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INCIDENCE SEMIRINGS OF GRAPHS AND VISIBLE BASES

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Abstract

We consider the incidence semirings of graphs and prove that every incidence semiring has convenient visible bases for its right ideals and for its left ideals, and that these visible bases can be used to determine the weights of all right ideals that have maximum weight and all left ideals that have maximum weight.

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1. Introduction

Visible bases for ideals have been considered in [4, 13], where more explanations of this concept and its applications are given. In particular, visible bases are convenient for determining the weights of ideals (see [2, 13]). It was shown in [4] that every structural matrix semiring has a nice visible basis for ideals.

The present paper is devoted to the more general construction of incidence semirings of graphs. We consider much larger classes of all right ideals and all left ideals in this more general construction. It is important to treat these larger classes, since they may lead to ideals with better properties essential for applications. Finally, the present paper handles the incidence semirings of graphs over a larger category of coefficients than that considered for structural matrix semirings in [4].

Our main theorem establishes that every incidence semiring has convenient visible bases for its right ideals and its left ideals, and that these visible bases can be used to determine the weights of all right ideals that have maximum weight and all left ideals that have maximum weight in the incidence semiring of graphs (see Theorem 3.2). Complete definitions of these terms are given in the next section.

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2. Preliminaries

We use standard terminology and refer readers to the monographs [5, 8, 9, 14] and research articles [6, 7, 10, 12] for more information. Throughout, the words 'graph' and 'digraph' mean a finite directed graph without multiple parallel edges but possibly with loops, and $G = (V, E)$ is a graph with the set V of vertices and the set E of edges.

Following [2, 4], we do not assume that all semirings have identity elements. This allows us to view every ideal of a semiring as a semiring and consider the incidence semirings for larger classes of graphs. Let R be a semiring. The *incidence semiring* of G over R is denoted by $I_G(R)$ and is defined as the set consisting of zero 0 and all finite sums $\sum_{i=1}^{n} r_i(g_i, h_i)$, where $n \geq 1$, $r_i \in R$, $(g_i, h_i) \in E$, endowed with the standard addition and multiplication defined by the distributive law and the rule

$$(g_1, h_1) \cdot (g_2, h_2) = \begin{cases} (g_1, h_2) & \text{if } h_1 = g_2 \text{ and } (g_1, h_2) \in E, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

for all $(g_1, h_1), (g_2, h_2) \in E$. Empty sums are assumed to be equal to zero. Incidence semirings are a natural generalisation of incidence rings (see [8, Section 3.15], [11] and [14]).

The graph G is said to be *balanced* if, for all $g_1, g_2, g_3, g_4 \in V$ with $(g_1, g_2), (g_2, g_3), (g_3, g_4), (g_1, g_4) \in E$, the following equivalence holds:

$$(g_1, g_3) \in E \Leftrightarrow (g_2, g_4) \in E.$$

It is easy to verify that the multiplication in $I_G(R)$ is associative if and only if G is balanced (see [14]). Therefore $I_G(R)$ is a semiring if and only if G is balanced.

The *weight* $\mathrm{wt}(r)$ of an element $r = \sum_{i=1}^{n} r_i(g_i, h_i) \in I_G(R)$ is the number of nonzero coefficients r_i in the sum. The *weight* of a subset S of $I_G(R)$ is defined as the minimum weight of a nonzero element in S. We refer to [8, 12, 13] for more details. Let $\mathbb{N}$ be the set of all positive integers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Here we deal with right and left ideals in the semiring $I_G(R)$. Right ideals and left ideals were considered, for example, in [1, 3]. Let us recall the definitions. Suppose that T is a subset of $I_G(R)$. An *ideal* generated by T in $I_G(R)$ is the set

$$\mathrm{id}(T) = \left\{ \sum_{i=1}^{k} \ell_i g_i r_i \;\middle|\; k \in \mathbb{N}_0,\, g_i \in T,\, \ell_i, r_i \in I_G(R) \cup \mathbb{N} \right\},$$

where it is assumed that the identity element 1 of $\mathbb{N}$ acts as an identity on the whole $I_G(R)$ too. A *right ideal* generated by T is the set

$$\mathrm{id}_r(T) = \left\{ \sum_{i=1}^{k} g_i r_i \;\middle|\; k \in \mathbb{N}_0,\, g_i \in T,\, r_i \in I_G(R) \cup \mathbb{N} \right\}, \tag{2.2}$$

and a *left ideal* generated by T is the set

$$\mathrm{id}_\ell(T) = \left\{ \sum_{i=1}^{k} \ell_i g_i \;\middle|\; k \in \mathbb{N}_0,\, g_i \in T,\, \ell_i \in I_G(R) \cup \mathbb{N} \right\}.$$

The set T is called a *generating set*.

3. Main results

The following definition introduces the concept of a visible basis for right ideals and left ideals in semirings. It is an exact analogue of the corresponding definition for ideals in semirings, given in [4] by analogy to a similar ring notion considered in [13].

DEFINITION 3.1. A subset S of a semiring R is called a *visible basis* of ideals (respectively, right ideals, left ideals) if, for every subset T of S, the weight of the ideal $\mathrm{id}(T)$ (respectively, right ideal $\mathrm{id}_r(T)$, left ideal $\mathrm{id}_\ell(T)$) generated by T in R is equal to the weight of T.

Let g be a vertex on V. We use the following notation for two sets of vertices:

$$\mathrm{In}(g) = \{h \in V \mid (h, g) \in E\},$$
$$\mathrm{Out}(g) = \{h \in V \mid (g, h) \in E\}.$$

Define two sets of edges of the graph $G = (V, E)$ by

$$E_r = \{(g, h) \in E \mid \mathrm{Out}(g) \cap \mathrm{Out}(h) = \emptyset\}, \tag{3.1}$$
$$E_\ell = \{(g, h) \in E \mid \mathrm{In}(g) \cap \mathrm{In}(h) = \emptyset\}.$$

For any subset T of $\mathrm{Out}(g)$, we put

$$\Delta(g, T) = \{h \in \mathrm{In}(g) \mid \mathrm{Out}(h) \cap \mathrm{Out}(g) = T\}, \tag{3.2}$$
$$d(g, T) = \sum_{h\in\Delta(g,T)} (h, g). \tag{3.3}$$

Let $\mathcal{M}$ be the maximum of the cardinalities $|\Delta(g, T)|$, for all $g \in G$, $T \subseteq \mathrm{Out}(g)$. Denote by $\mathcal{Z}_r$ and $\mathcal{B}_r$ the sets defined by

$$\mathcal{Z}_r = \left\{ \sum_{(g,h)\in E_r} r_{(g,h)}(g, h) \,\middle|\, 0 \neq r_{(g,h)} \in R \right\}, \tag{3.4}$$
$$\mathcal{B}_r = \{r \cdot d(g, T) \mid 0 \neq r \in R,\ g \in V,\ T \subseteq \mathrm{Out}(g),\ |\Delta(g, T)| = \mathcal{M}\}.$$

Likewise, for any subset T of $\mathrm{In}(g)$, let

$$\Theta(g, T) = \{h \in \mathrm{Out}(g) \mid \mathrm{In}(h) \cap \mathrm{In}(g) = T\},$$
$$e(g, T) = \sum_{h\in\Theta(g,T)} (h, g).$$

Let $\mathcal{N}$ be the maximum of the cardinalities $|\Theta(g, T)|$, for all $g \in G$, $T \subseteq \mathrm{In}(g)$. Denote by $\mathcal{Z}_\ell$ and $\mathcal{B}_\ell$ the sets defined by

$$\mathcal{Z}_\ell = \left\{ \sum_{(g,h)\in E_\ell} r_{(g,h)}(g, h) \,\middle|\, 0 \neq r_{(g,h)} \in R \right\},$$
$$\mathcal{B}_\ell = \{r \cdot e(g, T) \mid 0 \neq r \in R,\ g \in V,\ T \subseteq \mathrm{In}(g),\ |\Theta(g, T)| = \mathcal{N}\}. \tag{3.5}$$

Theorem 3.2. *Let G be a balanced graph, and let R be a semiring with identity element. Then the following conditions hold.*

(i) *The set $\mathcal{B}_r$ (respectively, $\mathcal{B}_\ell$) is a visible basis for right (respectively, left) ideals in $I_G(R)$.*

(ii) *For each right (respectively, left) ideal J that has a maximum weight among all right (respectively, left) ideals in $I_G(R)$, there exists an element x in $J \cap (B_r \cup \mathcal{Z}_r)$ (respectively, $J \cap (B_\ell \cup \mathcal{Z}_\ell)$) such that* $\mathrm{wt}(x) = \mathrm{wt}(J)$.

The following example shows that $\mathcal{M}$ may be different from the maximum outdegree and the maximum indegree of the graph.

Example 3.3. Let $G = (V, E)$ be the graph with the set $V = \{g, h, v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4\}$ of vertices and the set $E = \{(g, h), (v_1, g), (v_2, g), (v_3, g), (v_4, g), (u_1, g), (u_2, g), (u_3, g), (u_4, g), (v_1, h), (v_2, h), (v_3, h), (v_4, h)\}$ of edges. Then the set $\mathrm{Out}(g)$ has only two subsets $\emptyset$ and $\{g\}$. If $T = \emptyset$, then $\Delta(g, T) = \{u_1, u_2, u_3, u_4\}$. If $T = \{g\}$, then $\Delta(g, T) = \{v_1, v_2, v_3, v_4\}$. In both of these cases $|\Delta(g, T)| = 4$. Therefore $\mathcal{M} = 4$. However, the maximum indegree of a vertex of G is equal to 8, and the maximum outdegree of a vertex of G is equal to 1.

4. Proofs

Let us begin with a few lemmas required for the proof of the main theorem. For clarity, we include complete hypotheses in all lemmas.

Lemma 4.1. *Let $G = (V, E)$ be a balanced graph, R a semiring with identity element, $g \in V$, and let $T \subseteq \mathrm{Out}(g)$. If the product $d(g, T) \cdot r(g_1, g_2)$ is nonzero for some $r \in R$, $(g_1, g_2) \in E$, then it belongs to $\mathcal{B}_r$ and has the same weight as $d(g, T)$.*

Proof. Take any nonzero product $x = d(g, T) \cdot r(g_1, g_2)$, where $0 \neq r \in R$ and $(g_1, g_2) \in E$. Clearly, (2.1) and (3.3) imply that $g_1 = g$. Hence $g_2 \in \mathrm{Out}(g)$. Since $x \neq 0$, there exists $v \in \Delta(g, T)$ such that $(v, g)(g, g_2) \neq 0$. Hence $g_2 \in \mathrm{Out}(v)$ by (2.1). Thus $g_2 \in \mathrm{Out}(v) \cap \mathrm{Out}(g) = T$. By (3.2),

$$g_2 \in \mathrm{Out}(v) \quad \text{for all } v \in \Delta(g, T). \tag{4.1}$$

Therefore we can rewrite x as

$$x = r \sum_{v \in \Delta(g,T)} (v, g) \cdot (g_1, g_2) = r \sum_{v \in \Delta(g,T)} (v, g_2), \tag{4.2}$$

where all $(v, g_2) \in E$.

Denote by T_2 the set of all elements $v \in \mathrm{Out}(g_2)$ such that $x \cdot (g_2, v) \neq 0$. We claim that $x = r \sum_{v \in \Delta(g_2, T_2)} (v, g_2) \in \mathcal{B}_r$. It suffices to verify that $\Delta(g, T) = \Delta(g_2, T_2)$.

Let us first prove the inclusion $\Delta(g, T) \subseteq \Delta(g_2, T_2)$. Choose and fix any element w in $\Delta(g, T)$. By (3.2), we get $w \in \mathrm{In}(g)$ and $\mathrm{Out}(w) \cap \mathrm{Out}(g) = T$. In view of (4.1), we get $w \in \mathrm{In}(g_2)$. Therefore, in order to show that $w \in \Delta(g_2, T_2)$ it remains to prove that $\mathrm{Out}(w) \cap \mathrm{Out}(g_2) = T_2$.

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Pick any $v \in \mathrm{Out}(w) \cap \mathrm{Out}(g_2)$. We have $(w, v), (g_2, v) \in E$. Given that $w \in \Delta(g, T)$, we also have $(w, g_2) \in E$ by (4.1). Clearly, $(w, g_2), (g_2, v), (w, v) \in E$ imply that $(w, g_2)(g_2, v) = (w, v) \in I_D(R)$. Hence $x \cdot (g_2, v) \neq 0$ by (4.2). This means that $v \in T_2$. Therefore $\mathrm{Out}(w) \cap \mathrm{Out}(g_2) \subseteq T_2$.

Now choose any $v \in T_2$. Then we have $v \in \mathrm{Out}(g_2)$ and $x \cdot (g_2, v) \neq 0$. It follows from (4.2) that $(h, g_2)(g_2, v) \neq 0$, for some $h \in \Delta(g, T)$. Hence $v \in \mathrm{Out}(h)$. It follows from (4.2) that $v \in \mathrm{Out}(g)$. Therefore $v \in \mathrm{Out}(g) \cap \mathrm{Out}(h) = T$. Since $w \in \Delta(g, T)$, by (3.2) we get $v \in \mathrm{Out}(w)$. Thus $v \in \mathrm{Out}(w) \cap \mathrm{Out}(g_2)$; whence $\mathrm{Out}(w) \cap \mathrm{Out}(g_2) = T_2$. This completes the proof of inclusion $\Delta(g, T) \subseteq \Delta(g_2, T_2)$.

Now the reversed inclusion $\Delta(g, T) \supseteq \Delta(g_2, T_2)$ follows immediately from the maximality of $|\Delta(g, T)| = \mathcal{M}$. Thus, we get $\Delta(g, T) = \Delta(g_2, T_2)$. □

Lemma 4.2. *Let $G = (V, E)$ be a balanced graph, R a semiring, $g \in V$, and let $T_1, T_2 \subseteq \mathrm{Out}(g)$. If $T_1 \neq T_2$, then the sets $\Delta(g, T_1)$ and $\Delta(g, T_1)$ are disjoint.*

Proof. Suppose to the contrary that there exists a vertex $v \in \Delta(g, T_1) \cap \Delta(g, T_1)$. Then it follows from $v \in \Delta(g, T_1)$ and (3.2) that $\mathrm{Out}(v) \cap \mathrm{Out}(g) = T_1$. Similarly, $v \in \Delta(g, T_2)$ and (3.2) imply that $\mathrm{Out}(v) \cap \mathrm{Out}(g) = T_2$. This contradicts $T_1 \neq T_2$ and completes the proof. □

Lemma 4.3. *Let $G = (V, E)$ be a balanced graph, R a semiring with identity element, and let J be the right ideal generated in $I_G(R)$ by $d(g, T)$, where $g \in V$ and $T \subseteq \mathrm{Out}(v)$. Then the weight of J is equal to* $\mathrm{wt}(J) = \mathrm{wt}(d(g, T)) = |\Delta(g, T)|$.

Proof. The equality $\mathrm{wt}(d(g, T)) = |\Delta(g, T)|$ follows from (3.3). Therefore it remains to prove that $\mathrm{wt}(J) = \mathrm{wt}(d(g, T))$.

If $d(g, T) = 0$, then $\Delta(g, T) = \emptyset$ and $J = 0$; whence the equality holds true. Further, we assume that $d(g, T) \neq 0$ and $\Delta(g, T) \neq \emptyset$.

Since $d(g, T) \in J$, we get $\mathrm{wt}(J) \leq \mathrm{wt}(d(g, T))$. To prove the reversed inequality, pick a nonzero element x in J. It suffices to verify that $\mathrm{wt}(x) \geq \mathrm{wt}(d(g, T))$.

By (2.2), x can be written as

$$x = k \cdot d(g, T) + \sum_{i=1}^{n} d(g, T) \cdot r_i(g_i, h_i), \tag{4.3}$$

where $k, n \in \mathbb{N}_0$, $0 \neq r_i \in R$, $(g_i, h_i) \in E$. It follows from (3.3) that $d(g, T) \cdot r_i(g_i, h_i) = 0$ whenever $g_i \neq g$. We may assume that only nonzero summands have been included in (4.3), so that all g_i are equal to g and x is recorded as

$$x = k \cdot d(g, T) + \sum_{i=1}^{n} d(g, T) \cdot r_i(g, h_i). \tag{4.4}$$

If $h_i \notin T$, then it follows from (2.1), (3.5) and (3.3) that $d(g, T) \cdot (g, h_i) = 0$. Since it has been assumed that the sum (4.4) contains only nonzero summands, we get $h_i \in T$ for all $i = 1, \ldots, n$.

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If $h_i = g$, then it follows that $d(g, T) \cdot r_i(g, h_i) = r_i d(g, T)$. We can combine all such terms with $k \cdot d(g, T)$ in (4.4) and rewrite x as

$$x = r \cdot d(g, T) + \sum_{i=1}^{n} d(g, T) \cdot r_i(g, h_i), \tag{4.5}$$

where $r \in R$ and $g \neq h_i \in T$, for all $i = 1, \ldots, n$. Hence it follows that if $r \neq 0$, then we get $\mathrm{wt}(x) \geq \mathrm{wt}(r \cdot d(g, T)) = \mathrm{wt}(d(g, T))$, as required.

It remains to consider the case where $r = 0$. Substituting (3.3) in (4.5), we can rewrite x as

$$\begin{aligned} x &= \sum_{i=1}^{n} \sum_{h \in \Delta(g,T)} (h, g) \cdot r_i(g, h_i), \\ &= \sum_{i=1}^{n} \sum_{h \in \Delta(g,T)} r_i(h, h_i). \end{aligned} \tag{4.6}$$

Since $x \neq 0$, we get $n > 0$. We may assume that all summands in (4.6) with $h_{i_1} = h_{i_2}$ have been combined. This means that $h_{i_1} \neq h_{i_2}$ for any $1 \leq i_1 < i_2 \leq n$. We may also assume that only nonzero summands have been recorded in (4.6). In particular, $R - 1 \neq 0$. Hence it follows that all edges included in the sum $\sum_{h \in \Delta(g,T)} r_1(h, h_1)$ in (4.6) are different from all edges that occur in all remaining terms. Therefore we get $\mathrm{wt}(x) \geq \mathrm{wt}(d(g, T) \cdot r_1(g, h_1)) = \mathrm{wt}(d(g, T))$. This completes the proof. □

It is clear that the graph (V, Z_r) is a subgraph of $G = (V, E)$ and $I_{(V,Z_r)}(R)$ is a subsemiring of $I_G(R)$.

Lemma 4.4. *Let $G = (V, E)$ be a balanced graph, R a semiring with identity element, and let $y \in \mathcal{Z}_r$. Then the weight of the right ideal* $\mathrm{id}_r(y)$ *is equal to* $|E_r|$.

Proof. Take a nonzero element x that has the minimum weight among all nonzero elements in the right ideal $\mathrm{id}_r(y)$. Then $\mathrm{wt}(x) = \mathrm{wt}(\mathrm{id}_r(y)) = |E_r|$ by (3.4). It follows from (2.2) that $x = ky + yr$ for some $k \in \mathbb{N}_0$, $r \in I_G(R)$. By the definition of $I_G(R)$, there exist $n \geq 1$, $r_i \in R$, $(g_i, h_i) \in E$ such that $r = \sum_{i=1}^{n} r_i(g_i, h_i)$. By (3.4), we have $y = \sum_{(g,h)\in E_r} r_{(g,h)}(g, h)$ for some $0 \neq r_{(g,h)} \in R$. Therefore we can rewrite x as

$$x = ky + \sum_{(g,h)\in E_r} \sum_{i=1}^{n} r_{(g,h)} r_i(g, h)(g_i, h_i).$$

By (3.1), we get $\mathrm{Out}(g) \cap \mathrm{Out}(h) = \emptyset$ for all $(g, h) \in E_r$. Hence, for each $(g, h) \in E_r$ and every $1 \leq i \leq n$, either $g_i \neq h$ or $h_i \notin \mathrm{Out}(h)$. In any of these cases we get $(g, h)(g_i, h_i) = 0$. Therefore $x = ky$, and so $\mathrm{wt}(x) = \mathrm{wt}(y) = |E_r|$, as required. □

Proof of Theorem 3.2. The case of left ideals is dual to that of right ideals. This is why we only need to prove the main theorem for right ideals.

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First, let us prove condition (i). Take any subset T of $\mathcal{B}_r$ and consider the right ideal $\mathrm{id}_r(T)$ generated by T in $I_G(R)$. Since the weight of $\mathrm{id}_r(T)$ does not exceed the weight of any nonzero element in $\mathrm{id}_r(T)$, it suffices to verify that $\mathcal{B}_r \cap \mathrm{id}_r(T)$ contains an element of weight equal to $\mathrm{wt}(\mathrm{id}_r(T))$.

Choose a nonzero element w of minimum weight among all elements in $\mathrm{id}_r(T)$. Then $\mathrm{wt}(w) = \mathrm{wt}(\mathrm{id}_r(T))$. It follows from (2.2) that w can be written as

$$w = \sum_{i=1}^{k} t_i z_i,$$

for some $k \geq 0$, $t_i \in T$, $z_i \in I_G(R)$, because R is a semiring with identity element. Since every z_i is equal to a sum of some edges from E with coefficients in R, we can rewrite w in the form

$$w = \sum_{i=1}^{k} \sum_{j=1}^{k_i} r_{ij} t_i (g_{ij}, h_{ij}), \tag{4.7}$$

where $k_i \geq 0$, $r_{ij} \in R$, $(g_{ij}, h_{ij}) \in E$. Denote by S the set of all nonzero products $r_{ij} t_i (g_{ij}, h_{ij})$ in (4.7). Clearly, $S \subseteq \mathrm{id}_r(T)$; whence $\mathrm{id}_r(T \cup S) = \mathrm{id}_r(T)$. Lemma 4.1 shows that $S \subseteq \mathcal{B}_r$ and that the weight of every nonzero element $r_{ij} t_i (g_{ij}, h_{ij})$ is equal to the weight of $t_i \in T$. Therefore it is enough to prove that the weight of $\mathrm{id}_r(S \cup T) = \mathrm{id}_r(T)$ is equal to the weight of an element in $(S \cup T) \cap \mathrm{id}_r(S \cup T)$. To simplify notation we may replace T by $S \cup T$ and assume that from the very beginning every product $r_{ij} t_i (g_{ij}, h_{ij})$ in (4.7) belongs to T. This means that (4.7) simplifies to

$$w = \sum_{i=1}^{k} t_i. \tag{4.8}$$

Since $t_i \in \mathcal{B}_r$, we get $t_i = r_i d(g_i, T_i) \in \mathcal{B}_r$ for some $0 \neq r_i \in R$, $g_i \in V$, $T_i \subseteq \mathrm{Out}(g_i)$, $|\Delta(g_i, T_i)| = \mathcal{M}$. Substituting these expressions for the elements t_i in (4.8), by (3.3)

$$w = \sum_{i=1}^{k} r_i d(g_i, T_i) = \sum_{i=1}^{k} \sum_{h_i \in \Delta(g_i, T_i)} r_i (h_i, g_i). \tag{4.9}$$

We claim that the weight of w is equal to the sum of the weights of all the elements $d(g_i, T_i)$ in (4.9). Suppose that $g_{i_1} = g_{i_2}$ for some $i_1 \neq i_2$. Lemma 4.2 tells us that the sets $\Delta(g_{i_1}, T_{i_1})$ and $\Delta(g_{i_2}, T_{i_2})$ are disjoint, and so

$$\mathrm{wt}(r_{i_1} d(g_{i_1}, T_{i_1}) + r_{i_2} d(g_{i_2}, T_{i_2})) = \mathrm{wt}(r_{i_1} d(g_{i_1}, T_{i_1})) + \mathrm{wt}(r_{i_2} d(g_{i_2}, T_{i_2})). \tag{4.10}$$

On the other hand, if $g_{i_1} \neq g_{i_2}$ for some $i_1 \neq i_2$, then it is clear that all edges (h_{i_1}, g_{i_1}) that occur in $d(g_{i_1}, T_{i_1})$ are different from all edges (h_{i_2}, g_{i_2}) occurring in $d(g_{i_2}, T_{i_2})$. Therefore (4.10) holds true again.

It follows from (4.10) that $\mathrm{wt}(w) = \sum_{i=1}^{k} \mathrm{wt}(r_i d(g_i, T_i)) = \sum_{i=1}^{k} \mathrm{wt}(t_i)$. Therefore $\mathrm{wt}(w) \geq \mathrm{wt}(t_1)$. The minimality of the weight of w implies that $\mathrm{wt}(w) = \mathrm{wt}(t_1)$. This means that the set $\mathcal{B}_r$ is a visible basis for right ideals in $I_G(R)$, and so condition (i) holds.

Second, let us prove condition (ii). Consider a right ideal J that has a maximum weight among all right ideals in $I_G(R)$. Pick an element y of minimum weight in J, so that $\mathrm{wt}(y) = \mathrm{wt}(J)$. Since $y \in I_G(R)$, it can be recorded in the form $y = \sum_{i=1}^{n} y_i(g_i, h_i)$, for some $0 \neq y_i \in R$, $g_i, h_i \in V$. Consider two possible cases.

Case 1. $y(u, v) = 0$ for all $(u, v) \in E$. Since (V, Z_r) is a subgraph of $G = (V, E)$, we see that in this case (2.1) and (3.1) imply that $y \in I_{(V,E_r)}(R)$. Hence $\mathrm{wt}(y) \leq |E_r|$. Lemma 4.4 and the maximality of the weight of J show that $\mathrm{wt}(y) = |E_r|$. Hence (3.4) yields that $y \in \mathcal{Z}_r$. Thus $y \in J \cap \mathcal{Z}_r$, and so we are done in this case.

Case 2. There exists $(u, v) \in E$ such that $y(u, v) \neq 0$. Then $(g_i, h_i)(u, v) \neq 0$ for some $0 \leq i \leq n$. Without loss of generality we may assume that the elements (g_i, h_i) have been ordered so that $(g_i, h_i)(u, v) \neq 0$ for $i = 0, \ldots, m$ and $(g_i, h_i)(u, v) = 0$ for $i = m + 1, \ldots, n$, where $m \in \mathbb{N}_0$. By (2.1), we get $h_i = u$ and $(g_i, v) \in E$, for all $i = 0, \ldots, m$. Putting $z = y(u, v)$, by (2.1),

$$z = \sum_{i=1}^{m} y_i(g_i, v). \tag{4.11}$$

Clearly, $\mathrm{wt}(z) = m$ and $\mathrm{wt}(y) = n$. Since $z \in J$, the minimality of the weight of y implies that $m = n = \mathrm{wt}(J)$. Further, we may also assume that the elements $g_1, \ldots, g_m$ in (4.11) have been ordered so that

$$|\mathrm{Out}(g_1) \cap \mathrm{Out}(v)| \leq |\mathrm{Out}(g_2) \cap \mathrm{Out}(v)| \leq \cdots \leq |\mathrm{Out}(g_m) \cap \mathrm{Out}(v)|.$$

Suppose that there exist $w \in (\mathrm{Out}(g_2) \cap \mathrm{Out}(v)) \setminus (\mathrm{Out}(g_1) \cap \mathrm{Out}(v))$. Then $(g_1, v)(v, w) = 0$ and $(g_2, v)(v, w) = (g_2, w) \neq 0$. Hence $\mathrm{wt}(z(v, w)) < \mathrm{wt}(z) = \mathrm{wt}(y)$. Since $z(v, w) \in J$, this contradicts the minimality of the weight of y in J, and shows that $\mathrm{Out}(g_2) \cap \mathrm{Out}(v) = \mathrm{Out}(g_1) \cap \mathrm{Out}(v)$. Likewise,

$$|\mathrm{Out}(g_1) \cap \mathrm{Out}(v)| = |\mathrm{Out}(g_2) \cap \mathrm{Out}(v)| = \cdots = |\mathrm{Out}(g_m) \cap \mathrm{Out}(v)|.$$

Letting $T = \mathrm{Out}(g_1) \cap \mathrm{Out}(v)$, by (3.2) we get $g_1, \ldots, g_m \in \Delta(v, T)$. Lemma 4.3 shows that $\mathrm{wt}(\mathrm{id}_r(d(v, T))) = |\Delta(v, T)| \geq m = \mathrm{wt}(J)$. It follows from the maximality of the weight of J that $m = |\Delta(v, T)|$. Hence $\{g_1, \ldots, g_m\} = \Delta(v, T)$, and so $z = d(v, T)$. Thus, in this case the element z belongs to $J \cap B_r$ and satisfies $\mathrm{wt}(z) = \mathrm{wt}(J)$. We see that in both cases condition (ii) holds true. This completes the proof. □

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