18. A METHOD OF INTEGRATING THE EQUATIONS OF MOTION IN SPECIAL COORDINATES AND THE ELIMINATION OF A DISCONTINUITY IN THE THEORY OF THE MOTION OF PERIODIC COMET WOLF

E. I. KAZIMIRCHAK-POLONSKAYA
Institute for Theoretical Astronomy, Leningrad, U.S.S.R.

Abstract. From the integration formulae of Numerov and Subbotin we have developed and programmed for an electronic computer a particular method for integrating the differential equations of cometary motion in special rectangular coordinates, with a variable step and allowing for all planetary perturbations and nongravitational effects over a time interval of 400 yr. Application of this method and our set of programmes to the investigation of the motion of P/Wolf permits us to eliminate the discontinuity that has hitherto existed in the theory on account of the comet's close approach to Jupiter in 1922.

1. Introduction

Owing to the rapid development of electronic computers and their application to astronomical calculations, cometary astronomy, utilizing the methods of celestial mechanics, is coming to a new stage in its development. Large-scale problems may now be formulated and solved by new methods and using electronic computers. We shall point out here only four of the more important problems.

(1) Construction of continuous numerical theories for the motions of comets over the whole period spanned by their observations, with consideration given to the perturbations by the planets and to nongravitational effects.

(2) Determination of the mass of Jupiter from the large perturbations induced in the orbits of comets that pass through Jupiter's sphere of action.

(3) Investigation of the evolution of cometary orbits over a time scale of centuries and clarification of the role of the giant planets (Jupiter–Neptune) in defining this evolution.

(4) Study of the motion of meteor streams, particularly of the large transformations of their orbits in the sphere of action of Jupiter, and the determination of the effects of planetary perturbations upon the structure of these streams and on the conditions of visibility of the meteor showers.

Our aim is the development of a rigorous procedure for the investigation and solution of these problems, including the compilation of a set of constants and standard programmes.

This paper describes the procedure for investigation and the solution of the first of the above-mentioned problems. The other three problems are discussed in other papers (Kazimirchak-Polonskaya, 1972a, 1972b; Kazimirchak-Polonskaya *et al.*, 1972).

We have prepared punched cards (mainly in binary) containing the coordinates of the eight major planets, with full allowance for their mutual perturbations, and

Chebotarev et al. (eds.), The Motion, Evolution of Orbits, and Origin of Comets, 95–102.

they are given at 20-day intervals for a period of 400 yr (1660–2060). These coordinates are based both on classical and more recent theories for the motions of the major planets (Newcomb, 1895a, 1895b; Clemence, 1943, 1949, 1954, 1961; Duncombe, 1958; Eckert *et al.*, 1951; Morgan, 1945). The computations were performed using programmes by Subbotina (1965) and by the author.

2. Method of Integration

In developing a method for integrating the differential equations of motion for a minor body in the solar system:

$$w^2 \frac{d^2x}{dt^2} = -\frac{k^2w^2x}{r^3} + R_x, \qquad x \to y, z, \tag{1}$$

where

$$R_x = k^2w^2 \sum_{i=2}^{l} m_i \left(\frac{x_i - x}{\Delta_i^3} - \frac{x_i}{r_i^3}\right); \qquad l = 6, 7, 8, 9, \tag{2}$$

we took as our basis the theory proposed by Myachin (1962), which provides an estimate of the total error in the calculations, and also two methods of integration devised by Numerov (1923) and Subbotin (1927, 1928).

According to Myachin, the total linearized error of the integration of Equations (1) is, after n steps, the sum of the following:

(1) an error, proportional to n, arising from errors in the initial data;

(2) a truncation error depending on the number of derivatives or differences taken into account;

(3) errors due to multiple iterations at each integration step;

(4) an error, proportional to $n^{3/2}$, arising from the rounding at each step.

We have aimed at developing a method where each of the partial errors, and hence the total error, would be reduced to a minimum.

Let us consider each of the points in detail.

(1) On the one hand, the error arising from errors in the initial data is dependent upon the accuracy of observations. It is thus of primary importance to increase the accuracy of cometary observations. On the other hand, the error is also dependent upon the theoretical development of the method. This error may be reduced in two ways:

(a) Firstly, considering that the error increases directly as the number of integration steps, we utilize, following Numerov and Subbotin, both the ordinary coordinates x, y, z, and special coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$, that are related to the ordinary ones by

$$\bar{x}_k = x_k \left(1 + \frac{k^2w^2}{12r_k^3}\right)$$

$$\bar{r}_k^2 = \bar{x}_k^2 + \bar{y}_k^2 + \bar{z}_k^2. \tag{3}$$

By introducing auxiliary functions λ and σ, defined by

https://doi.org/10.1017/S0074180900006331 Published online by Cambridge University Press

$$\lambda_k = \frac{k^2 w^2}{\bar{r}_k^3}; \qquad \sigma_k = \lambda_k + \tfrac{1}{6}\lambda_k^2 + \tfrac{7}{144}\lambda_k^3 + \tfrac{5}{288}\lambda_k^4 + \cdots \tag{4}$$

(where it is usually sufficient to ignore further terms in σ_k), Subbotin arrived at the following formulae for integrating Equations (1) in the special coordinates:

$$\bar{x}_k = f_{x_k}^{-2} + \overline{\mathrm{Red}}_{x_k} \tag{5}$$

$$\overline{\mathrm{Red}}_{x_k} = \tfrac{1}{12}R_{x_k} - \tfrac{1}{240}f_{x_k}^2 + \tfrac{31}{60480}f_{x_k}^4 - \tfrac{289}{3628800}f_{x_k}^6 + \cdots, \tag{6}$$

where the function f_{x_k} denotes, as in Cowell's method, the right-hand members of Equations (1),

$$f_{x_k} = -\frac{k^2 w^2 x_k}{r_k^3} + R_{x_k} = -\sigma_k \bar{x}_k + R_{x_k}. \tag{7}$$

The functions $f_{x_k}^{-2}, f_{x_k}^2, f_{x_k}^4, f_{x_k}^6, \ldots$ stand for the second sum and even differences of f_{x_k}. For computing Equations (2) the special coordinates are changed to the ordinary ones by the very simple formula

$$x_k = \bar{x}_k - \bar{x}_k\sigma_k/12. \tag{8}$$

Comparing Equations (5) and (6) with the well-known formulae for integration by Cowell's method,

$$x_k = f_{x_k}^{-2} + \mathrm{Red}_{x_k} \tag{9}$$

$$\mathrm{Red}_{x_k} = \tfrac{1}{12}f_{x_k} - \tfrac{1}{240}f_{x_k}^2 + \tfrac{31}{60480}f_{x_k}^4 - \tfrac{289}{3628800}f_{x_k}^6 + \cdots, \tag{10}$$

we see, that while the first terms in Equations (6) and (10) are appreciably different, the rest of the terms are identical. Near perihelion, the first term in Cowell's method becomes large and changes rapidly, making it necessary to reduce the integration step. In contrast, the corresponding term in Subbotin's method remains small and changes slowly. Therefore, for any specified accuracy, one may use a larger step-size in Subbotin's method than in Cowell's.

Introduction of special coordinates thus reduces the number of integration steps, and this reduces the error arising from errors in the initial data.

(b) Secondly, we may appreciably increase the accuracy of the initial integration table. Ever since it became customary to refer the tables of the coordinates of the major planets to standard dates, astronomers have imposed the requirement that the epoch of osculation in the initial integration table should also be a standard date. Nevertheless, the initial orbital elements for a comet are frequently referred to arbitrary epochs; this is particularly true for the earlier studies, when the epochs might have been reckoned in Greenwich Mean Time, Berlin Mean Time, or Paris Mean Time, etc. When adopting a new standard epoch of osculation it was usual to adjust only the mean anomaly, the other elements being considered invariable since the perturbations by the planets are small.

In order to refine the initial data we derived and programmed formulae enabling the rectangular coordinates of the body to be obtained, with significantly higher accuracy, for seven initial standard dates (Kazimirchak-Polonskaya, 1967, p. 35). The initial epoch of osculation could have been any date, standard or nonstandard, and we allowed for all planetary perturbations (and nongravitational effects, if necessary). The initial integration table is computed from these coordinates in double precision, thus providing an appreciable reduction in the errors of the initial data.

(2) In order to reduce the truncation error we use Subbotin's quadrature method and perform the integration in double precision (18 significant figures) using Equations (1)–(8). Owing to the rapid decrease of the coefficients, Equation (6) may be restricted to the sixth differences. If the eighth differences become significant, within the limits of the specified accuracy, the step-size is immediately decreased.

On the basis of our theoretical investigations a new criterion has been formulated for automatic choice of the integration step, depending on the specified accuracy and on the distance of the body under study from the Sun and all disturbing planets at any time (Kazimirchak-Polonskaya, 1967). It is interesting to note that the standard programme for automatic choice of the integration step is based exclusively on logical operations.

In connection with this criterion, we have found the most efficient way of doing the computation, with some operations performed in single and others in double precision; as a result, we attain a high degree of accuracy of the coordinates of the body under investigation, even on a time scale of centuries, and this requires a minimum of operation time.

(3) The third partial error arises after multiple iterations at each integration step, the number of iterations being considerably increased when we use double precision.

As is well known, the necessity for successive approximations arises because the expressions in the right-hand members of Equations (6) or (10) are initially unknown. It can be seen from comparison of Equations (9) and (10) with Equations (3)–(6) that the number of iterations in Cowell's method is greater than in Subbotin's method, specifically because of the different first terms in Equations (6) and (10).

We suggest a new way of applying Subbotin's method directly, without approximations. For this purpose we write Equation (6) in the form

$$\overline{\text{Red}_{x_k}} = \tfrac{1}{12}R_{x_k} + a_{-3}f_{x_{k-3}} + a_{-2}f_{x_{k-2}} + a_{-1}f_{x_{k-1}} + a_0 f_{x_k} + a_1 f_{x_{k+1}} + a_2 f_{x_{k+2}} + a_3 f_{x_{k+3}}, \tag{11}$$

where the coefficients $a_{-3}, a_{-2}, \ldots, a_3$ are calculated to the required accuracy. Values of the functions $f_{x_{k-3}}, f_{x_{k-2}}, f_{x_{k-1}}$ in Equation (11) are known at the kth step to a high degree of accuracy. To compute the values for $f_{x_k}, f_{x_{k+1}}, f_{x_{k+2}}, f_{x_{k+3}}$ we proceed from the integration formula in the method of differences by Numerov:

$$\bar{x}_{k+1} = (2 - \sigma_k)\bar{x}_k - \bar{x}_{k-1} + R_{x_k} + \overline{\text{red}_{x_k}} \tag{12}$$

$$\overline{\text{red}_{x_k}} = \tfrac{1}{12}\Delta^2 R_{x_k} - \tfrac{1}{240}f^4_{x_k} + \tfrac{31}{60480}f^6_{x_k} - \cdots, \tag{13}$$

where R_{x_k}, σ_k and f_{x_k} are calculated as in Subbotin's method by Equations (2), (4) and (7).

Since Equation (6) in Subbotin's method always involves a very restricted number of significant figures, it is permissible to ignore Equation (13) and obtain the f-functions for dates t_k, t_{k+1}, t_{k+2}, t_{k+3} to appropriate accuracy by Equations (2), (3), (4), (7) and (8) and by Numerov's reduced formula

$$\bar{x}_{k+1} = (2 - \sigma_k)\bar{x}_k - \bar{x}_{k-1} + R_{x_k}; \tag{14}$$

this requires no iteration. The values obtained for f_{x_k}, $f_{x_{k+1}}$, $f_{x_{k+2}}$, $f_{x_{k+3}}$ are then substituted into Equation (11), enabling Subbotin's method to be applied to full accuracy without iteration, and thereby reducing the iteration error to zero.

(4) Since the integration is performed in double precision the rounding error does not become significant until after 10 000 steps or more.

As a result of the expressions and procedures developed by us the total error of our integration method is reduced to a minimum, and this is a necessary condition for solving the four problems defined in the Introduction.

3. Application to the Study of the Motion of P/Wolf

All astronomers who work on the motions of comets know how difficult it is to construct an appropriately accurate numerical theory linking the motion of a comet over a large number of apparitions covering a long interval of time. In addition, if the comet passes well inside Jupiter's sphere of action within this time interval, the calculation of the great changes in its orbit becomes so increasingly complicated that up to now it has been considered impossible to link all the apparitions before and after the close approach to Jupiter. Therefore, the investigators had recourse to an artificial solution: they excluded the critical revolution containing the comet's passage through the sphere of action of Jupiter and constructed numerical theories for separate time intervals on the two sides of the encounter. There was inevitably a discontinuity in the theory of the motion and this was considered irremovable. Elimination of such discontinuities and the construction of continuous numerical theories of the motions of comets for the whole period covered by observations is an urgent necessity and our first aim.

Such a discontinuity existed even in the very skillful theory, constructed by Kamieński (1959), of the motion of P/Wolf. The theory comprises two isolated series of linked apparitions of the comet. The first series covers its motion from discovery to the last apparition before the close approach to Jupiter in 1922, i.e., the interval 1884–1918, and it represents 50 normal places (about 2000 observations) with the very low mean error $\varepsilon = \pm 1''.77$ (Kamieński, 1933). The perturbations by six planets, Venus through Uranus, were considered, and the effect of nongravitational forces was taken into account, to a high degree of accuracy, in the form of secular deceleration in the motion, using the formulae

$$\Delta\mu_t = -0''.0000\quad 0042\,(t - t_0)$$
$$\Delta M_t = -0''.0000\quad 0021\,(t - t_0)^2,$$

where $\Delta\mu_t$ and ΔM_t are the secular variations in the mean daily motion and mean anomaly over the time interval (measured in days) from the initial epoch t_0 to the running time t.

In the second series Kamieński (1948) linked three apparitions over the interval 1925–1942, after the comet's passage through the sphere of action of Jupiter in 1922, with a still higher degree of accuracy ($\varepsilon = \pm 1''.21$). Allowance was again made for the perturbations by the same six planets and nongravitational effects, the only difference being that the coefficients for the secular deceleration were diminished to

$$\Delta\mu_t = -0''.0000 \quad 00067\,(t - t_0)$$
$$\Delta M_t = -0''.0000 \quad 00034\,(t - t_0)^2.$$

Kamieński believes that these changes in the nongravitational effects result from the fact that the comet, through the large perturbations induced by Jupiter, was deflected into a new orbit with considerably changed parameters, particularly the perihelion distance (increasing from 1.6 to 2.4 AU).

Using the second series of elements Kamieński extended his integration of the equations of the motion of P/Wolf to the present time. The comet was observed in 1950–1951, 1959 and 1967, and the agreement between the observations and calculations was quite satisfactory.

Thus there exists a numerical theory of the motion of P/Wolf for two prolonged but isolated time intervals, 1884–1918 and 1925–1967. The critical revolution of 1918–1925 was excluded since Kamieński and Bielicki (1935, 1936) could not get a good representation of the 1925 observations in terms of the elements for 1918 (obtained in the first series). The maximum (O – C) residuals in 1925 were as high as $5^s.2$ in right ascension and $4''.8$ in declination. Until now the residuals have not been reduced by any of the methods employed (Kamieński, 1959).

At Kamieński's request we endeavoured to eliminate the discontinuity in the theory of motion of P/Wolf during the revolution 1918–1925. Having thoroughly analysed the causes that prevented the solution of the problem, we developed the above method of integration in special coordinates and elaborated with great care heliocentric and jovicentric methods for computing the large perturbations on cometary orbits in the sphere of action of Jupiter (Kazimirchak-Polonskaya, 1961).

The application of our procedure and set of programmes enabled us to solve this difficult problem. In spite of very large perturbations in the orbital elements of P/Wolf in Jupiter's sphere of action in 1922, amounting to more than 20° in mean anomaly, about $13^\circ.5$ in longitude of perihelion, about 10° in eccentricity angle, and more than 96″ in mean daily motion, we obtained a quite satisfactory representation of the normal places of the comet in 1925. The results are given in Table I and compared with those of Kamieński and Bielicki (1936, p. 11).

It should be noted that we have allowed for the perturbations by the six planets Venus-Uranus and the secular deceleration, together with nongravitational effects in other orbital elements, which were detected by Kamieński (1933) but not taken into account by him. They have been calculated by the formulae

$$\Delta\Omega_t = -0''.000345\,(t - t_0)$$
$$\Delta\omega_t = -0.000612\,(t - t_0),$$

https://doi.org/10.1017/S0074180900006331 Published online by Cambridge University Press

where the coefficients have been determined according to Kamieński's (1933, p. 44) method.

In order to obtain an additional check on our results, we have compared our set of elements for P/Wolf at the epoch of osculation 1925 July 12.5 UT with the set of elements derived by Kamieński for the same epoch from linking the 1925–1942

TABLE I

Representation of normal places of P/Wolf in 1925

1925 UT	Kamieński and Bielicki		Kazimirchak-Polonskaya	
	$\Delta\alpha \cos\delta$	$\Delta\delta$	$\Delta\alpha \cos\delta$	$\Delta\delta$
July 18.0	$-3^s.88$	$+2''.0$	$-0^s.03$	$+0''.7$
Aug. 19.0	-4.90	0.0	-0.34	$+1.2$
Sept. 14.0	-5.16	-3.3	-0.30	$+0.4$
Oct. 12.0	-5.04	-3.1	-0.33	-0.5
Nov. 11.0	-4.34	$+3.5$	-0.14	$+2.1$
Dec. 19.0	-3.43	$+4.8$	$+0.19$	$+0.9$
m_J^{-1}	1047.400 (de Sitter)		1047.355 (Hill)	
Nongravitational effects included	$\Delta\mu, \Delta M$		$\Delta\mu, \Delta M, \Delta\Omega, \Delta\omega$	
ε	$\pm 67''.6$		$\pm 3''.9$	

apparitions. The differences, in the sense Kamieński minus Kazimirchak-Polonskaya are very small:

$$\Delta T = -0^d.0152, \quad \Delta\Omega = -1''.31, \quad \Delta\varphi = -1''.17$$

$$\Delta\omega = +3''.83, \quad \Delta i = +2''.23, \quad \Delta\mu = +0''.0015.$$

Our orbital elements give a good representation of observations of the comet at subsequent apparitions considerably separated from the approach to Jupiter.

We can conclude from Table I that the discontinuity in the theory of motion of P/Wolf has been eliminated and the revolution 1918–1925 covered by the numerical theory of the motion of comet. Thus the theory has become continuous for the whole period of observations of P/Wolf, from discovery in 1884 up to the present time. However, we still consider our results to be preliminary and hope to arrive at a somewhat better agreement between the observations and calculations when solving the second problem, that of determining the mass of Jupiter from the large perturbations on P/Wolf in 1922 (Kazimirchak-Polonskaya, 1972a).

Acknowledgments

We wish to express our deep and sincerest thanks to N. A. Bokhan, who constructed the numerous double-precision standard programmes (Bokhan, 1972) that have been utilized in our set of programmes. We also thank N. A. Belyaev for providing his programme for comparison of calculations and observations (Belyaev, 1972).

References

Belyaev, N. A: 1972, this Symposium, p. 90.
Bokhan, N. A: 1972, this Symposium, p. 86.
Clemence, G. M.: 1943, *Astron. Pap. Washington* **11**, part 1.
Clemence, G. M.: 1949, *Astron. Pap. Washington* **11**, part 2.
Clemence, G. M.: 1954, *Astron. Pap. Washington* **13**, part 5.
Clemence, G. M.: 1961, *Astron. Pap. Washington* **16**, part 2.
Duncombe, R. L.: 1958, *Astron. Pap. Washington* **16**, part 1.
Eckert, W. J., Brouwer, D., and Clemence, G. M.: 1951, *Astron. Pap. Washington* **12**.
Kamieński, M.: 1933, *Acta Astron. Ser. a.* **3**, 1.
Kamieński, M.: 1948, *Bull. Acad. Polon. Sci. Lettres Ser. A* 1.
Kamieński, M.: 1959, *Acta Astron.* **9**, 53.
Kamieński, M. and Bielicki, M.: 1935, *Repr. Astron. Obs. Warsaw Univ.* **30**, 270.
Kamieński, M. and Bielicki, M.: 1936, *Repr. Astron. Obs. Warsaw Univ.* **32**, 1.
Kazimirchak-Polonskaya, E. I.: 1961, *Trudy Inst. Teor. Astron.* **7**, 191.
Kazimirchak-Polonskaya, E. I.: 1967, *Trudy Inst. Teor. Astron.* **12**, 24.
Kazimirchak-Polonskaya, E. I.: 1972a, this Symposium, p. 227.
Kazimirchak-Polonskaya, E. I.: 1972b, this Symposium, p. 373.
Kazimirchak-Polonskaya, E. I., Belyaev, N. A., and Terent'eva, A. K.: 1972, this Symposium, p. 462.
Morgan, H. R.: 1945, *Astron. J.* **51**, 127.
Myachin, V. F.: 1962, *Byull. Inst. Teor. Astron.* **8**, 537.
Newcomb, S.: 1895a, *Astron. Pap. Washington* **6**, part 1, part 3.
Newcomb, S.: 1895b, *The Elements of the Four Inner Planets and the Fundamental Constants of Astronomy*, Washington.
Numerov, B. V.: 1923, *Trudy Gl. Ross. Astrofiz. Obs.* **2**, 188.
Subbotin, M. F.: 1927, *Byull. Sredne-Aziat. Univ. Tashkent* **16**, 273.
Subbotin, M. F.: 1928, *Byull. Sredne-Aziat. Univ. Tashkent* **17**, 21.
Subbotina, N. S.: 1965, *Byull. Inst. Teor. Astron.* **10**, 143.

Discussion

M. Bielicki: What are the causes of the differences between your results and those by Kamieński and myself? In order to obtain the most accurate elements for P/Wolf in 1918 (before the approach to Jupiter) I developed a method of linking two consecutive apparitions with the aim of determining nongravitational effects. These consecutive links enabled me to obtain changes in all the orbital elements as a function of time. According to the tradition then existing I took into consideration the nongravitational secular changes only in the mean motion and mean anomaly. I then calculated the perturbations (by hand, of course) in Jupiter's sphere of action three times, twice using the heliocentric method and once by the more precise jovicentric method. Very great difficulties arose in the determination of accurate initial values of the position and velocity components in the jovicentric method. In the heliocentric calculations, I allowed for the perturbations from Venus through Uranus, and in the jovicentric calculations I allowed for those by the Sun and Saturn only.

E. I. Kazimirchak-Polonskaya: Firstly, the value you adopted for the mass of Jupiter (1/1047.400) is not consistent with the coordinates of Jupiter, which you took from the *Berliner Astronomisches Jahrbuch* and which correspond to Newcomb's value of 1/1047.35. I used Hill's value for the mass of Jupiter (1/1047.355) and the highly accurate coordinates in the *Astronomical Papers*, Volume 12 (plus the corrections in Volume 13) which correspond to Hill's value. Secondly, we developed a more accurate procedure for the calculation of the large perturbations experienced by comets in Jupiter's sphere of action; further, the calculations were performed in double precision and the total integration error reduced to a minimum. Finally, I allowed also for the nongravitational effects in the longitude of the ascending node and longitude of perihelion.