Torsion properties of modified diagonal classes on triple products of modular curves

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Abstract. Consider three normalized cuspidal eigenforms of weight 2 and prime level $p$. Under the assumption that the global root number of the associated triple product $L$-function is $+1$, we prove that the complex Abel–Jacobi image of the modified diagonal cycle of Gross–Kudla–Schoen on the triple product of the modular curve $X_0(p)$ is torsion in the corresponding Hecke isotypic component of the Griffiths intermediate Jacobian. The same result holds with the complex Abel–Jacobi map replaced by its étale counterpart. As an application, we deduce torsion properties of Chow–Heegner points associated with modified diagonal cycles on elliptic curves of prime conductor with split multiplicative reduction. The approach also works in the case of composite square-free level.

1 Introduction

The study of diagonal cycles on triple products of Shimura curves has its origins in the work of Gross, Kudla, and Schoen [11, 12]. They introduced a null-homologous modification of the diagonal embedding of the curve in its triple product, referred to as the modified diagonal cycle, or more commonly today as the Gross–Kudla–Schoen cycle. Given three cuspidal newforms of weight 2 and square-free level $N$ such that the sign of the functional equation of the associated triple product $L$-function is $-1$, Gross and Kudla [11] conjectured that the central value at $s = 2$ of the derivative of this $L$-function is given by a complex period times the Beilinson–Bloch height of the corresponding Hecke isotypic component of the modified diagonal cycle on the triple product of an indefinite Shimura curve determined by the local triple product root numbers. A proof of this conjecture was announced in work of Yuan et al. [28], but has yet to be published. The Shimura curve in question is the modular curve $X_0(N)$ precisely when the local triple product root numbers are $+1$ at all finite places.

1.1 Main results

In this article, we exhibit certain torsion properties of modified diagonal classes on the triple product of the modular curve $X := X_0(p)$ defined over $\mathbb{Q}$ and of prime
level $p$. The results hold more generally for composite square-free level $N$ (see Section 1.4). Since the prime level case already contains all relevant ingredients of the proof, we have chosen to focus on this case.

The modified diagonal cycle depends on a base point $e$ in $X(\mathbb{Q})$. It will be denoted by $\Delta_{GKS}(e)$ and viewed as an element of the Chow group $\text{CH}^2(X^3)_0(\mathbb{Q})$ of null-homologous codimension 2 algebraic cycles on $X^3$ over $\mathbb{Q}$ modulo rational equivalence (see Section 1.7 for our slightly unconventional definition of Chow groups as functors). Let $f_1, f_2, f_3$ be three normalized cuspidal eigenforms of weight 2 and level $\Gamma_0(p)$, and denote by $F := f_1 \otimes f_2 \otimes f_3$ their triple product. We place ourselves in the setting where the global root number $W(F)$ of the triple product $L$-function $L(F,s)$ associated with $F$ is $+1$. This assumption forces $L(F,s)$ to vanish to even order at its centre $s = 2$. Comparing with the more classical situation of Heegner points studied in the seminal work of Gross and Zagier [13], it seems reasonable to expect that the “$F$-isotypic Hecke component” $\Delta^F_{GKS}(e)$ (see Remark 3.2) of the modified diagonal cycle, with $e \in X(\mathbb{Q})$, is trivial in the Chow group $\text{CH}^2(X^3)_0(\mathbb{Q}) \otimes_{\mathbb{Z}} K_F$ of cycles defined over $\mathbb{Q}$ with coefficients in the Hecke field of $F$, in line with the predictions of the Beilinson–Bloch conjectures [4]. While it appears difficult to prove a torsion statement directly in the Chow group, we can prove the corresponding result for the image of $\Delta^F_{GKS}(e)$ under the complex Abel–Jacobi map

$$\text{AJ} : \text{CH}^2(X^3)_0(\mathbb{C}) \longrightarrow j^2(X_C^3),$$

whose target is the Griffiths intermediate Jacobian of $X_C^3$ viewed as a complex manifold.

**Theorem 1.1** Let $f_1, f_2, f_3$ be three normalized eigenforms of weight 2 and level $\Gamma_0(p)$, denote by $F := f_1 \otimes f_2 \otimes f_3$ their triple product, and suppose that the global root number of $L(F,s)$ is $+1$. Then $\text{AJ}(\Delta^F_{GKS}(e))$ is trivial in $j^2(X_C^3) \otimes_{\mathbb{Z}} K_F$, for all $e \in X(\mathbb{Q})$.

The kernel of the complex Abel–Jacobi map (restricted to cycles defined over $\mathbb{Q}$) is conjectured to be torsion [16, Conjecture 9.12]. Conditional on this conjecture, Theorem 1.1 implies that $\Delta^F_{GKS}(e)$ is trivial in $\text{CH}^2(X^3)_0(\mathbb{Q}) \otimes_{\mathbb{Z}} K_F$. The same statement as in Theorem 1.1 holds with the complex Abel–Jacobi map replaced by its $\ell$-adic étale counterpart [4]

$$\text{AJ}^\ell_{X^3} : \text{CH}^2(X^3)_0(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, H^3_{et}(X^3_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(2))),$$

with $\ell$ a rational prime (see Remark 4.4).

In the special case $p = 37$, using numerical results due to Stein [5, Appendix], we deduce the following, where $\xi_\infty$ denotes the cusp of $X$ at infinity.

**Theorem 1.2** Let $f$ and $g$ be the normalized eigenforms of weight 2 and level $\Gamma_0(37)$ corresponding to the elliptic curves with Cremona labels 37b and 37a, and let $F := g \otimes g \otimes f$. Then $\text{AJ}_{X_0(37)}(8\Delta^F_{GKS}(\xi_\infty))$ is a nontrivial 6-torsion element of $j^2(X_0(37)_C^3)$. 

https://doi.org/10.4153/5000843952200011X Published online by Cambridge University Press
1.2 Application to Chow–Heegner points

Chow–Heegner points were introduced by Bertolini et al. [2] as a generalization of the construction of Heegner points. The idea is to produce rational points on elliptic curves by pushing forward algebraic cycles on higher dimensional varieties using suitable correspondences, or generalized modular parametrizations, as they are referred to in [2].

Let $f$ be a normalized cuspidal eigenform of weight 2 and level $\Gamma_0(p)$ with rational Fourier coefficients. Denote by $E_f$ the optimal elliptic curve over $\mathbb{Q}$ of conductor $p$ associated with $f$ by Eichler and Shimura [27]. Using an auxiliary normalized cuspidal eigenform $g$ of weight 2 and level $\Gamma_0(p)$, it is possible to construct a correspondence $\Pi^f_g \in \text{CH}^2(X^3 \times E_f)(\mathbb{Q})$, which gives rise via push-forward to a generalized modular parametrization, that is, a natural transformation

$$\Pi^f_g \ast : \text{CH}^2(X^3)_0 \rightarrow \text{CH}^1(E_f)_0 = E_f.$$

The Chow–Heegner point associated with the modified diagonal cycle based at a point $e \in X(\hat{\mathbb{Q}})$ is then defined as

$$P^F_{g}(e) := \Pi^f_g(\Delta_{\text{GKS}}(e)) \in E_f(\hat{\mathbb{Q}}).$$

Darmon et al. [7] have studied such points, in the broader context of Shimura curves over totally real fields, notably by computing their images under the complex Abel–Jacobi map in terms of iterated integrals. Methods have been developed by Darmon et al. [5] to numerically calculate such points in the case of modular curves.

Let $F := g \otimes g \otimes f$. We exhibit a correspondence mapping $\Delta_{\text{GKS}}^F(e)$ to $P^F_{g}(e)$. When the global root number $W(F)$ is $-1$, Darmon et al. [7] have studied the nontorsion properties of $P^F_{g}(\xi_{\infty})$, building on [28]. In the complementary situation when $W(F) = +1$, we use Theorem 1.1 and functoriality of Abel–Jacobi maps with respect to correspondences to deduce the following:

**Theorem 1.3** Let $f$ and $g$ be as above, and let $F = g \otimes g \otimes f$. If $W(F) = +1$, then the Chow–Heegner point $P^F_{g}(e)$ is torsion in $E_f(\mathbb{Q})$, for all $e \in X(\mathbb{Q})$.

Theorem 1.3 with $e = \xi_{\infty}$ recovers a result of Daub [8, Theorem 3.3.8] by a different method in the case of prime level. Similar arguments should work for $f$ not rational.

1.3 Strategy of the proof

The key ingredient in the proof of Theorem 1.1 is the Atkin–Lehner involution $w_p$ of $X$. The global root number of $W(F)$ is the product of the global root numbers of $f_1$, $f_2$, and $f_3$, which are each equal to the negative of their $w_p$-eigenvalue. As a consequence, the assumption that $W(F)$ equals $+1$ translates into information about the action of $w_p \times w_p \times w_p$ on $F$, and consequently on $A_{X^3}(\Delta_{\text{GKS}}^F(e))$, as the latter lies in the $F$-isotypic Hecke component of the intermediate Jacobian by functoriality of the Abel–Jacobi map with respect to correspondences. The work of Mazur [22]
provides necessary information about the rational points \( X(\mathbb{Q}) \) and the action of \( w_p \) on them.

### 1.4 Composite square-free level

The arguments of this paper carry over to the more general setting where the level \( N \) is composite, but square-free. This is the situation initially considered in the work of Gross and Kudla [11]. It becomes necessary to replace eigenforms by newforms.

Let \( f_1, f_2, f_3 \) be three normalized newforms of weight 2 and level \( \Gamma_0(N) \), and let \( F := f_1 \otimes f_2 \otimes f_3 \). The level being square-free guarantees that the local root numbers \( W_p(F) \) for \( p \mid N \) are the products of the local root numbers at \( p \) of \( f_1, f_2, \) and \( f_3 \), which are each the negative of their \( w_p \)-eigenvalue. The Atkin–Lehner correspondences \( w_p, p \mid N \), commute with the good Hecke correspondences \( T_n \) (i.e., with \( (n, N) = 1 \)), and this is sufficient for our purposes (see Remark 2.1). Assume that there exists \( p \mid N \) for which \( W_p(F) = -1 \). Using multiplicity one for newforms, this assumption can be parlayed into information about the torsion properties of the images of modified diagonal cycles under Abel–Jacobi maps, as long as one has sufficient understanding of the action of the Atkin–Lehner involution \( w_p \) on the rational points of \( X_0(N) \). The only rational points on composite level modular curves \( X_0(N) \) of genus \( \geq 2 \) are the rational cusps [18]. It is known that the subgroup of the Jacobian \( J_0(N) \) generated by the cusps is torsion by the Manin–Drinfeld theorem [20]. It follows that Theorem 1.1 remains true for normalized newforms \( f_1, f_2, f_3 \) of composite square-free level under the assumption \( W_p(F) = -1 \) for some \( p \mid N \).

The proof of Theorem 1.3 adapts verbatim to the setting of composite square-free level, provided that the eigenforms are newforms and \( W_p(F) = -1 \) for some \( p \mid N \). This recovers [8, Theorem 3.3.8] by a different approach.

Examining Stein's Table 2 in [5, Appendix], we obtain results similar to Theorem 1.2, e.g., in the following cases:

- \( N = 57 \): \( f \) corresponds to the elliptic curve with Cremona label 57c, and \( g \) corresponds to the curves with labels 57a or 57b.
- \( N = 58 \): \( f \) corresponds to the elliptic curve with Cremona label 58b, and \( g \) corresponds to the curve with label 58a.

### 1.5 Related work

The approach taken in this paper is explicit and elementary, exploiting the connection between triple product root numbers and eigenvalues of Atkin–Lehner involutions. A more powerful approach is considered in the work of Yuan et al. [28], using Prasad's dichotomy for the existence of trilinear forms on automorphic representations. Forthcoming work of Qiu and Zhang [24] further develops this approach and gives applications.

### 1.6 Outline

Background on cusp forms of weight 2 is recalled in Section 2. Section 3 recalls facts about the triple product \( L \)-function and states the Beilinson–Bloch conjecture in this
setting. Section 4 constitutes the proof of Theorem 1.1. The application to Chow–Heegner points is given in Section 5. Theorem 1.2 is proved in Section 6.

1.7 Notational conventions

Fix a complex embedding \( \mathbb{Q} \hookrightarrow \mathbb{C} \), as well as \( p \)-adic embeddings \( \mathbb{Q} \hookrightarrow \mathbb{C}_p \) for each rational prime \( p \). In this way, all finite extensions of \( \mathbb{Q} \) are viewed simultaneously as subfields of \( \mathbb{C} \) and \( \mathbb{C}_p \). For a field extension \( F \) of \( \mathbb{Q} \), the subscript \( F \) on a group (resp. \( \mathbb{Q} \)-algebra) will denote the tensor product with \( F \) over \( \mathbb{Z} \) (resp. \( \mathbb{Q} \)). For any field \( K \), we fix an algebraic closure \( \bar{K} \).

By a variety \( X \) over \( K \), we shall mean an integral separated scheme of finite type over \( K \). A subvariety is an integral separated closed subscheme. If \( F \) is a field extension of \( K \), \( X_F \) will denote the base change of \( X \) to Spec\( (F) \).

An algebraic cycle of codimension \( r \) on \( X \) is a finite \( \mathbb{Z} \)-linear combination of subvarieties of \( X \) of codimension \( r \). The Chow group of codimension \( r \) algebraic cycles modulo \( \mathbb{Z} \)-rational equivalence will be denoted \( \text{CH}_r(X) \).

This convention is borrowed from [7] and differs from the more classical notation of [10]. Given two varieties \( X \) and \( Y \) over \( K \), we write \( \text{Corr}^r(X, Y) := \text{CH}^{\dim X + r}(X \times Y) \).

2 Cusp forms

Let \( p > 3 \) be a rational prime. Let \( Y := Y_0(p) \) be the modular curve over \( \mathbb{Q} \) for the congruence subgroup \( \Gamma_0(p) \subset \text{SL}_2(\mathbb{Z}) \) consisting of matrices which are upper-triangular modulo \( p \). It admits a canonical proper desingularization \( Y_0(p) \hookrightarrow X_0(p) \), obtained over the complex numbers by adjoining the cusps. The curve \( X := X_0(p) \) is a geometrically connected, smooth, and proper curve over \( \mathbb{Q} \). It is the coarse moduli scheme representing pairs \((E, H)\) consisting of a generalized elliptic curve \( E \) over a \( \mathbb{Q} \)-scheme \( S \), together with a cyclic subgroup scheme \( H \) of order \( p \). It admits a uniformization by the extended Poincaré upper half-plane

\[
\mathcal{H}^* \longrightarrow X(\mathbb{C}), \quad \tau \mapsto (\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}, \langle 1/p + \mathbb{Z} \oplus \tau\mathbb{Z} \rangle),
\]

which identifies \( X(\mathbb{C}) \) with the quotient \( \Gamma_0(p) \backslash \mathcal{H}^* \). There are two cusps \( \xi_{\infty} \) and \( \xi_0 \) on \( X \), which correspond via (2.1) to the points \( i\infty \) and \( 0 \) of \( \mathcal{H}^* \). The genus \( g_X \) of \( X \) is given by the formula

\[
g_X = \begin{cases} \left\lfloor \frac{p+1}{12} \right\rfloor - 1, & \text{if } p \equiv 1 \pmod{12}, \\ \left\lfloor \frac{p+1}{12} \right\rfloor, & \text{otherwise}. \end{cases}
\]

The space \( S_2(\Gamma_0(N)) \) of weight 2 cusp forms of level \( \Gamma_0(p) \) is naturally identified with the space of global sections of the sheaf of regular differential 1-forms on \( X \) via
the isomorphism

\[(2.3) \quad S_2(\Gamma_0(p)) \cong H^0(X_C, \Omega^1_X), \quad f \mapsto \omega_f := 2\pi i f(\tau)d\tau.\]

In particular, the dimension of \(S_2(\Gamma_0(p))\) is equal to \(g_X\).

## 2.1 Hecke operators

The curve \(X\) is equipped with the usual collection of Hecke correspondences, which act on cohomology and give rise to operators on \(S_2(\Gamma_0(p))\) via (2.3). These correspondences and their induced operators are denoted by \(U_p\) and \(T_n\), for integers \(n \geq 1\) coprime to \(p\). Their precise definition can be found in [1, (3.1)].

The curve \(X\) also comes equipped with the Atkin–Lehner involution \(w_p\). It is defined, following the moduli description, by mapping a \(p\)-isogeny \(\phi : E \rightarrow E'\) of elliptic curves to its dual isogeny \(\phi^\vee : E' \rightarrow E\). In terms of covering spaces, using (2.1), it is given by \(\tau \mapsto -\frac{1}{p}\tau\), where \(\tau \in \mathcal{H}\). This involution is defined over \(\mathbb{Q}\) and thus maps \(\mathbb{Q}\)-rational points of \(X\) to \(\mathbb{Q}\)-rational points. It induces, via (2.3), an operator on \(S_2(\Gamma_0(p))\), which we also denote by \(w_p\).

The operators \(T_m, \) with \((m, p) = 1\), on \(S_2(\Gamma_0(p))\) commute with the operators \(T_n, U_p\) and \(w_p\) [1, Lemma 17]. Let \(\mathbb{T} := \mathbb{T}(p)\) denote the \(\mathbb{Q}\)-algebra generated by the operators \(T_n,\) with \((n, p) = 1\). The space of cusp forms \(S_2(\Gamma_0(p))\) admits a basis of eigenforms for \(\mathbb{T}\) [1, Theorem 2].

Let \(f = \sum_{n \geq 1} a_n(f) q^n \in S_2(\Gamma_0(p))\) be a normalized eigenform, in the sense that \(a_1(f) = 1\). Because the level is prime, there are no oldforms. As a consequence, we have the equality of operators \(U_p = -w_p\). In particular, the operators \(U_p\) and \(w_p\) commute. Note that this is only the case for general composite level when restricting to newforms [1, Lemma 17]. It follows that \(w_p(f) = -a_p(f) f\). In particular, we have \(a_p(f) \in \{\pm 1\}\).

The normalized eigenform \(f\) determines a surjective homomorphism \(\lambda_f : \mathbb{T} \rightarrow K_f\) of algebras by sending \(T_n\) to \(a_n(f)\). Here, \(K_f\) is the totally real finite extension of \(\mathbb{Q}\) generated by the Fourier coefficients \(a_n(f)\) of \(f\).

Let \(S_2(\Gamma_0(p))_f\) denote the \(f\)-isotypic component of \(S_2(\Gamma_0(p))\) consisting of cusp forms \(f'\) in \(S_2(\Gamma_0(p))\) such that \(T(f') = \lambda_f(T)f'\), for all \(T \in \mathbb{T}\). By the theorem of multiplicity one [1, Lemma 20 and 21] of Atkin and Lehner for newforms, the space \(S_2(\Gamma_0(p))_f\) is one-dimensional over \(\mathbb{C}\). We have the spectral decomposition

\[S_2(\Gamma_0(p)) = \bigoplus_h S_2(\Gamma_0(p))_h,\]

where the sum is taken over all normalized eigenforms \(h \in S_2(\Gamma_0(p))\). Since the dual space \(S_2(\Gamma_0(p))^\vee\) is a free \(\mathbb{T}_\mathbb{C}\)-module of rank one by multiplicity one, we similarly obtain a decomposition

\[\mathbb{T}_\mathbb{C} = \bigoplus_h \mathbb{T}_{\mathbb{C}, h},\]

where \(\mathbb{T}_{\mathbb{C}, h}\) denotes the algebra of Hecke operators \(T_n,\) with \((n, p) = 1\), acting on \(S_2(\Gamma_0(p))_h\), which is again a \(\mathbb{C}\)-vector space of dimension one.

Let \([f]\) denote the \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) orbit of \(f\). Form the complex vector space \(\bigoplus_{g \in [f]} S_2(\Gamma_0(p))_g\) of dimension \(d_f := [K_f : \mathbb{Q}]\), and consider the \(\mathbb{Q}\)-subspace

\(\mathbb{T}_\mathbb{Q} = \bigoplus_h \mathbb{T}_{\mathbb{Q}, h},\)
2.2 Hecke projectors

Let \( f = \sum_{n \geq 1} a_n(f) q^n \in S_2(\Gamma_0(p)) \) be a normalized eigenform. Let \( V := S_2(\Gamma_0(p))^\vee \) be the \( \mathbb{C} \)-dual of \( S_2(\Gamma_0(p)) \). The complex points of the Jacobian \( J_C \) are

\[
J_C(\mathbb{C}) = H^0(X_C, \Omega^1_X)^\vee / \text{Im } H_1(X_C(\mathbb{C}), \mathbb{Z}),
\]

where \( \Lambda := \text{Im } H_1(X_C(\mathbb{C}), \mathbb{Z}) \) is viewed as a lattice via integration of differential forms. By (2.3), we thus have an identification \( J_C(\mathbb{C}) = V/\Lambda \) as a \( g_X \)-dimensional complex torus, where we recall that \( g_X \) is the genus of \( X \). Let \( V_f \) be the subspace of \( V \) on which \( T \) acts via the homomorphism \( \lambda_f : T \rightarrow K_f \), and let \( \text{pr}_f : V \rightarrow V_f \) be the orthogonal projection with respect to the Petersson scalar product. The projector \( \text{pr}_f \) naturally belongs to \( T \otimes \mathbb{Q} K_f \), and by (2.5) and (2.6) we may view \( \text{pr}_f \) as an idempotent element

\[
[t_f] \in (\text{CH}^1(X^2)(\mathbb{Q})_{K_f})/(\text{pr}_f^* \text{CH}^1(X)(\mathbb{Q})_{K_f} + \text{pr}_f^* \text{CH}^1(X)(\mathbb{Q})_{K_f}),
\]

where \( t_f \) denotes some lift of \( \text{pr}_f \) to \( \text{CH}^1(X^2)(\mathbb{Q})_{K_f} \). The correspondence \( t_f \) is some choice of \( K_f \)-linear combination of Hecke correspondences which induces the projection on cohomology onto the \( f \)-isotypic component.
3 Triple products

Let \( f_i = \sum_{n \geq 1} a_n(f_i) q^n \), \( f_2 = \sum_{n \geq 1} a_n(f_2) q^n \), and \( f_3 = \sum_{n \geq 1} a_n(f_3) q^n \) be three normalized cuspidal eigenforms of weight 2 and level \( \Gamma_0(p) \), and let \( F := f_1 \otimes f_2 \otimes f_3 \) be the associated cusp form of weight \((2,2,2)\) for \( \Gamma_0(p) \). Let \( K_F = K_{f_1} \cdot K_{f_2} \cdot K_{f_3} \) denote the compositum of the Hecke fields of the forms \( f_1, f_2, \) and \( f_3 \).

3.1 Triple product \( L \)-functions

For \( i \in \{1, 2, 3\} \) and a prime \( \ell \), let \( \lambda_i \) be the prime ideal of \( K_{f_i} \) above \( \ell \) determined by the embeddings fixed in Section 1.7. Denote by \( K_{f_i, \lambda_i} \) the completion of \( K_{f_i} \) with respect to \( \lambda_i \), and let \( V_\ell(f_i) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(K_{f_i, \lambda_i}) \) be the two-dimensional \( \ell \)-adic Galois representation associated to \( f_i \) [6, Theorem 3.1]. We remark that given any choice of correspondence \( t_{f_i} \) as in (2.7), the representation \( V_\ell(f_i) \) admits a description as \( (t_{f_i})_* \cdot H^1_{\text{et}}(X_{\overline{Q}, \mathbb{Q}_\ell}) \) followed by the map induced by the projection \( K_{f_i} \otimes \mathbb{Q}_\ell \to K_{f_i, \lambda_i} \).

The triple product \( L \)-function \( L(F,s) = L(f_1, f_2, f_3, s) \) is the \( L \)-function associated with the compatible family of eight-dimensional \( \ell \)-adic representations

\[
V_\ell(F) := V_\ell(f_1) \otimes V_\ell(f_2) \otimes V_\ell(f_3).
\]

It admits a description as an Euler product converging absolutely for \( \Re(s) > 5/2 \). The Euler factors are given explicitly in [11, (1.7) and (1.8)].

Define the local \( L \)-factor at infinity following the general recipe of [9] by

\[
L_\infty(F,s) = 2^4(2\pi)^{3-4s} \Gamma(s-1)^3 \Gamma(s).
\]

The completed \( L \)-function \( \Lambda^*(F,s) := (p^5)^n L_\infty(F,s) L(F,s) \) admits an analytic continuation to the entire complex plane and satisfies the functional equation

\[
\Lambda^*(F,s) = W(F) \cdot \Lambda^*(F,4-s),
\]
where \( W(F) \in \{\pm 1\} \) is the global root number of \( F \) \cite[Proposition 1.1]{11}. The global root number, as stated in \cite[Section 1]{11}, is given by

\[
W(F) = a_p(f_1)a_p(f_2)a_p(f_3).
\]

(3.2)

A detailed proof of this can for instance be found in \cite[Proposition 4.5]{19}.

3.2 The Beilinson–Bloch conjecture

The center of symmetry of the functional equation (3.1) is the point \( s = 2 \) at which \( L(F,s) \) has no pole. Moreover, \( L_\infty(F,s) \) has neither zero nor pole at \( s = 2 \), so the center is a critical point, and we have

\[
W(F) = (-1)^{\text{ord}_{s=2}L(F,s)}.
\]

(3.3)

For \( i \in \{1, 2, 3\} \), let \( t_{f_i} \) be a choice of self-correspondence of \( X \) lifting the \( f_i \)-Hecke projector \( \text{pr}_{f_i} \). Define a self-correspondence of \( X^3 \) by

\[
t_F := t_{f_1} \otimes t_{f_2} \otimes t_{f_3} = \text{pr}_{14}^*(t_{f_1}) \cdot \text{pr}_{25}^*(t_{f_2}) \cdot \text{pr}_{36}^*(t_{f_3}) \in \text{Corr}^0(X^3, X^3)(\mathbb{Q})_{K_F},
\]

where \( \text{pr}_{ij} : X^6 \to X^2 \) denotes the natural projection to the \( i \)th and \( j \)th components.

Remark 3.1 The correspondence \( t_F \) is some choice of \( K_F \)-linear combination of tensor products of Hecke correspondences projecting to the 1-dimensional \( F \)-isotypic component of the \((T^{\otimes 3} \otimes \mathbb{R})\)-module \( H^0(X^3, \Omega^3_{X^3}) \otimes \mathbb{R} = H^0(X, \Omega^1_X)^{\otimes 3} \otimes \mathbb{R} \).

The Beilinson–Bloch conjecture \cite{4} predicts in this setting that

\[
\text{ord}_{s=2}L(F,s) = \dim_{K_F}(t_F)_*(CH^2(X^3)_0(\mathbb{Q})_{K_F}).
\]

(3.5)

In the case when \( W(F) = +1 \), Gross and Kudla proved a formula for the central value \( L(F,2) \), expressing it as a product of a complex period and an algebraic number \cite[Proposition 10.8]{11}. This algebraic number admits an explicit description in terms of the coefficients of the Jacquet–Langlands transfers of \( f_1, f_2, \) and \( f_3 \) to the definite quaternion algebra ramified at \( p \) and \( \infty \).

In the case when \( W(F) = -1 \), the \( L \)-function \( L(F,s) \) vanishes to odd order at its centre \( s = 2 \). By (3.5), we expect \( (t_F)_*(CH^2(X^3)_0(\mathbb{Q})_{K_F}) \) to have dimension greater or equal to 1. A natural element of \( CH^2(X^3)_0(\mathbb{Q}) \) to consider is the modified diagonal cycle, also referred to as the Gross–Kudla–Schoen cycle, which we now define.

Let \( \Delta \) denote the image of \( X \) under the diagonal embedding \( X \to X^3 \), i.e.,

\[
\Delta = \{(x,x,x) \mid x \in X\} \subset X^3.
\]

(3.6)

In order to get a null-homologous cycle, we apply a projector to \( \Delta \) following \cite{11, 12}.

Definition 3.1 Let \( C \) be a smooth, projective, and geometrically connected curve over a number field \( k \), and let \( e \) be a point in \( X(\bar{k}) \). For any nonempty subset \( T \) of \( \{1, 2, 3\} \), let \( T' \) denote the complementary set. Write \( p_T : C^3 \to C^{[T]} \) for the natural projection map and let \( q_T(e) : C^{[T]} \to C^3 \) denote the inclusion obtained by filling in the missing coordinates using the point \( e \). Let \( P_T(e) \) denote the graph of \( q_T(e) \circ p_T \).
viewed as a codimension 3 cycle on the product $C^3 \times C^3$. Define the Gross–Kudla–Schoen projector

$$P_{\text{GKS}}(e) := \sum_{T} (-1)^{|T|} P_T(e) \in \text{CH}^3(C^3 \times C^3)(\hat{k}),$$

where the sum is taken over all subsets of $\{1, 2, 3\}$. This is an idempotent in the ring of correspondences of $C^3$ with the property that it annihilates the cohomology groups $H^i(C^3_{\mathbb{C}}(\mathbb{C}), \mathbb{Z})$, for $i \in \{4, 5, 6\}$, and maps $H^3(C^3_{\mathbb{C}}(\mathbb{C}), \mathbb{Z})$ onto the Künneth summand $H^3(C^3_{\mathbb{C}}(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Q}$ [12, Corollary 2.6].

Given a point $e \in X(\mathbb{Q})$, the Gross–Kudla–Schoen cycle with base point $e$ is

$$\Delta_{\text{GKS}}(e) := P_{\text{GKS}}(e) \cdot (\Delta) \in \text{CH}^2(X^3)_0(\mathbb{Q}).$$

Note that the cycle $\Delta_{\text{GKS}}(e)$ is in fact null-homologous since $P_{\text{GKS}}(e)$ annihilates $H^4(X^3_{\mathbb{C}}(\mathbb{C}), \mathbb{Z})$, the target of the cycle class map. Define the “$F$-isotypic component” of the Gross–Kudla–Schoen cycle by $(t_F)_* \Delta_{\text{GKS}}(e) \in \text{CH}^2(X^3)_0(\mathbb{Q})_{K_F}$.

**Remark 3.2** Although $t_F$ is not unique, the difference $t_F - t'_F$ of two such projectors annihilates $H^0(X^3_{\mathbb{C}}, \Omega^3_{X_{\mathbb{C}}})$. Conditional on the nondegeneracy of the Beilinson–Bloch height pairing for $X^3$, this implies that $(t_F)_* \Delta_{\text{GKS}}(e) = (t'_F)_* \Delta_{\text{GKS}}(e)$. Unconditionally, the Beilinson–Bloch height of $(t_F)_* \Delta_{\text{GKS}}(e)$ is independent of the choice of $t_F$ [12, Proposition 8.3, Notes 8.5 and 8.6].

Gross and Kudla [11, Conjecture 13.2] conjectured the formula

$$\frac{L'(F, 2)}{\Omega_F} = \langle (t_F)_* (\Delta_{\text{GKS}}(\xi_\infty)), (t_F)_* (\Delta_{\text{GKS}}(\xi_\infty)) \rangle^{BB},$$

where $\langle , , \rangle^{BB} : \text{CH}^2(X^3)_0(\mathbb{Q})_\mathbb{R} \times \text{CH}^2(X^3)_0(\mathbb{Q})_\mathbb{R} \to \mathbb{R}$ denotes the Beilinson–Bloch height pairing [11, (13.9)], and $\Omega_F := \|\omega_{f_1}\|^2 \cdot \|\omega_{f_2}\|^2 \cdot \|\omega_{f_3}\|^2 / (4\pi p)$ is the complex period of $F$ with $\| \cdot \|$ denoting the Petersson norm. A proof of (3.8) due to Yuan et al. was announced in [28].

### 4 Abel–Jacobi maps

Let $f_1, f_2, f_3$ be three normalized eigenforms in $S_2(\Gamma_0(p))$, and let $F = f_1 \otimes f_2 \otimes f_3$. We work under the following assumption on the sign of the functional equation (3.1).

**Assumption 4.1** $W(F) = +1$.

Under Assumption 4.1, the $L$-function $L(F, s)$ vanishes to even order at the central critical point $s = 2$, by (3.3), and the Beilinson–Bloch conjecture (3.5) predicts that the algebraic rank of the $F$-isotypic component of $\text{CH}^2(X^3)_0(\mathbb{Q})$ is even. Comparing with the situation of Heegner points on modular curves studied in [13], it seems reasonable to expect that the $F$-isotypic component of $\Delta_{\text{GKS}}(e)$ is trivial, for all $e \in X(\mathbb{Q})$. While this appears to be difficult to show directly in the Chow group, we...
can prove the corresponding statement for the image of the cycle under the complex Abel–Jacobi map

\[(4.1) \quad \text{AJ}_{X^3} : \text{CH}^2(X_3^3) \rightarrow \text{J}^2(X_3^3) := \frac{(\text{Fil}^2 H^3_{dR}(X_3^3))^\vee}{\text{Im} H_3(X_3^3(\mathbb{C}), \mathbb{Z})},\]

whose target is the second Griffiths intermediate Jacobian of $X_3^3$. This map is a higher dimensional generalization of the familiar Abel–Jacobi isomorphism for curves. It is defined by the integration formula

\[\text{AJ}_{X^3}(Z)(\alpha) := \int_{\partial^{-1}(Z)} \alpha, \quad \text{for all } \alpha \in \text{Fil}^2 H^3_{dR}(X_3^3),\]

where $\partial^{-1}(Z)$ denotes any continuous 3-chain in $X_3^3(\mathbb{C})$ whose image under the boundary map $\partial$ is $Z$.

**Definition 4.1** Given a point $e$ in $X(\bar{\mathbb{Q}})$ and a choice of correspondence $t_F$ (3.4) projecting to the $F$-isotypic component of $H^0(X^3, \Omega^3_{X^3}) \otimes \mathbb{R}$, define the $F$-isotypic component of the Abel–Jacobi image of the Gross–Kudla–Schoen cycle by

\[\text{AJ}^F_{X^3}(\Delta_{GKS}(e)) := \text{AJ}_{X^3}((t_F)_*(\Delta_{GKS}(e))) \in J^2(X_3^3)_{K_F}.\]

**Remark 4.2** Definition 4.1 is independent of the choice of $t_F$, as $\text{AJ}_{X^3}$ is functorial and any two such projectors act similarly on cohomology.

Henceforth, we fix a choice of projector $t_F = t_{f_1} \otimes t_{f_2} \otimes t_{f_3}$, the aim of this section is to prove the main result:

**Theorem 4.3** Let $f_1$, $f_2$, and $f_3$ be three normalized eigenforms in $S_2(\Gamma_0(p))$, denote by $F = f_1 \otimes f_2 \otimes f_3$ their triple product, and suppose that $F$ satisfies Assumption 4.1. Then $\text{AJ}^F_{X^3}(\Delta_{GKS}(e)) = 0$ in $J^2(X_3^3)_{K_F}$, for all $e \in X(\mathbb{Q})$.

**Remark 4.4** Similar arguments to the ones presented in the proof of Theorem 4.3 below can be used to prove that the image of $(t_F)_*(\Delta_{GKS}(e))$ under Bloch’s [4] $\ell$-adic étale Abel–Jacobi map

\[(4.2) \quad \text{AJ}^\text{et}_{X_3} : \text{CH}^2(X_3) \rightarrow \text{H}^1(\mathbb{Q}, H^3_{\text{et}}(X^3_{\mathbb{Q}}, \mathbb{Q}_{\ell}(2)))\]

is torsion, when the global root number is $W(F) = +1$. It is conjectured that for any smooth proper variety over a number field, and for any prime $\ell$, the $\ell$-adic Abel–Jacobi maps in any codimension are injective up to torsion [16, Conjecture 9.15]. Thus, conditional on this conjecture, $(t_F)_*(\Delta_{GKS}(e))$ is trivial in the Chow group $\text{CH}^2(X^3)_{K_F}$.

The rest of this section constitutes the proof of Theorem 4.3. We distinguish different situations depending on the genus $g_X$ of $X$, which we recall is given by the formula (2.2). The curve $X$ has genus zero exactly when $p \in \{2, 3, 5, 7, 13\}$. In this case, the space of cusp forms $S_2(\Gamma_0(p))$ is trivial, and there is no triple product $L$-function to consider in the first place. We have $\Delta_{GKS}(e) = 0$ in $\text{CH}^2(X^3)_{0}(\mathbb{Q})$, as the cycle class map is injective in this case [12, Proposition 4.1].
4.1 The genus one case

Suppose that \( g_X = 1 \), i.e., \( p \in \{11, 17, 19\} \). In this case, \( X \) is an elliptic curve over \( \mathbb{Q} \) of Mordell–Weil rank 0. For all \( e \in X(\mathbb{Q}) \), we have \( 6\Delta_{\text{GKS}}(e) = 0 \) in \( \text{CH}^2(X^3)_0(\mathbb{Q}) \) [12, Corollary 4.7]. On the \( L \)-function side, \( f_1 = f_2 = f_3 = f \) is the normalized eigenform corresponding to the elliptic curve \( X \). By [11, (11.8)] the triple product \( L \)-function decomposes as

\[
L(F, s) = L(\text{Sym}^3 f, s)L(f, s - 1)^2.
\]

Note that \( W(F) = a_p(f)^3 = a_p(f) = W(f) = +1 \) by (3.2) and the fact that the sign of the functional equation of \( L(f, s) \) centered at \( s = 1 \) is equal to +1, since \( X \) has Mordell–Weil rank 0. For each \( p \in \{11, 17, 19\} \), we have \( L(F, 2) \neq 0 \) [11, Tables 12.5–12.7]. In other words, \( \text{ord}_{s=2}(L(F, s)) = 0 \). The fact that \( \Delta_{\text{GKS}}(e) \) is torsion in the Chow group is therefore consistent with the Beilinson–Bloch conjecture (3.5).

4.2 The higher genus case

Suppose that \( g_X \geq 2 \). It will be convenient to sometimes view the Atkin–Lehner involution \( w_p \) of Section 2.1 as a correspondence by taking its graph. By slight abuse of notation, we will write \( w_p \in \text{Corr}^0(X, X)(\mathbb{Q}) \). The operator \( w_p \) naturally belongs to the Hecke algebra \( \mathbb{T} \) by (2.5), and commutes with the Hecke operators. The modular forms \( f_j \), with \( j \in \{1, 2, 3\} \), are eigenforms for the operator \( w_p \) with eigenvalues given by \( -a_p(f_j) \) respectively (see Section 2.1).

Consider the involution \( u_p := w_p \times w_p \times w_p \) of \( X^3 \). By taking its graph, it may be viewed as a correspondence, and we write again \( u_p \in \text{Corr}^0(X^3, X^3)(\mathbb{Q}) \), by slight abuse of notation. Note that, as correspondences, we have

\[
u_p = w_p \otimes w_p \otimes w_p := \text{pr}^*_1(w_p) \cdot \text{pr}^*_2(w_p) \cdot \text{pr}^*_3(w_p) \in \text{Corr}^0(X^3, X^3)(\mathbb{Q}).
\]

The map \( u_p \) induces an involution on cohomology via pull-back, hence an involution on the space of cusp forms of weight \((2, 2, 2)\) for \( \Gamma_0(p)^3 \). By (3.2), we see that

\[
u_p^*(F) = -W(F) \cdot F.
\]

Lemma 4.5 We have \( (u_p)_*(\Delta_{\text{GKS}}(e)) = \Delta_{\text{GKS}}(w_p(e)) \), for any \( e \in X(\mathbb{Q}) \).

Proof The induced map \((u_p)_* : \text{CH}^2(X^3) \to \text{CH}^2(X^3)\) on Chow groups simply maps a cycle to its image under \( u_p \). We have \( u_p(\Delta) = \Delta \), since \( u_p \) is an automorphism of \( X^3 \). However, \( u_p(P_T(e)_*(\Delta)) = P_T(w_p(e))_*(\Delta) \) for any proper subset \( T \) of \( \{1, 2, 3\} \).

Proposition 4.6 Let \( f_1, f_2, \) and \( f_3 \) be three normalized eigenforms in \( S_2(\Gamma_0(p)) \), denote by \( F = f_1 \otimes f_2 \otimes f_3 \) their triple product, and suppose that \( F \) satisfies Assumption 4.1. For any point \( e \in X(\mathbb{Q}) \), we have \( \text{AJ}_X^F(\Delta_{\text{GKS}}(e)) = -\text{AJ}_X^F(\Delta_{\text{GKS}}(w_p(e))) \).
Proof By functoriality of Abel–Jacobi maps with respect to correspondences, we have
\begin{equation}
\label{eq:modular_curve}
\text{AJ}_{X^3}((u_p)_*(t_F)_*(\Delta_{\text{GKS}}(e))) = (u_p^*)^\vee \text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(e)).
\end{equation}
For \(i \in \{1, 2, 3\}\), \(w_p\) commutes with \(t_f\) as self-correspondences of \(X\) up to vertical and horizontal divisors, by (2.5) and (2.6). This implies that
\[
(u_p \circ t_F = (w_p \circ t_f) \otimes (w_p \circ t_f') \otimes (u_p \circ t_f) = (t'_f \circ w_p) \otimes (t'_f \circ w_p) \otimes (t'_f \circ w_p) = t'_f \circ u_p,
\]
where \(t'_f = t'_f \otimes t'_f \otimes t'_f\) is possibly another \(F\)-isotypic projector. In particular, using Lemma 4.5, we obtain
\[
(u_p)_*(t_F)_*(\Delta_{\text{GKS}}(e)) = (t'_F)_*(u_p)_*(\Delta_{\text{GKS}}(e)) = (t'_F)_*(\Delta_{\text{GKS}}(w_p(e))).
\]
The left hand side of (4.4) is thus equal to \(\text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(w_p(e)))\) by Remark 4.2.

On the other hand, \(\text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(e))\) lies in \((t'_F)^\vee (J^2(X^3))\) by functoriality of the complex Abel–Jacobi map with respect to correspondences, that is, in the \(F\)-isotypic Hecke component of the intermediate Jacobian. The triple product Hecke algebra \(T^{\otimes 3}\) acts via correspondences on the latter by multiplication by the Hecke eigenvalues of \(F\). For any \(\alpha \in \text{Fil}^2 H_{\text{dR}}^3(X^3_\mathbb{C})\), we have the equality
\[
(u_p^*)^\vee \text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(e))(\alpha) = \text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(e))(u_p^*(t'_F(\alpha))).
\]
The operator \(u_p\) in \(T^{\otimes 3}\) acts via pull-back on the \(F\)-isotypic component \((t'_F)^* H_{\text{dR}}^3(X^3)\) as multiplication by \(-W(F)\) by (4.3). In particular, \(u_p^*(t'_F(\alpha)) = -W(F) t'_F(\alpha)\). By Assumption 4.1, the right hand side of (4.4) is thus \(- \text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(e))\).

Mazur proved, for \(g_X \geq 2\) and \(p \not\in \{37, 43, 67, 163\}\), that \(X(\mathbb{Q}) = \{\xi_\infty, \xi_0\}\), where we recall that \(\xi_\infty\) and \(\xi_0\) denote the two cusps of \(X\) [22, Theorem 1]. Moreover, the modular curve \(X_0(37)\) has two noncuspidal \(\mathbb{Q}\)-rational points, while \(X_0(p)\) has a unique noncuspidal \(\mathbb{Q}\)-rational point, for \(p \in \{43, 67, 163\}\).

Corollary 4.7 Let \(f_1, f_2,\) and \(f_3\) be three normalized eigenforms in \(S_2(\Gamma_0(p))\), denote by \(F = f_1 \otimes f_2 \otimes f_3\) their triple product, and suppose that \(F\) satisfies Assumption 4.1. If \(p\) belongs to \(\{43, 67, 163\}\), and \(e\) denotes the unique noncuspidal \(\mathbb{Q}\)-rational point of \(X\), then \(\text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(e)) = 0\).

Proof The involution \(w_p\) maps \(\mathbb{Q}\)-rational points to \(\mathbb{Q}\)-rational points and permutes the two cusps \(\xi_\infty\) to \(\xi_0\). It therefore fixes the noncuspidal point \(e\), and the result follows from Proposition 4.6.

Corollary 4.8 Let \(f_1, f_2,\) and \(f_3\) be three normalized eigenforms in \(S_2(\Gamma_0(p))\), denote by \(F = f_1 \otimes f_2 \otimes f_3\) their triple product, and suppose that \(F\) satisfies Assumption 4.1. If \(g_X \geq 2\), then \(\text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(\xi_\infty)) = \text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(\xi_0)) = 0\).

Proof Gross and Schoen [12, Proposition 3.6] have constructed a correspondence \(\Xi\) in \(\text{Corr}^1(X, X^3)(\mathbb{Q})\) with the property that the natural transformation induced by
The involution \( S \) permutes the cusps \( \xi \) push-forward
\[
\Xi_* : \text{CH}^4(X) = \text{Pic}(X) \to \text{CH}^2(X^3)
\]
sends the rational equivalence class of a divisor \( \sum m(e)e \) to \( \sum m(e)\Delta_{\text{GKS}}(e) \).
In particular, the cycle \( \Delta_{\text{GKS}}(\xi_\infty) - \Delta_{\text{GKS}}(\xi_0) \) in \( \text{CH}^2(X^3)_0(\mathbb{Q}) \) depends only on the class of the degree zero divisor \( (\xi_\infty) - (\xi_0) \) in \( \text{CH}^4(X)_0(\mathbb{Q}) = \mathcal{I}(\mathbb{Q}) \).

By Manin–Drinfeld [20], the divisor \( \Delta_{\text{GKS}}(\xi_\infty) - \Delta_{\text{GKS}}(\xi_0) \) is torsion in the Jacobian \( J \).
It follows that \( \Delta_{\text{GKS}}(\xi_\infty) - \Delta_{\text{GKS}}(\xi_0) \) is torsion in \( \text{CH}^2(X^3)_0(\mathbb{Q}) \), and in particular \( \text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(\xi_\infty)) - \text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(\xi_0)) = 0 \) in \( \text{J}^2(X^3)_{K_F} \).
The involution \( w_p \) permutes the cusps \( \xi_\infty \) and \( \xi_0 \). By Proposition 4.6, we thus have the equality
\[
\text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(\xi_\infty)) = -\text{AJ}_{X^3}^F(\Delta_{\text{GKS}}(\xi_0)),
\]
and the proof is complete.

4.3 The case \( p = 37 \)

To complete the proof of Theorem 4.3, the only remaining case is the one where \( p = 37 \) and the chosen base point is a noncuspidal \( \mathbb{Q} \)-rational point. The curve \( X_0(37) \) has been extensively studied by Mazur and Swinnerton-Dyer [23, Section 5]. It has genus 2 and is therefore hyperelliptic. Its hyperelliptic involution will be denoted by \( S \).

In particular, for all points \( e \) in \( X_0(37)(\mathbb{Q}) \), we have \( \Delta_{\text{GKS}}(e) = 0 \) in the Griffiths group \( \text{Gr}^2(X_0(37))^3 \) of null-homologous algebraic cycles modulo algebraic equivalence [12, Corollary 4.9]. The involution \( S \) is distinct from the Atkin–Lehner involution \( w_{37} \), as the quotient \( X_0(37)/w_{37} \) has genus 1. Since \( S \) commutes with every automorphism of \( X_0(37) \) [23, p. 27], it commutes in particular with \( w_{37} \), and we can define another involution \( T = S \circ w_{37} = w_{37} \circ S \).

Let \( y_0 = T(\xi_0) \) and \( y_\infty = T(\xi_\infty) \) be the images of the two cusps by \( T \). By [23, Proposition 2], we have
\[
X_0(37)(\mathbb{Q}) = \{ \xi_0, \xi_\infty, y_0, y_\infty \} \quad \text{and} \quad w_{37}(y_0) = y_\infty.
\]

We now complete the proof of Theorem 4.3.

**Corollary 4.9** Let \( f_1, f_2, \) and \( f_3 \) be three normalized eigenforms in \( S_2(\Gamma_0(37)) \), denote by \( F = f_1 \otimes f_2 \otimes f_3 \) their triple product, and suppose that \( F \) satisfies Assumption 4.1. Then
\[
\text{AJ}_{X_0(37)}^F(\Delta_{\text{GKS}}(y_0)) = \text{AJ}_{X_0(37)}^F(\Delta_{\text{GKS}}(y_\infty)) = 0.
\]

**Proof** By (4.6), the Atkin–Lehner involution \( w_{37} \) interchanges \( y_0 \) and \( y_\infty \). By Proposition 4.6, we have \( \text{AJ}_{X_0(37)}^F(\Delta_{\text{GKS}}(y_0)) = -\text{AJ}_{X_0(37)}^F(\Delta_{\text{GKS}}(y_\infty)) \).

The element
\[
2\text{AJ}_{X_0(37)}^F(\Delta_{\text{GKS}}(y_0)) = \text{AJ}_{X_0(37)}^F((t_F)_* (\Delta_{\text{GKS}}(y_0) - \Delta_{\text{GKS}}(y_\infty)))
\]
in \( J^2(X_0(37))_{K_F} \) depends only on the class of \( (y_0) - (y_\infty) \) in \( J_0(37)(\mathbb{Q}) \) by the existence of (4.5). But this class is the image of the class of \( (\xi_0) - (\xi_\infty) \) by the involution of \( J_0(37) \) obtained from \( T \) by push-forward. The latter class is torsion by the Manin–Drinfeld theorem [20].

https://doi.org/10.4153/S000843952200011X Published online by Cambridge University Press
5 Chow–Heegner points

Let $f$ be a normalized eigenform in $S_2(\Gamma_0(p))$ with rational coefficients, and let $E_f$ be the optimal elliptic curve quotient of $J$ associated with $f$ by the Eichler–Shimura construction [27]. Following Section 2.2, denote by $\pi_f : J \to E_f$ the natural quotient map with connected kernel. It is induced by the element

$$\left[ m_f \tau_f \right] \in \text{CH}^1(X^2)(\mathbb{Q})/(\text{pr}_1^* \text{CH}^1(X)(\mathbb{Q}) + \text{pr}_2^* \text{CH}^1(X)(\mathbb{Q})),$$

where $m_f \in \mathbb{N}$ denotes the denominator of $pr_f \in \mathbb{T}$.

**Remark 5.1** To the best of the author’s knowledge, it is unknown whether there are finitely or infinitely many elliptic curves over $\mathbb{Q}$ of prime conductor. It is a result of Setzer [26, Theorem 2] that, given a prime $p$ distinct from 2, 3, and 17, there is an elliptic curve of conductor $p$ over $\mathbb{Q}$ with a rational 2-torsion point if and only if $p = u^2 + 64$ for some rational integer $u$. A conjecture of Hardy and Littlewood [14, Conjecture F] implies that there are infinitely many values of $u$ such that $u^2 + 64$ is prime. Thus, conditional on this conjecture of Hardy and Littlewood, there are infinitely many primes $p$ which occur as the conductor of an elliptic curve over $\mathbb{Q}$. This is explained in detail in the preprint [15].

Let $g$ be an auxiliary normalized eigenform in $S_2(\Gamma_0(p))$. Following the notations of Section 2.2, recall that $\text{pr}_{[g]} \in \mathbb{T}$ denotes the $[g]$-isotypic Hecke projector. Define the $[g]$-isotypic component $\text{End}_Q^0(J)[g] := \text{pr}_{[g]} : \text{End}_Q^0(J)$ and let $\text{CH}^1(X^2)[g]_\mathbb{Q}$ be the group of cycles mapping to $\text{End}_Q^0(J)[g]$ under (2.6) modulo vertical and horizontal divisors. Let $t_{[\mathbb{g}]}$ be an element of $\text{CH}^1(X^2)[g]_\mathbb{Q}$ mapping to $\text{pr}_{[\mathbb{g}]}$.

For any correspondence $Z \in \text{CH}^1(X^2)(\mathbb{Q})$, define

$$\Pi_Z := \text{pr}_{12}^*(Z) \cdot \text{pr}_{34}^*(\Delta) \in \text{CH}^2(X^4)(\mathbb{Q}),$$

where $\Delta \in \text{CH}^1(X^2)(\mathbb{Q})$ is the diagonal cycle. It induces a push-forward map

$$\Pi_{Z,*} : \text{CH}^2(X^3)_0(L) \to \text{CH}^1(X)_0(L) = J(L)$$

for any field extension $L$ of $\mathbb{Q}$. For $e \in X(\widehat{\mathbb{Q}})$, define the point

$$P_Z(e) := \Pi_{Z,*}(\Delta_{\text{GKd}(e)}) \in J(\widehat{\mathbb{Q}}).$$

**Remark 5.2** The association of a point in $J$ to a self-correspondence is well-defined modulo vertical and horizontal divisors [8, Ex. 3.1.7]. Associate to $Z \in \text{CH}^1(X^2)(\mathbb{Q})_\mathbb{Q}$ a point $P_Z(e) := P_{mZ}(e) \otimes 1/m \in J(\widehat{\mathbb{Q}})_\mathbb{Q}$, where $m \in \mathbb{N}$ such that $mZ \in \text{CH}^1(X^2)(\mathbb{Q})$.

By composing correspondences, we can define

$$\Pi_{Z,t_f} := (m_f \tau_f) \circ \Pi_Z = \text{pr}_{12}^*(Z) \cdot \text{pr}_{34}^*(m_f \tau_f) \in \text{Corr}^{-1}(X^3, X)(\mathbb{Q}).$$

This induces, in the terminology of [2], a generalized modular parametrization

$$\Pi^f_Z := \Pi_{Z,t_f,*} = \pi_f \circ \Pi_{Z,*} : \text{CH}^2(X^3)_0(L) \to E_f(L),$$

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independent of the choice of $t_f$. Given $e \in X(\bar{\mathbb{Q}})$, we define the Chow–Heegner point

$$P^f_Z(e) := \Pi^f_Z(\Delta_{\text{GKS}}(e)) = \pi^f_f(e) \in E_f(\bar{\mathbb{Q}}).$$

By Remark 5.2, we can define the Chow–Heegner point associated with $f$ and $[g]$ by

$$P^f_{[g]}(e) := P^f_{[t_\xi]}(e) \in E_f(\bar{\mathbb{Q}}).$$

Concretely, we have

$$P^f_{[g]}(e) = \pi^f_f(\Pi^f_{m_{[g]}}(\xi_\infty)) \otimes \frac{1}{m_{[g]}} \in E_f(\bar{\mathbb{Q}}),$$

where $m_{[g]}$ is the denominator of $\text{pr}_{[g]}$.

Building on the work of Yuan et al. [28], Darmon et al. proved the following in [7]:

**Theorem 5.3** Assume that $g \neq f$, $W(f) = -1$, and $W(\text{Sym}^2 g \otimes f) = +1$. The subspace

$$\langle P^f_{\xi_\infty} : T \in \text{CH}^1(X)[g] \rangle \subset E_f(\mathbb{Q})$$

is nonzero if and only if

$$\text{ord}_{s=1} L(f, s) = 1 \quad \text{and} \quad \text{ord}_{s=2} L(\text{Sym}^2 (g^\sigma) \otimes f, s) = 0, \quad \forall \sigma : K \hookrightarrow \mathbb{C}.$$

**Proof** This is a particular case of [7, Theorem 3.7].

**Remark 5.4** The triple product $L$-function attached to $(g, g, f)$ decomposes as

$$L(g, g, f, s) = L(f, s - 1) L(\text{Sym}^2 g \otimes f, s),$$

and therefore the assumptions of Theorem 5.3 imply in particular that $W(g, g, f) = -1$.

**Remark 5.5** When $g$ equals $f$, $(t^\otimes_3 f)_* (\Delta_{\text{GKS}}(e))$ is the Gross–Kudla–Schoen cycle in $\text{CH}^2(E^3_f)_0(\mathbb{Q})$ based at $\pi^f_f(e)$, which is torsion by [12, Corollary 4.7]. The resulting Chow–Heegner point is then trivial by (5.2), whence the assumption in Theorem 5.3.

In the complementary setting where $W(g, g, f) = +1$, we now prove the following:

**Theorem 5.6** If $E_f$ admits split multiplicative reduction at $p$, then $P^f_{[g]}(e)$ is trivial in $E_f(\mathbb{Q})$, for all $e \in X(\mathbb{Q})$. Equivalently, $m_{[g]}^2 P^f_{[g]}(e)$ is torsion in $E_f(\mathbb{Q})$, for all $e$ in $X(\mathbb{Q})$.

**Proof** Following Section 2.2, we have $t_{[g]} = \sum_{h\in[g]} t_h$, and thus

$$t_{[g]} \otimes t_{[g]} \otimes t_f = \sum_{h_1, h_2 \in [g]} t_{h_1} \otimes t_{h_2} \otimes t_f.$$
By (3.2), for any \( h_1, h_2 \in [g] \), the global root number of the triple product \( L\)-function \( L(h_1, h_2, f, s) \) is given by \( W(h_1, h_2, f) = a_p(h_1)a_p(h_2)a_p(f) \). The \( p \)-th Fourier coefficient of a normalized cuspidal eigenform is the negative of the \( w_p \)-eigenvalue of the form, hence it belongs to \( \{ \pm 1 \} \). In particular, since this coefficient belongs to \( \mathbb{Q} \), it is fixed by the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), and thus \( a_p(g) = a_p(h_1) = a_p(h_2) \in \{ \pm 1 \} \). It follows that \( W(h_1, h_2, f) = a_p(f) = a_p(E_f) \). We have \( a_p(E_f) = 1 \), since \( E_f \) admits split multiplicative reduction at \( p \), and the triple \( (h_1, h_2, f) \) satisfies Assumption 4.1. By Theorem 4.3, for any \( e \in X(\mathbb{Q}) \), \( \text{AJ}_X((t_{h_1} \otimes t_{h_2} \otimes t_f)_*(\Delta_{\text{GKS}}(e))) \) is trivial in the intermediate Jacobian. Thus, \( \text{AJ}_X((t_{[g]} \otimes t_{[g]} \otimes t_f)_*(\Delta_{\text{GKS}}(e))) \) is trivial in \( J^2(X^3_\mathbb{C})_\mathbb{Q} \), or equivalently, \( \text{AJ}_X((m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f)_*(\Delta_{\text{GKS}}(e))) \) is torsion in \( J^2(X^3_\mathbb{C})_\mathbb{Q} \).

Define the cycle \( \Pi := \text{pr}_{12}^*(\Delta) \cdot \text{pr}_{34}^*(\Delta) \in \text{CH}^2(X^4(\mathbb{Q})) \). Viewing \( m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f \) in \( \text{Corr}^0(\mathbb{X}^3, \mathbb{X}^3)(\mathbb{Q}) \) and \( \Pi \) in \( \text{Corr}^{-1}(\mathbb{X}^3, \mathbb{X})(\mathbb{Q}) \), we compute

\[
\Pi \circ (m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f) = \text{pr}_{12}^*(m_{[g]}t_{[g]} \circ m_{[g]}t_{[g]}) \cdot \text{pr}_{34}^*(m_f t_f) = m_{[g]} \Pi m_{[g]}t_{[g]}t_{[g]},
\]

as elements of \( \text{Corr}^{-1}(\mathbb{X}^3, \mathbb{X})(\mathbb{Q}) \), where \( t'_{[g]} \) is possibly another \( [g] \)-projector arising from the fact that \( t_{[g]} \) is an idempotent element of the ring of self-correspondences modulo vertical and horizontal divisors. We deduce the equality of points in \( E_f(\mathbb{Q}) \)

\[
(5.2) \quad \Pi_*(m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f)_*(\Delta_{\text{GKS}}(e)) \otimes 1/m_{[g]}^2 = P^f_{g}(e).
\]

By functoriality of Abel–Jacobi maps with respect to correspondences, the diagram

\[
\begin{array}{ccc}
\text{CH}^2(X^3_\mathbb{C})_0(\mathbb{C}) & \xrightarrow{\text{AJ}_X} & J^2(X^3_\mathbb{C}) \\
\downarrow \Pi_* & & \downarrow (\Pi^*)^\vee \\
E_f(\mathbb{C}) & \xrightarrow{\sim} & J^1(E_f, \mathbb{C})
\end{array}
\]

commutes. Here, \( J^1(E_f, \mathbb{C}) = H^0(E_f(\mathbb{C}), \Omega^1_{E_f})/\text{Im} H_1(E_f(\mathbb{C}), \mathbb{Z}) \) is the Jacobian of \( E_f \), and \( \text{AJ}_{E_f} \) is the classical Abel–Jacobi isomorphism for the elliptic curve \( E_f \) given by

\[
\text{AJ}_{E_f}(P)(\alpha) := \int_O^P \alpha, \quad \text{for all } \alpha \in H^0(E_f(\mathbb{C}), \Omega^1_{E_f}),
\]

where \( O \) is the origin of \( E_f \). By (5.2) and (5.3), we have the equality in \( J^1(E_f, \mathbb{C}) \)

\[
(5.4) \quad \text{AJ}_{E_f}(m_{[g]}^2P^f_{g}(e)) = (\Pi^*)^\vee \text{AJ}_X((m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f)_*(\Delta_{\text{GKS}}(e)))).
\]

The result follows from the facts that \( \text{AJ}_X((m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f)_*(\Delta_{\text{GKS}}(e))) \) is torsion and \( \text{AJ}_{E_f} \) is an isomorphism. 

\[\square\]
Remark 5.7  Theorem 5.6 with \( e = \xi_\infty \) is a special case of [8, Theorem 3.3.8]. In his thesis [8], Daub proved more generally for composite level \( N \) that if the local root number \( W_p(g, g, f) = -1 \) for some \( p | N \), then the resulting Chow–Heegner points based at \( \xi_\infty \) are torsion. His proof relies on an identification of these points with Zhang points [29]. As explained in the introduction (Section 1.4), our method works for composite level \( N \).

6 Example of a nontrivial torsion element

Techniques were developed in [5] to numerically calculate Chow–Heegner points associated with modified diagonal cycles. The algorithms are based on a formula for the image of these cycles under the complex Abel–Jacobi map (4.1) proved in [7]. Most of the examples calculated in [5] concern the situation where the elliptic curve \( E_f \) has algebraic rank equal to 1. In particular, the global root number of \( E_f \) is \(-1\), and this is not the setting studied in the present paper. However, in the appendix of [5] by Stein, some examples are computed for which the rank of \( E_f \) is 0. In particular, we deduce the following:

**Theorem 6.1**  Let \( f \) and \( g \) be the normalized eigenforms of weight 2 and level \( \Gamma_0(37) \) corresponding to the elliptic curves with Cremona labels 37b and 37a, and define \( F := g \otimes g \otimes f \). Then \( AJ_{X_0(37)}^3((2t_g \otimes 2t_g \otimes 2t_f)_* (\Delta_{GKS}(\xi_\infty))) \) is a nontrivial 6-torsion element of \( J^2(X_0(37))^3 \).

**Proof**  In [5, Appendix], it is verified numerically in this case that \( m_g P_g^f(\xi_\infty) \) is a point of order 3 in \( E_f(\mathbb{Q}) \). By inspecting the first few Fourier coefficients of \( f \) and \( g \), we see that \( m_g = m_f = 2 \) (see [5, Section 5.1]). The point \( 4P_g^f(\xi_\infty) \in E_f(\mathbb{Q}) \) has order 3, and by (5.4) \( AJ_{X_0(37)}^3((2t_g \otimes 2t_g \otimes 2t_f)_* (\Delta_{GKS}(\xi_\infty))) \) is thus nontrivial in \( J^2(X_0(37))^3 \).

The element \( 2AJ_{X_0(37)}^3((2t_g \otimes 2t_g \otimes 2t_f)_* (\Delta_{GKS}(\xi_\infty))) \) is equal by Proposition 4.6 to \( AJ_{X_0(37)}^3((2t_g \otimes 2t_g \otimes 2t_f)_* (\Delta_{GKS}(\xi_\infty) - \Delta_{GKS}(\xi_0))) \), and depends only on the class of \((\xi_\infty) - (\xi_0) \) in \( J_0(37)(\mathbb{Q}) \) by existence of (4.5). The latter has order 3 [21, Theorem 1].

Acknowledgment  The author thanks H. Darmon, B. H. Gross, and A. Shnidman for helpful comments, as well as C. Qiu and W. Zhang for answering questions related to their work. The author was supported by the Institut des Sciences Mathématiques at McGill University, and by an Emily Erskine Endowment Fund Postdoctoral Fellowship at the Hebrew University. The author thanks the anonymous referee for their valuable feedback and suggestions.

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https://doi.org/10.4153/50008439522000011X Published online by Cambridge University Press
Torsion properties of modified diagonal classes


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