AN INTEGRAL INVOLVING A MODIFIED BESSEL FUNCTION OF THE SECOND KIND AND AN E-FUNCTION

by FOUAD M. RAGAB

(Received 30th April, 1951)

§ 1. *Introductory.* The formula to be established is

$$2^{m+1}\, m\pi^{m-1}\int_0^\infty K_{mn}(2m\lambda)\,\lambda^{mk-1}\, E\left(p\,;\;\alpha_r:q\,;\;\rho_s:\frac{x}{\lambda^{2m}}\right)d\lambda = E(p+2m\,;\;\alpha_r:q\,;\;\rho_s:x), \quad \ldots(1)$$

where m is a positive integer,

$$\left.\begin{aligned}\alpha_{p+2\nu+1} &= \tfrac{1}{2}k+\tfrac{1}{2}n+\frac{\nu}{m}\\ \alpha_{p+2\nu+2} &= \tfrac{1}{2}k-\tfrac{1}{2}n+\frac{\nu}{m}\end{aligned}\right\}\nu=0,\,1,\,2,\,\ldots,\,m-1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1')$$

and the constants are such that the integral converges.

This formula was proved by induction for the case $m=2^l$, $l=1, 2, 3, \ldots$, (**1**).

The following formulae are required in the proof:

$$4\int_0^\infty K_n(2\lambda)\,\lambda^{k-1}\,E\left(p\,;\;\alpha_r:q\,;\;\rho_s:\frac{x}{\lambda^2}\right)d\lambda = E(p+2\,;\;\alpha_r:q\,;\;\rho_s:x), \quad \ldots\ldots\ldots\ldots(2)$$

where $\alpha_{p+1}=\frac{1}{2}k+\frac{1}{2}n$, $\alpha_{p+2}=\frac{1}{2}k-\frac{1}{2}n$, $R(k\pm n)>0$ (**2**);

$$\prod_{\nu=1}^{m-1}\int_0^\infty K_n(2t)\,t_\nu^{2\nu/m-1}\,dt_\nu\,K_n\left(\frac{2b}{t_1t_2\ldots t_{m-1}}\right) = \left(\frac{\pi}{2}\right)^{m-1}K_{mn}(2mb^{1/m}), \quad \ldots\ldots\ldots\ldots\ldots(3)$$

where $b>0$, (**3**).

§ 2. *Proof of the Integral.* Apply formula (2) repeatedly to itself, taking for k the values $k+2\nu/m$, $\nu=1, 2, \ldots, m-1$: then

$$\begin{aligned}4^m\int_0^\infty K_n(2\lambda)\,\lambda^{k-1}\,d\lambda\prod_{\nu=1}^{m-1}\int_0^\infty K_n(2t_\nu)\,t_\nu^{k+2\nu/m-1}dt_\nu\,E\left(p\,;\;\alpha_r:q\,;\;\rho_s:\frac{x}{\lambda^2 t_1^{\,2}\ldots t^2_{m-1}}\right)\\ = E(p+2m\,;\;\alpha_r:q\,;\;\rho_s:x),\end{aligned}$$

where the α's after α_p are given by (1′).

Here, on the left, put the first integral last and then put $\lambda=\mu/(t_1t_2\ldots t_{m-1})$; the expression becomes

$$4^m\prod_{\nu=1}^{m-1}\int_0^\infty K_n(2t_\nu)\,t_\nu^{2\nu/m-1}\,dt_\nu\int_0^\infty K_n\left(\frac{2\mu}{t_1t_2\ldots t_{m-1}}\right)\mu^{k-1}\,E\left(p\,;\;\alpha_r:q\,;\;\rho_s:\frac{x}{\mu^2}\right)d\mu.$$

Now change the order of integration so that the last integral becomes the first integral, and apply (3); this gives

$$2^{m+1}\,\pi^{m-1}\int_0^\infty K_{mn}(2m\mu^{1/m})\,\mu^{k-1}\,E\left(p\,;\;\alpha_r:q\,;\;\rho_s:\frac{x}{\mu^2}\right)d\mu.$$

On putting $\mu=\lambda^m$, formula (1) is obtained.

Corollary. On replacing λ by λ/i and x by $xe^{-im\pi}$ in (1) and making use of the formula

$$K_n(t)=i^n\,G_n(it), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

https://doi.org/10.1017/S2040618500035589 Published online by Cambridge University Press

it is found that

$$2^{m+1}\, m\pi^{m-1} \int_0^\infty G_{mn}(2m\lambda)\, \lambda^{mk-1}\, E\left(p\, ;\ \alpha_r : q\, ;\ \rho_s : \frac{x}{\lambda^{2m}}\right) d\lambda$$
$$= i^{m(k-n)}\, E(p+2m\, ;\ \alpha_r : q\, ;\ \rho_s : xe^{-im\pi}), \quad \ldots\ldots\ldots\ldots(5)$$

where $k \pm n > 0$, $m(2\alpha_r - k) > -\frac{3}{2}$, $r = 1, 2, \ldots, p$.

Similarly, on replacing λ by λi and x by $xe^{im\pi}$, it is found that

$$2^{m+1}\, m\pi^{m-1} \int_0^\infty G_{mn}(2m\lambda e^{i\pi})\, \lambda^{mk-1}\, E\left(p\, ;\ \alpha_r : q\, ; \rho_s : \frac{x}{\lambda^{2m}}\right) d\lambda$$
$$= i^{-m(k+n)} E(p+2m\, ;\ \alpha_r : q\, ;\ \rho_s : xe^{im\pi}), \quad \ldots\ldots\ldots\ldots(6)$$

where $k \pm n > 0$, $m(2\alpha_r - k) > -\frac{3}{2}$, $r = 1, 2, \ldots, p$.

From (5) and (6), and making use of the formula

$$\pi i J_n(x) = G_n(x) - i^{2n}\, G_n(xe^{i\pi}), \quad \ldots\ldots\ldots\ldots(7)$$

it follows that

$$2^{m+1}\, m\pi^m\, i \int_0^\infty J_{mn}(2m\lambda)\, \lambda^{mk-1}\, E\left(p\, ;\ \alpha_r : q\, ;\ \rho_s : \frac{x}{\lambda^{2m}}\right) d\lambda$$
$$= i^{m(k-n)}\, E(p+2m\, ;\ \alpha_r : q\, ;\ \rho_s : xe^{-im\pi})$$
$$- i^{-m(k-n)}\, E(p+2m\, ;\ \alpha_r : q\, ;\ \rho_s : xe^{im\pi}), \quad \ldots\ldots\ldots\ldots(8)$$

where the α's are given by (1′) and $k+n>0$, $m(2\alpha_r - k) > -\frac{3}{2}$, $r = 1, 2, \ldots, p$.

REFERENCES

(1) Ragab, F. M., *Proc. Glasg. Math. Ass.*, **1,** 74 (1952).
(2) MacRobert, T. M., *Phil. Mag.* (vii), **31,** 258 (1941).
(3) Ragab, F. M., *Proc. Glasg. Math. Ass.*, **1,** 115 (1953).

University of Glasgow

https://doi.org/10.1017/S2040618500035589 Published online by Cambridge University Press