, i.e., iff $S(A)\in\mathscr{C}_R$. Since $A/S(A)\in\mathscr{C}_S$, $S(A)\in\mathscr{C}_R$ iff $A\in\mathscr{C}_R\circ\mathscr{C}_S$. Conversely, if $0\to K\to A\xrightarrow{\eta} A/K\to 0$, with $K\in\mathscr{C}_R$ and $A/K\in\mathscr{C}_S$, then $\eta(S(A))\subset S(A/K)=0$, i.e., $S(A)\subset K$, and so $RS(A)\subset R(K)=0$. Hence $A\in\mathscr{C}_{RS}$.

The operation $\circ$ in $\mathbb{C}$ generalises the composition of varieties in groups. Therefore the collection of varieties is a submonoid of $\langle\mathbb{C},\circ\rangle$.

Given a set $\mathscr{R}$ of radicals we define the intersection $S=\bigcap_{R\in\mathscr{R}} R$ by $S(X)=\bigcap_{R\in\mathscr{R}} R(X)$.

Proposition 2.2 *The intersection $S=\bigcap_{R\in\mathscr{R}} R$ is a radical;* $\langle\text{Rad},\cap\rangle$ *is a monoid.*

Proof. For $f: X \to Y$, $f(S(X)) \subset \cap f(R(X)) \subset \cap R(Y) = S(Y)$. Now $R(X/S(X)) \subset R(X)/S(X)$ for all $R \in \mathscr{R}$ since $X/S(X) \to X/R(X)$ (epimorphism with kernel $R(X)/S(X)$) takes $R(X/S(X))$ into 0. So $S(X/S(X)) \subset \cap(R(X)/S(X)) = 0$.

The intersection was employed in [1] to construct an idempotent radical $\bar{R}$ from a given radical R, such that $\mathscr{B}_{\bar{R}} = \mathscr{B}_R$. For ordinals v, denote $R^{v+1} = R \circ R^v$ and $R^v = \bigcap_{\iota < v} R^\iota$ for limit ordinals v. Then put $\bar{R} = R^v$, v the first ordinal such that $R^v = R^{v+1}$.

We denote by $p\,\text{Rad}$ the collection of *pre-radicals* on (Λ, Σ)-mod, namely normal subfunctors of the identity on (Λ, Σ)-mod. Evidently $\langle p\,\text{Rad}, \circ \rangle$ is a monoid. With $\mathbb{B} = \{\mathscr{B}_R \mid R \in p\,\text{Rad}\}$ we have an obvious isomorphism of monoids $\langle p\,\text{Rad}, \circ \rangle$ and $\langle \mathbb{B}, \cap \rangle$.

The following additional operation was defined on $p\,\text{Rad}$, in [1]. For $R, S \in p\,\text{Rad}$ and $A \in (\Lambda, \Sigma)$-mod, $(R \times S)(A)/S(A) = R(A/S(A))$. This operation was employed to construct a radical $\tilde{R}$ from a pre-radical R as follows. For every ordinal v, $R_{v+1} = R \times R_v$ and $R_v = \bigcup_{\iota < v} R_\iota$ for limit ordinals. Then put $\tilde{R} = R_v$, v the first ordinal for which $R_v = R_{v+1}$. Then $\tilde{R}$ is a radical, and $\tilde{R}$ is idempotent if R is. (The classes $\mathscr{B}_R, \mathscr{C}_R$ are defined identically for pre-radicals, as they were for radicals.)

Proposition 2.3. *Let R be an idempotent pre-radical. Then $\mathscr{B}_{R_n} \circ \mathscr{B}_{R_m} \subset \mathscr{B}_{R_{n+m}}$ for all positive integers n, m.*

Proof. Let $B \lhd A$, $B \in \mathscr{B}_{R_n}$, $A/B \in \mathscr{B}_{R_m}$. Now $B = R_n(B) \subset R_n(A)$, so $R_m(A/R_n(A)) = A/R_n(A)$. Therefore, for $m = 1$ we obtain $R_{n+1}(A)/R_n(A) = R(A/R_n(A)) = A/R_n(A)$, i.e., $R_{n+1}(A) = A$. For $m > 1$, put $R_{m-1}(A/R_n(A)) = K/R_n(A)$. Then clearly $R_{m-1}(K/R_n(A)) = K/R_n(A)$, and $R_n(R_n(A)) = R_n(A)$. Therefore we may inductively assume that $R_{n+m-1}(K) = K$. Now

$$
\begin{aligned}
A/K &\cong (A/R_n(A))/(K/R_n(A)) = R_m(A/R_n(A))/R_{m-1}(A/R_n(A)) \\
&= R((A/R_n(A))/R_{m-1}(A/R_n(A))) = R((A/R_n(A))/(K/R_n(A))) \\
&\cong R(A/K).
\end{aligned}
$$

Hence $R(A/K) = A/K$, and $R_{n+m-1}(K) = K$. Therefore $R_{n+m}(A) = A$.

Proposition 2.4. *Let $R \in p\,\text{Rad}$. Then $\mathscr{C}_{R^n} \circ \mathscr{C}_{R^m} \subset \mathscr{C}_{R^{n+m}}$ for all positive integers, n, m.*

Proof. Let $K \lhd A$, with $R^n(K) = R^m(A/K) = 0$. Then $(R^m(A) + K)/K \subset R^m(A/K) = 0$, and so $R^m(A) \subset K$. Therefore $R^{n+m}(A) = R^n(R^m(A)) \subset R^n(K) = 0$.

A well-known example in group theory: Let $K \lhd A$, K a group nilpotent of class $\leqq n$, A/K nilpotent of class $\leqq m$. Then A is nilpotent of class $\leqq n + m$.

The previous example suggests the importance of extending beyond the classes $\mathscr{C}_{R^n}$, R a pre-radical, or radical, in order to obtain a theory which would include the class of nilpotent groups, and the class of solvable groups. This may be done as follows:

Lemma 2.5. *Let $\mathscr{R}$, $\mathscr{S}$, $\mathscr{T}$ be subsets of* Rad. *Put*

$$
{}_{\mathscr{R}}\mathscr{C} = \bigcup_{R \in \mathscr{R}} \mathscr{C}_R, \quad {}_{\mathscr{S}}\mathscr{C} = \bigcup_{S \in \mathscr{S}} \mathscr{C}_S, \quad {}_{\mathscr{T}}\mathscr{C} = \bigcup_{T \in \mathscr{T}} \mathscr{C}_T.
$$

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Then

$$({}_{\mathscr{R}}\mathscr{C} \circ {}_{\mathscr{S}}\mathscr{C}) \circ {}_{\mathscr{T}}\mathscr{C} = {}_{\mathscr{R}}\mathscr{C} \circ ({}_{\mathscr{S}}\mathscr{C} \circ {}_{\mathscr{T}}\mathscr{C}).$$

Proof. Let $A \in ({}_{\mathscr{R}}\mathscr{C} \circ {}_{\mathscr{S}}\mathscr{C}) \circ {}_{\mathscr{T}}\mathscr{C}$. Then there exists $B \lhd A$ such that $B \in {}_{\mathscr{R}}\mathscr{C} \circ {}_{\mathscr{S}}\mathscr{C}$ and $A/B \in {}_{\mathscr{T}}\mathscr{C}$. Also there exists $C \lhd B$ such that $C \in {}_{\mathscr{R}}\mathscr{C}$ and $B/C \in {}_{\mathscr{S}}\mathscr{C}$. Therefore there exist $R \in \mathscr{R}$, $S \in \mathscr{S}$ and $T \in \mathscr{T}$ such that $C \in \mathscr{C}_R$, $B/C \in \mathscr{C}_S$, and $A/B \in \mathscr{C}_T$. Hence $A \in (\mathscr{C}_R \circ \mathscr{C}_S) \circ \mathscr{C}_T = \mathscr{C}_R \circ (\mathscr{C}_S \circ \mathscr{C}_T)$, Proposition 2.1. Clearly $\mathscr{C}_R \circ (\mathscr{C}_S \circ \mathscr{C}_T) \subset {}_{\mathscr{R}}\mathscr{C} \circ ({}_{\mathscr{S}}\mathscr{C} \circ {}_{\mathscr{T}}\mathscr{C})$. The proof of the opposite inclusion is similar.

Consequence 2.6. Let $\mathscr{R}$ be a subset of Rad, and put ${}_{\mathscr{R}}\mathscr{C} = \bigcup_{R \in \mathscr{R}} \mathscr{C}_R$. Then for every positive integer n, $({}_{\mathscr{R}}\mathscr{C})^n = {}_{\mathscr{R}}\mathscr{C} \circ {}_{\mathscr{R}}\mathscr{C} \circ \ldots \circ {}_{\mathscr{R}}\mathscr{C}$ is independent of parenthesisation.

For example, let $\mathscr{N}$ denote the class of nilpotent groups. Then $\mathscr{N}^n$ is well defined for every positive integer n.

Consequence 2.7. Let ${}_{\mathscr{R}}\mathscr{C}$ be as in 2.6 and let $0 \neq G \in (\Lambda, \Sigma)$-mod. If all the factors of the finite subnormal series $0 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ belong to ${}_{\mathscr{R}}\mathscr{C}$, then G possesses a non-trivial normal submodule belong to ${}_{\mathscr{R}}\mathscr{C}$.

In view of 2.6, Consequence 2.7 in effect states that if $G \in ({}_{\mathscr{R}}\mathscr{C})^n$, then G possesses a non-trivial, normal submodule belonging to ${}_{\mathscr{R}}\mathscr{C}$.

For a (Λ, Σ)-module A and a pre-radical R we call a series of (Λ, Σ)-modules $0 \lhd A_1 \lhd \ldots \lhd A_\alpha = A$ an *ascending R-series* if $A_{\beta+1}/A_\beta \in \mathscr{B}_R$ for every ordinal β and $A_\beta = \bigcup_{v<\beta} A_v$ for every limit ordinal β. A *descending R-series* is a series $0 = A_\alpha \lhd \ldots \lhd A_1 \lhd A$ which satisfies $A_\beta/A_{\beta+1} \in \mathscr{C}_R$ for every ordinal β and $A_\beta = \bigcap_{v<\beta} A_v$ for every limit ordinal β.

Proposition 2.8. *Let R be an idempotent pre-radical. Then* $A \in \mathscr{B}_{\tilde{R}}$ *iff there exists an ascending R-series for A. In this case the sequence* $0 \lhd R(A) \lhd \ldots \lhd \tilde{R}(A) = A$ *is the unique upper R-series for A.*

Proof. If $A \in \mathscr{B}_{\tilde{R}}$ then clearly $0 \lhd R(A) \lhd \ldots \lhd R_\alpha(A) = \tilde{R}(A) = A$ is an ascending R-sequence for A. Conversely, let $0 \lhd A_1 \lhd \ldots \lhd A_\alpha = A$ be such a sequence. We claim: $A_\beta \subset R_\beta(A)$ for every index ordinal β. Assume $A_v \subset R_v(A)$ for all $v < \beta$. First take β not a limit ordinal, say $\beta = v + 1$. Since A_β and $R_v(A)$ are normal submodules it follows (since Λ is distributively generated) that $Y = A_\beta + R_v(A)$ is a normal submodule and

$$Y/R_v(A) \cong A_\beta/A_\beta \cap R_v(A) \cong (A_\beta/A_v)/((A_\beta \cap R_v(A))/A_v),$$

and since $A_\beta/A_v \in \mathscr{B}_R$ it follows that $Y/R_v(A) \in \mathscr{B}_R$, [1] 4.2. Therefore

$$Y/R_v(A) = R(Y/R_v(A)) \subset R(A/R_v(A)) = R_\beta(A)/R_v(A).$$

Thus $Y \subset R_\beta(A)$ and so $A_\beta \subset R_\beta(A)$. Finally if β is a limit ordinal then

$$A_\beta = \bigcup_{v<\beta} A_v \subset \bigcup_{v<\beta} R_v(A) = R_\beta(A).$$

Proposition 2.9. *Let R be a radical. Then $A \in \mathscr{C}_R$ iff there exists a descending R-series for A. In this case the series $0 = \bar{R}(A) \lhd \ldots \lhd R(A) = A$ is the unique lower R-series for A.*

Proof. If $\beta = \nu + 1$ and Y is the submodule generated by $R^{\nu}(A) + A_{\beta}$ then $Y/A_{\beta} \in \mathscr{C}_R$ and under the natural map $Y \to Y/A_{\beta}$ the submodule $R^{\beta}(A)$ goes to 0. So $R^{\beta}(A) \subset A_{\beta}$. The rest is similar to the proof of the preceding proposition.

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