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THE FIXED POINT SET OF REAL MULTI-VALUED CONTRACTION MAPPINGS

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1. **Introduction.** Let (X, d_1) and (Y, d_2) be metric spaces. A mapping $f:X\to Y$ is said to be a Lipschitz mapping if there exists a real number λ such that

$$d_2(f(x), f(y)) \leq \lambda d_1(x, y)$$

for each $x, y \in X$. We call λ a Lipschitz constant for f. If $\lambda \in [0, 1)$, f is called a contraction mapping. Throughout this note $CB(Y)$ denotes the set of closed and bounded subsets of Y equipped with the Hausdorff metric induced by d_2. Letting R^1 be the set of real numbers and

$$N(\alpha, A) = \{y \in Y : d_2(y, a) < \alpha \text{ for some } a \in A\}$$

for each $\alpha \in R^1$ and each $A \in CB(Y)$, we recall that this metric, say D, is defined as follows:

$$D(A, B) = \inf\{\alpha \in R^1 : A \subset N(\alpha, B) \text{ and } B \subset N(\alpha, A)\}$$

for each $A, B \in CB(Y)$. A mapping $g:X\to CB(Y)$ is called a multi-valued mapping from X to Y, and if $X=Y$, $x \in X$ is called a fixed point of g provided $x \in g(x)$. If f and g are multi-valued mappings from X to Y, it is clear that $f \cup g:X\to CB(Y)$ defined by $f \cup g(x)=f(x) \cup g(x)$ for all $x \in X$ is also a multi-valued mapping from X to Y.

In this note, we show that if X is a connected subset of R^1 and Y is R^1, then a multi-valued Lipschitz mapping f from X to Y, with the property that $f(x)$ has n components for each $x \in X$, can be characterized as the union of n connected-valued Lipschitz mappings. Thence we deduce that if f is a multi-valued contraction mapping from R^1 to R^1, with the property that $f(x)$ has n components for each $x \in R^1$, then its set of fixed points has exactly n components. At the same time we answer in the affirmative the following question of Helga Schirmer [2, p. 170]:

"If the image of the contractive function $\varphi:R^1\to R^1$ consists of exactly n points for all $x \in R^1$, does the fixed point set of φ consist of n points?" (where "contractive function" means multi-valued contraction mapping in our terminology).

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2. **Results.**

DEFINITION. Let (X, d_1) and (Y, d_2) be metric spaces. We say that a mapping $f:X \to Y$ is a local radial Lipschitz mapping with Lipschitz constant λ (≥ 0) if for each $x \in X$ there exists $\varepsilon(x)>0$ such that

$$d_1(x, y) < \varepsilon(x) \Rightarrow d_2(f(x), f(y)) \leq \lambda d_1(x, y)$$

LEMMA 1. *Let X be a connected subset of R^1 and (Y, d) be a metric space. The function $f:X \to CB(Y)$ is a local radial Lipschitz mapping with Lipschitz constant λ if and only if f is a Lipschitz mapping with Lipschitz constant λ.*

Proof. Suppose f is a local radial Lipschitz mapping with Lipschitz constant λ, and let D be the Hausdorff metric for $CB(Y)$ induced by d.

To show that f is a Lipschitz mapping with Lipschitz constant λ, let $x' \leq x''$ be arbitrary elements of X and $I=[x', x'']$. Since X is connected, $I \subset X$. For each $x \in I$, let $\varepsilon(x)$ be such that

$$|x-y| < \varepsilon(x) \Rightarrow d(f(x), f(y)) \leq \lambda |x-y|$$

and then set $O_x=\{y:|x-y|<\varepsilon(x)\}$. Since $\{O_x:x \in I\}$ is an open cover of the compact set I, there is a finite subset of I, say $\{x_1, x_2, \ldots, x_n\}$, such that $\{O_{x_i}:i \in \{1, 2, \ldots, n\}\}$ is a cover of I. We may assume:

$$x_1 < x_2 < \cdots < x_n,$$

$$x' \in O_{x_1}, \qquad x'' \in O_{x_n},$$

and

$$O_{x_1} \cap O_{x_{i+1}} \neq \phi \quad \text{for } i \in \{1, 2, \ldots, n-1\}$$

Now we define $y_0=x'$, $y_{2n}=x''$ and $y_{2i-1}=x_i$ for $i \in \{1, 2, \ldots, n\}$. Finally, for each $i \in \{1, 2, \ldots, n-1\}$, we choose $y_{2i} \in O_{x_i} \cap O_{x_{i+1}}$ satisfying $x_i \leq y_{2i} \leq x_{i+1}$. Then it is clear that we have

$$y_0 \leq y_1 \leq \cdots \leq y_{2n}$$

and

$$D(f(y_i), f(y_{i+1})) \leq \lambda |y_i - y_{i+1}| \quad \text{for each } i \in \{0, 1, \ldots, 2n-1\}.$$

Hence

$$\begin{aligned} D(f(x'), f(x'')) &\leq \sum_{i=0}^{2n-1} D(f(y_i), f(y_{i+1})) \\ &\leq \sum_{i=0}^{2n-1} \lambda |y_i - y_{i+1}| = \lambda \sum_{i=0}^{2n-1} (y_{i+1} - y_i) \\ &= \lambda(y_{2n} - y_0) = \lambda |x' - x''|. \end{aligned}$$

The proof is thus completed since the converse follows directly from the definitions.

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The following lemma, stated previously by S. B. Nadler, Jr. [1, Theorem 3], is an immediate consequence of the definitions:

LEMMA 2. *Let X and Y be metric spaces. If* $f:X\to CB(Y)$ *is a Lipschitz mapping with Lipschitz constant* α *and* $g:X\to CB(Y)$ *is a Lipschitz mapping with Lipschitz constant* β, *then* $f\cup g$ *is a multi-valued Lipschitz mapping with Lipschitz constant* $\max\{\alpha, \beta\}$.

THEOREM. *Let A be a connected subset of* R^1. *Then a necessary and sufficient condition for a multi-valued function* $f:A\to CB(R^1)$ *to be a Lipschitz mapping with Lipschitz constant* $\lambda\geq 0$ *and to be such that* $f(x)$ *has n components for each* $x\in A$ *is that there exist n connected-valued Lipschitz mappings*

$$f_i:A\to CB(R^1), \qquad i\in\{1,2,\ldots,n\},$$

with nonintersecting graphs such that λ *is a Lipschitz constant for each* f_i *and* $f=\bigcup_{i=1}^n f_i$.

Proof. Necessity follows directly from Lemma 2 and the hypothesis that the graphs of the f_i are nonintersecting. To prove sufficiency, let $x\in A$. By hypothesis,

$$f(x) = [a_1(x), b_1(x)]\cup[a_2(x), b_2(x)]\cup\cdots\cup[a_n(x), b_n(x)]$$

where

$$a_1(x)\leq b_1(x) < a_2(x)\leq b_2(x) <\cdots< a_n(x)\leq b_n(x).$$

We set $f_i(x)=[a_i(x), b_i(x)]$ for each $x\in A$ and each $i\in\{1,2,\ldots,n\}$. The proof is finished by showing that $\{f_i:i\in\{1,2,\ldots,n\}\}$ forms the desired decomposition of f.

Clearly $f=\bigcup_{i=1}^n f_i$ and the f_i are connected-valued and have nonintersecting graphs. Now if $\lambda=0$, then f is a constant function and thence, so is each f_i, $i\in\{1,2,\ldots,n\}$, showing that each f_i is a Lipschitz mapping with Lipschitz constant $\lambda=0$. Otherwise, let $\varepsilon(x)=\frac{1}{2}\min\{|a_{i+1}(x)-b_i(x)|:i\in\{1,2,\ldots,n-1\}\}$ for each $x\in A$. By Lemma 1, we are done if we show that for each $i\in\{1,2,\ldots,n\}$, f_i is a local radial Lipschitz mapping with Lipschitz constant λ. Therefore, we complete the proof by showing

$$|x-x_0|<\frac{\varepsilon(x_0)}{\lambda}\Rightarrow D(f_i(x), f_i(x_0))\leq\lambda\,|x-x_0|$$

for each $x_0\in A$ and each $i\in\{1,2,\ldots,n\}$.

To this end, let $x_0\in A$ and let $x\in A$ be such that $|x-x_0|<\varepsilon(x_0)/\lambda$. Since $D(f(x_0), f(x))\leq\lambda\,|x-x_0|$, it follows that $f(x)\subset N(\lambda\,|x-x_0|, f(x_0))$ by the definition of the Hausdorff metric. Also, since $\lambda\,|x-x_0|<\varepsilon(x_0)$, $N(\lambda\,|x-x_0|, f(x_0))$ has the n components $N(\lambda\,|x-x_0|, f_i(x_0))$, $i\in\{1,2,\ldots,n\}$. Hence, for each $i\in\{1,2,\ldots,n\}$ there exists a unique $k_i\in\{1,2,\ldots,n\}$ such that

$$f_i(x)\subset N(\lambda\,|x-x_0|, f_{k_i}(x_0)).$$

Setting $I=\{1, 2, \ldots, n\}$, we may thus define a mapping $k:I\to I$ by letting $k(i)=k_i$ for each $i\in I$.

Now if we can show that k is the identity mapping on I, then the following argument will complete the proof: for each $i\in I$,

$$f_i(x)\subset N(\lambda\,|x-x_0|, f_{k_i}(x_0)) = N(\lambda\,|x-x_0|, f_i(x_0)).$$

On the other hand, since $\lambda\,|x-x_0|<\varepsilon(x_0)$ and $f_i(x)\subset C(\lambda\,|x-x_0|, f_i(x_0))$, it follows that

$$\begin{aligned} N(\lambda\,|x-x_0|, f_i(x)) &\subset N(\lambda\,|x-x_0|, N(\lambda\,|x-x_0|, f_i(x_0))) \\ &\subset N(2\varepsilon(x_0), f_i(x_0)) \end{aligned}$$

and hence

$$N(\lambda\,|x-x_0|, f_i(x))\cap f_j(x_0)=\phi \quad \text{for } i\neq j.$$

However, since $D(f(x), f(x_0))\leq\lambda\,|x-x_0|$ implies $f(x_0)\subset N(\lambda\,|x-x_0|, f(x))$, it now follows that $f_i(x_0)\subset N(\lambda\,|x-x_0|, f_i(x))$ for each $i\in I$. Thus, by the definition of the Hausdorff metric, we have $D(f_i(x), f_i(x_0))\leq\lambda\,|x-x_0|$ for each $i\in I$.

Now in order to show that k is the identity mapping on I, we first show that k is a bijection. For this it is sufficient to show that k is a surjection because I is finite. If k is not surjective, then there exists $\bar{i}\in I$ such that

$$f_i(x)\not\subset N(\lambda\,|x-x_0|, f_{\bar{i}}(x_0))$$

for each $i\in I$. Thus, setting $I'=I\sim\{\bar{i}\}$, we have

$$f(x)\subset\bigcup_{i\in I'} N(\lambda\,|x-x_0|, f_i(x_0)),$$

and hence

$$\begin{aligned} N(\lambda\,|x-x_0|, f(x)) &\subset N\Big(\lambda\,|x-x_0|, \bigcup_{i\in I'} N(\lambda\,|x-x_0|, f_i(x_0))\Big) \\ &\subset \bigcup_{i\in I'} N(2\varepsilon(x_0), f_i(x_0)) \end{aligned}$$

since $\lambda\,|x-x_0|<\varepsilon(x_0)$. However, by the definition of $\varepsilon(x_0)$,

$$f_{\bar{i}}(x_0)\cap\Big(\bigcup_{i\in I'} N(2\varepsilon(x_0), f_i(x_0))\Big)=\phi$$

and thence $f_{\bar{i}}(x_0)\not\subset N(\lambda\,|x-x_0|, f(x))$. Thus $f(x_0)\not\subset N(\lambda\,|x-x_0|, f(x))$ and then by the definition of the Hausdorff metric, $D(f(x), f(x_0))>\lambda\,|x-x_0|$. This contradicts the hypotheses and hence we deduce that k is a bijection.

Now, suppose k is not the identity, and let i be the first integer in I such that $k(i)\neq i$. Let $k(i)=i'$. Since k is a bijection, $i'>i$ and there exists $j\in I$, $j>i$, such that $k(j)=i$. Then we have

$$f_i(x)\subset N(\lambda\,|x-x_0|, f_{i'}(x_0))\subset N(\varepsilon(x_0), f_{i'}(x_0))$$

and similarly

$$f_j(x)\subset N(\varepsilon(x_0), f_i(x_0)).$$

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Thus, $a_i(x) \in N(\varepsilon(x_0), f_{i'}(x_0))$ and $a_j(x) \in N(\varepsilon(x_0), f_i(x_0))$. However, since $i' > i$ and since

$$N(\varepsilon(x_0), f_i(x_0)) \cap N(\varepsilon(x_0), f_{i'}(x_0)) = \phi,$$

it follows from the definition of the f_p, $p \in I$, that for each $x \in N(\varepsilon(x_0), f_i(x_0))$ and each $y \in N(\varepsilon(x_0), f_{i'}(x_0))$, $x < y$. Hence we have $a_j < a_i$ which implies $j < i$ by the definition of the a_p, $p \in I$. This contradiction shows that k is the identity mapping on I and thus completes the proof of the theorem.

Helga Schirmer has shown [2, Theorem 2] that a connected-valued contraction mapping from R^1 to R^1 has a compact connected set of fixed points. From this we deduce the following corollary:

COROLLARY 1. *Let $f: R^1 \to CB(R^1)$ be a contraction mapping such that $f(x)$ has n components for each $x \in R^1$. Then the set of fixed points of f is a compact set with n components.*

Proof. By the theorem, we have $f = \bigcup_{i=1}^n f_i$ where f_i is a connected-valued contraction mapping for each $i \in \{1, 2, \ldots, n\}$, and the graphs of the f_i are nonintersecting. We note that the set of fixed points of f is the union of those of the f_i. Hence, since the graphs of the f_i are nonintersecting, we are done by the preceding remark.

Finally, we answer in the affirmative the question of Helga Schirmer stated in the introduction.

COROLLARY 2. *Let $f: R^1 \to CB(R^1)$ be a contraction mapping such that $f(x)$ consists of exactly n points for each $x \in R^1$. Then the set of fixed points of f consists of exactly n points.*

Proof. By the theorem, $f = \bigcup_{i=1}^n f_i$ where f_i is a connected-valued contraction mapping for each $i \in \{1, 2, \ldots, n\}$, and the graphs of the f_i are nonintersecting. Clearly each f_i must be single-valued, and thus, by Banach's contraction principle, f_i has a unique fixed point, $i \in \{1, 2, \ldots, n\}$. The corollary thus follows since the graphs of the f_i are nonintersecting.

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