Forum of Mathematics, Sigma (2022), **Vol. 10**:e56 1–31
doi:10.1017/fms.2022.52

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Calabi–Yau properties of Postnikov diagrams

Matthew Pressland

School of Mathematics & Statistics, University of Glasgow, University Place, Glasgow G12 8QQ, United Kingdom;
E-mail: Matthew.Pressland@glasgow.ac.uk.

Received: 1 October 2021; **Revised:** 5 May 2022; **Accepted:** 18 June 2022

2020 Mathematics Subject Classification: *Primary* – 16G20; *Secondary* – 13F60, 14M15, 18E10, 18G80

Abstract
We show that the dimer algebra of a connected Postnikov diagram in the disc is bimodule internally 3-Calabi–Yau in the sense of the author's earlier work [43]. As a consequence, we obtain an additive categorification of the cluster algebra associated to the diagram, which (after inverting frozen variables) is isomorphic to the homogeneous coordinate ring of a positroid variety in the Grassmannian by a recent result of Galashin and Lam [18]. We show that our categorification can be realised as a full extension closed subcategory of Jensen–King–Su's Grassmannian cluster category [28], in a way compatible with their bijection between rank 1 modules and Plücker coordinates.

1. Introduction

1.1. Main results

A Postnikov diagram, or alternating strand diagram [41], is a combinatorial object consisting of a collection of strands in the disc. It encodes a great deal of further combinatorial, algebraic and geometric data, mostly connected to questions regarding total positivity in the Grassmannian. From the point of view of this paper, the key piece of data encoded by a Postnikov diagram D is that of an ice quiver with potential (Q_D, F_D, W_D). Here Q_D is a quiver with a distinguished subquiver F_D, which we call frozen, so that the pair (Q_D, F_D) is an ice quiver, and the potential W_D is, loosely speaking, a linear combination of cycles in Q_D (see Definition 2.3 for a precise definition). The quiver Q_D and the vertex set of F_D determine a cluster algebra $\mathscr{A}_D$ with frozen variables; the reader unfamiliar with this construction can find details in, for example, Keller's survey [29]. In this paper, we will explain how to use the triple (Q_D, F_D, W_D) to construct an additive categorification of this cluster algebra, in the case that D is connected.

The construction proceeds via the frozen Jacobian algebra A_D associated to (Q_D, F_D, W_D), also known as the dimer algebra of D. This algebra is a quotient of the complete path algebra of Q_D by the closure of an ideal generated by relations computed using the data of the arrow set of F_D (or, more directly, the complement of this set) and the potential W_D; again, precise definitions are found in Definition 2.3. This algebra has a distinguished idempotent e, given by the sum of vertex idempotents over the vertices of F_D, from which we obtain a *boundary algebra* $B_D = eA_De$. We will show that the category $\mathrm{GP}(B_D)$ of Gorenstein projective B_D-modules is our desired categorification.

Theorem 1 (compare to Theorem 4.3 and Theorem 6.11). *Let D be a connected Postnikov diagram, and let $B_D = eA_De$ be the boundary algebra of its dimer algebra. Then the category* $\mathrm{GP}(B_D)$ *is an additive categorification of the cluster algebra $\mathscr{A}_D$.*

© The Author(s), 2022. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

While we use the term 'additive categorification' for brevity in the introduction, we acknowledge that it does not have a widely accepted formal definition in this context and refer the reader to the more precise statements below, which we hope justify our use of this terminology.

The main step in proving Theorem 1 is to establish a certain Calabi–Yau property for the algebra A_D, relative to the boundary idempotent e, as referred to in the title. This result is likely to be of independent interest and is analogous to Broomhead's theorem [3, Thm. 7.1] concerning Calabi–Yau properties of algebraically consistent dimer models on the torus (see also Davison [13] for higher genus closed surfaces).

Theorem 2 (compare to Theorem 3.7). *Let D be a connected Postnikov diagram with dimer algebra A_D. Then A_D is bimodule internally* 3*-Calabi–Yau with respect to the idempotent e given by summing the vertex idempotents for the vertices of F_D.*

The general definition of a bimodule internally 3-Calabi–Yau algebra, and the role of this property in constructing categorifications, is explained in [43]. We will not give this general definition here; since A_D is presented as a frozen Jacobian algebra, we can work instead with a sufficient condition implying the Calabi–Yau property, which appears in Theorem 3.7.

1.2. Positroid varieties

Postnikov diagrams and their associated cluster algebras are important in studying positroid varieties in the Grassmannian—while this connection has only a minor logical role in the paper, we recall it here to help contextualise and motivate our results. Positroid varieties were introduced by Knutson, Lam and Speyer [32] and are closely related to various stratifications of the Grassmannian [4, 21, 36, 46]; see [32, §5.3] for a discussion of this relationship. Intersecting a positroid variety with the totally nonnegative Grassmannian produces a cell in Postnikov's decomposition of this space, the study of which is one of the main motivations of the work in [41].

In this context, a positroid is a subset $\mathcal{P}$ of the set $\binom{[n]}{k}$ of k-element subsets of $\{1, \dots, n\}$ (see [38, §1.4] for the defining properties). Each positroid is uniquely determined by a *necklace*, a cyclically ordered set of n elements of $\binom{[n]}{k}$ (see [39, Defn. 4.2]). Strictly speaking, it is more compatible with our conventions to use the reverse necklace (see [38, §2.1] and [11, §8]), but we will abuse terminology by referring to the set $\mathcal{I} \subseteq \mathcal{P}$ of elements of this reverse necklace as simply 'the necklace $\mathcal{I}$ of $\mathcal{P}$'. Recall that the Grassmannian Gr_k^n of k-dimensional subspaces of $\mathbb{C}^n$ has as a standard generating set of homogeneous coordinates the Plücker coordinates Δ_I for $I \in \binom{[n]}{k}$.

Definition 1.1. Given a positroid $\mathcal{P} \subseteq \binom{[n]}{k}$, the (closed) *positroid variety* $\Pi(\mathcal{P})$ is the subvariety of the Grassmannian Gr_k^n on which $\Delta_I = 0$ for $I \notin \mathcal{P}$. The *open positroid variety* $\Pi^\circ(\mathcal{P})$ is the subvariety of $\Pi(\mathcal{P})$ on which additionally $\Delta_I \neq 0$ for $I \in \mathcal{I}$. We denote the cones on these varieties, which are defined by the same conditions inside the affine cone $\widehat{\mathrm{Gr}}_k^n$ on the Grassmannian, by $\widehat{\Pi}(\mathcal{P})$ and $\widehat{\Pi}^\circ(\mathcal{P})$, respectively.

A Postnikov diagram D determines a positroid $\mathcal{P}_D$ with necklace $\mathcal{I}_D$ and a maximal weakly separated (or maximal noncrossing) collection [39, Defn. 4.3] $\mathcal{C}_D$ with $\mathcal{I}_D \subseteq \mathcal{C}_D \subseteq \mathcal{P}_D$. In particular, D determines the numbers k and n such that $\mathcal{P}_D$, $\mathcal{I}_D$ and $\mathcal{C}_D$ consist of k-element subsets of $\{1, \dots, n\}$; see Definition 2.1. The elements of $\mathcal{C}_D$ are attached to the alternating regions of D, or equivalently to the vertices of the associated ice quiver Q_D; the precise construction is given in Remark 2.7. While many different Postnikov diagrams can determine the same positroid and necklace, it is rarer for two diagrams to determine the same weakly separated collection—indeed, given a positroid $\mathcal{P}$ with necklace $\mathcal{I}$, Oh–Postnikov–Speyer [39, Thm. 1.5] show that the maximal weakly separated collections $\mathcal{C}$ with $\mathcal{I} \subseteq \mathcal{C} \subseteq \mathcal{P}$ are in bijection with Postnikov diagrams D such that $\mathcal{P}_D = \mathcal{P}$ (equivalently, $\mathcal{I}_D = \mathcal{I}$), up to isotopy and certain moves (see Figure 4).

We may interpret the collection $\mathcal{C}_D$ as a set of functions on $\widehat{\Pi}^\circ(\mathcal{P}_D)$ by restricting the Plücker coordinates Δ_I for $I \in \mathcal{C}_D$ to this subvariety of $\widehat{\mathrm{Gr}}_k^n$. Since these functions are attached to the vertices

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

of Q_D, we thus obtain a map $\mathscr{A}_D \to \mathbb{C}(\widehat{\Pi}^\circ(\mathcal{P}_D))$, from the abstract cluster algebra $\mathscr{A}_D$ associated to D to the ring of rational functions on $\widehat{\Pi}^\circ(\mathcal{P}_D)$, sending each initial cluster variable to the restricted Plücker coordinate corresponding to the same quiver vertex. A long-standing conjecture, formalised by Muller and Speyer [37, Conj. 3.4] and recently verified by Galashin and Lam [18] (see also [48] for the special case of Schubert varieties), is the following.

Theorem 1.2 [18]. *The specialisation map $\mathscr{A}_D \to \mathbb{C}(\widehat{\Pi}^\circ(\mathcal{P}_D))$ above is injective, and its image is precisely the coordinate ring $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$.*

Thus the coordinate ring $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$ has a cluster algebra structure, with initial seed $(Q_D, \mathcal{C}_D)$, isomorphic to the cluster algebra we categorify in Theorem 1. While $\Pi^\circ(\mathcal{P}_D)$ is an affine variety, and so it is slightly unusual to consider its affine cone, it is this cone on which restricted Plücker coordinates are functions. By contrast, the coordinate ring $\mathbb{C}[\Pi^\circ(\mathcal{P}_D)]$ is the degree zero part of $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$ in the $\mathbb{Z}$-grading in which Plücker coordinates have degree 1. Said differently, the grading determines a $\mathbb{C}^\times$-action on $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$ by $\lambda \cdot f = \lambda^d f$ when f has degree d, and $\mathbb{C}[\Pi^\circ(\mathcal{P}_D)]$ is the invariant subring $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]^{\mathbb{C}^\times}$ for this action.

Remark 1.3. For Theorem 1.2, it is necessary to adopt the convention that the frozen variables of $\mathscr{A}_D$ are invertible, since the images of these frozen variables in $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$ are precisely the Plücker coordinates Δ_I for $I \in \mathcal{I}_D$, which are nonzero on $\widehat{\Pi}^\circ(\mathcal{P}_D)$ by definition. This convention is not relevant to Theorem 1, however, so we will freely compare our results with those of authors who do not invert frozen variables, such as [28].

As already observed, the positroid $\mathcal{P}_D$ and its corresponding variety $\Pi^\circ(\mathcal{P}_D)$ do not determine D uniquely; choosing a different Postnikov diagram with positroid $\mathcal{P}_D$ amounts to choosing a different initial seed in the cluster algebra structure on $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$. This is reflected in our categorical picture by the fact (Corollary 4.7) that if $\mathcal{P}_D = \mathcal{P}_{D'}$ for two connected Postnikov diagrams D and D', then the two algebras B_D and B'_D are isomorphic, and hence the categories $\mathrm{GP}(B_D)$ and $\mathrm{GP}(B_{D'})$ from Theorem 1 are equivalent.

The best understood case of these constructions is that in which $\mathcal{P}_D = \binom{[n]}{k}$ is the *uniform positroid*, in which the notion of a maximal weakly separated collection from [39] coincides with that of Leclerc–Zelevinsky [34]. In this case, we have $\Pi(\mathcal{P}_D) = \mathrm{Gr}_k^n$, the open subvariety $\Pi^\circ(\mathcal{P}_D)$ is determined by the nonvanishing of those Plücker coordinates labelled by cyclic intervals, and Muller–Speyer's conjecture was verified much earlier by Scott [47]. (The case $k = 2$ even appears in Fomin–Zelevinsky's first paper on cluster algebras [16].) Scott in fact proves more, namely that a completely analogous specialisation map to that of Theorem 1.2 is an isomorphism from the cluster algebra with noninvertible frozen variables determined by Q_D to the homogeneous coordinate ring of Gr_k^n. However, this depends on special properties of the uniform positroid, and there is no corresponding result for arbitrary Postnikov diagrams.

1.3. The Grassmannian cluster category

When $\mathcal{P}_D$ is the uniform positroid, a categorification of $\mathscr{A}_D$ is provided by Jensen–King–Su [28] (see also Demonet–Luo [14] for $k = 2$). This *Grassmannian cluster category*, denoted $\mathrm{CM}(C)$, consists of Cohen–Macaulay modules over a certain algebra C (depending on k and n), defined directly via a quiver with relations. Some of the results in [28] rely on the categorification by Geiß–Leclerc–Schröer [19] of a cluster algebra obtained from $\mathscr{A}_D$ by specialising a single frozen variable to 1 or, equivalently, removing a frozen vertex and all adjacent arrows from the quiver.

Jensen–King–Su's categorification is connected back to the combinatorics of Postnikov diagrams by work of Baur–King–Marsh [2], which is our main inspiration. They show [2, Cor. 10.4] that for any Postnikov diagram D for which $\mathcal{P}_D$ is the uniform positroid, there is an isomorphism $B_D \cong C$ between the boundary algebra of D, with which we describe our categorification, and the algebra C describing

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

the Grassmannian cluster category. Since C has the property that $\mathrm{CM}(C) = \mathrm{GP}(C)$, our categorification coincides with Jensen–King–Su's in this case.

In our more general situation, because B_D is defined as an idempotent subalgebra of the dimer algebra A_D, we do not typically know a presentation of B_D via a quiver with relations. Our approach therefore differs significantly from that of Jensen–King–Su, since we must deduce properties of the category $\mathrm{GP}(B_D)$ from the combinatorics of the diagram D and the representation theory of A_D, for which we do have such a presentation. This is done primarily via the internal Calabi–Yau property of Theorem 3.7, which, in the full strength established here, is also a new result in the case that $\mathcal{P}_D$ is the uniform positroid.

For more general Postnikov diagrams D, a categorification analogous to that of Geiß–Leclerc–Schröer's in the uniform case is provided by Leclerc [33] in work on Richardson varieties; again, he categorifies a cluster algebra obtained from $\mathscr{A}_D$ by specialising a frozen variable to 1. Leclerc's category plays a crucial role in the construction of a cluster structure on $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$ in [18, 48]. In this paper, we will always consider the full quiver Q_D and its associated cluster algebra—in particular, this means that in the case of the uniform positroid, Theorem 1 recovers Jensen–King–Su's category [28], not Geiß–Leclerc–Schröer's [19].

Our category $\mathrm{GP}(B_D)$ is closely connected to the coordinate ring $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$ (and its cluster algebra structure). To demonstrate this, we exhibit a cluster character on $\mathrm{GP}(B_D)$, in the sense of Fu–Keller [17], providing a bijection between reachable rigid indecomposable objects of $\mathrm{GP}(B_D)$ and cluster variables of $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$, and a bijection between reachable cluster-tilting objects of $\mathrm{GP}(B_D)$ and clusters of $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$. We also recall results from joint work with Çanakçı and King [11] demonstrating even stronger combinatorial connections between the representation theory of B_D and the geometry of $\widehat{\Pi}^\circ(\mathcal{P}_D)$. Precisely, there is a fully faithful functor $\rho\colon \mathrm{GP}(B_D) \to \mathrm{CM}(C)$ from the categorification described here to the Grassmannian cluster category (for the appropriate Grassmannian). For each $I \in \binom{[n]}{k}$, there is a combinatorially defined C-module M_I corresponding to the Plücker coordinate Δ_I, and we show (Theorem 7.6) that if $\rho(M) = M_I$, then the cluster character on $\mathrm{GP}(B_D)$ takes M to the restriction of Δ_I to $\widehat{\Pi}^\circ(\mathcal{P}_D)$.

Remark 1.4. Leclerc's categories [33] can be used to categorify a cluster algebra structure on $\mathbb{C}[\Pi^\circ(\mathcal{P}_D)]$—that is, on the coordinate ring of the open positroid variety itself rather than its affine cone—but in a way that depends on choosing a frozen variable Δ_I with $I \in \mathcal{I}_D$. Indeed, the construction involves an isomorphism of $\Pi^\circ(\mathcal{P}_D)$ with the subvariety of $\widehat{\Pi}^\circ(\mathcal{P}_D)$ on which $\Delta_I = 1$; this is the graph of a global section of the line bundle $\widehat{\Pi}^\circ(\mathcal{P}_D) \to \Pi^\circ(\mathcal{P}_D)$. In ring-theoretic terms, this isomorphism corresponds to splitting the inclusion

$$\mathbb{C}[\Pi^\circ(\mathcal{P}_D)] = \mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]^{\mathbb{C}^\times} \subseteq \mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$$

using the map taking $f \in \mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$ of degree d to the $\mathbb{C}^\times$-invariant function $f\Delta_I^{-d}$, recalling that Δ_I is invertible on $\Pi^\circ(\mathcal{P}_D)$.

By avoiding this arbitrary choice, our categorification better reflects the symmetries of $\Pi^\circ(\mathcal{P}_D)$. For example, in the case of the uniform positroid, both $\Pi^\circ(\mathcal{P}_D)$ and our category (which coincides with Jensen–King–Su's [28] in this case) carry an action of the order n cyclic group coming from natural action of this group on the cyclically ordered set $[n]$, and by extension on the set $\binom{[n]}{k}$ labelling Plücker coordinates. Since this action does not fix the frozen variables, but rather cyclically permutes them, choosing a frozen vertex breaks the symmetry, which indeed is no longer present in Geiß–Leclerc–Schröer's category [19].

In the case of the uniform positroid, the distinction between Jensen–King–Su's category and Geiß–Leclerc–Schröer's is even more critical, since in this case the homogeneous coordinate ring $\mathbb{C}[\widehat{\Pi}(\mathcal{P}_D)] = \mathbb{C}[\widehat{\mathrm{Gr}}_k^n]$ of the closed positroid variety carries its own cluster algebra structure, in which the n frozen variables are noninvertible. For more general positroids $\mathcal{P}$, the ring $\mathbb{C}[\widehat{\Pi}(\mathcal{P})]$ is not a cluster algebra, but we still hope that the avoidance of symmetry-breaking in our constructions will allow us to use

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

the algebra B_D constructed here, and its category of Cohen–Macaulay modules, to study the closed varieties $\Pi(\mathcal{P})$ and $\widehat{\Pi}(\mathcal{P})$—see [11, Prop. 8.6] for a first step in this direction.

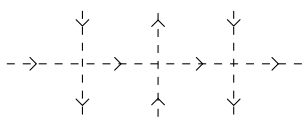
The structure of the paper is as follows. In Section 2, we recall the definition of a Postnikov diagram and the construction of its dimer algebra. The necessary Calabi–Yau property of this algebra is established in Section 3 and used to construct the categorification $\mathrm{GP}(B_D)$ in Section 4. In Section 5, we show that $\mathrm{GP}(B_D)$ has a mutation class of cluster-tilting objects, again determined by D, whose quivers have no loops or 2-cycles. This allows us to show in Section 6 that there is a cluster character on $\mathrm{GP}(B_D)$ inducing bijections between the cluster-tilting objects in this class and the clusters of $\mathscr{A}_D$, and between their indecomposable summands and the cluster (and frozen) variables of $\mathscr{A}_D$. We close in Section 7 by explaining how our categorification is related to Jensen–King–Su's Grassmannian cluster category [28] and how its cluster character relates to Galashin and Lam's isomorphism of $\mathscr{A}_D$ with the coordinate ring $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$.

2. Postnikov diagrams and their dimer algebras

In this section, we introduce Postnikov diagrams in the disc and their associated dimer algebras, which will be defined over a field $\mathbb{K}$ of characteristic zero.

Definition 2.1 [41, §14] (see also [47, §3] and [2, §2]). Let Σ be an oriented disc, with $n \geq 3$ marked points on its boundary. A *Postnikov diagram* D in Σ consists of n oriented strands in Σ, starting and ending at the boundary marked points, subject to the following constraints:

(P0) Each boundary marked point is the start point of exactly one strand and the end point of exactly one strand.
(P1) The strands intersect in finitely many points, and each such crossing is transverse and involves only two strands.
(P2) Moving along a strand, the signs of its crossings with other strands alternate.



(P3) A strand may not intersect itself in the interior of the disc.
(P4) If two strands intersect twice, the strands are oriented in opposite directions between these intersection points. In other words, the configuration

is not permitted, but the configuration obtained from this by reversing the orientation of one of the strands is legal.

For axioms (P2) and (P4), the intersection of two strands at the boundary, with its natural sign, is also treated as a crossing.

The diagram D divides Σ into *regions*, the connected components of the complement of the strands, which have three types: the orientations of the strands around the boundary of such a region can be inconsistent (in which case we call the region *alternating*), oriented anticlockwise (i.e. consistent with the orientation of Σ) or oriented clockwise. We will use the terminology *anticlockwise* and *clockwise* for regions with these boundary orientations. The alternating regions are further subdivided into *boundary regions*, those incident with the boundary of Σ, and *internal regions*.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

We say that D is *connected* if the union of its strands is connected. This forces each boundary region to meet the boundary of Σ in a single arc, so there are exactly n such regions.

There is an integer $k \in \{1, \dots, n\}$ such that each alternating region of D is on the left of exactly k strands. We say that the *type* of D is (k, n), recalling that n is the number of strands. One can also compute k as the average number of boundary regions to the left of a strand. If D has the property that every strand has exactly k boundary regions on its left, then each strand must terminate at a marked point k steps clockwise from its source, and in this case D is sometimes [2] called a (k, n)-*diagram*. Diagrams of this kind are always connected.

Remark 2.2. It is possible to consider Postnikov diagrams on $n < 3$ strands, but they do not lead to interesting cluster algebras. For example, their associated ice quivers (see Definition 2.4 below) have only frozen vertices and so describe a cluster algebra with only frozen variables: that is, a polynomial ring. Thus the assumption that $n \geq 3$, which will be useful in Section 5, only excludes a small number of highly degenerate examples. Both Theorems 1 and 2 do hold when $n < 3$, but they reduce to the claim that B_D is an algebra of global dimension at most 3 with n indecomposable projective modules up to isomorphism, which follows immediately from observing that B_D is the complete path algebra of an n-cycle in this case (or from the results of Section 3, which do not use the assumption $n \geq 3$).

A Postnikov diagram in the disc has an associated algebra called its *dimer algebra*, which, when D is connected, is an instance of a frozen Jacobian algebra defined via an ice quiver with potential. We first give the necessary definitions for this general construction and then explain how to read off the relevant data from a Postnikov diagram.

Definition 2.3 (compare to [44, §2]). An *ice quiver* is a pair (Q, F), where Q is a quiver and F is a (not necessarily full) subquiver of Q. We call F_0, F_1 and F the *frozen* vertices, arrows and subquiver, respectively. Vertices in $Q_0^{\mathrm{m}} := Q_0 \setminus F_0$ will be called *mutable*, whereas arrows in $Q_1^{\mathrm{m}} := Q_1 \setminus F_1$ will be simply called *unfrozen*. We assume for simplicity that Q has no loops.

Let $\mathbb{K}\langle\langle Q\rangle\rangle$ denote the complete path algebra of Q (see [44, Defn. 2.6]). This algebra is graded by path length, which makes its quotient $\mathbb{K}\langle\langle Q\rangle\rangle_{\mathrm{cyc}}$ by the vector subspace spanned by commutators into a graded vector space. A *potential* on Q is an element $W \in \mathbb{K}\langle\langle Q\rangle\rangle_{\mathrm{cyc}}$ expressible as a linear combination of homogeneous elements of degree at least 2. We usually think of W as a linear combination of cyclic paths in Q (of length at least 2), considered up to cyclic equivalence: that is, the finest equivalence relation on such paths such that

$$a_n a_{n-1} \cdots a_1 \sim a_{n-1} \cdots a_1 a_n.$$

We call the triple (Q, F, W) an *ice quiver with potential*.

Let $p = a_n \cdots a_1$ be a cyclic path, with each $a_i \in Q_1$, and let $a \in Q_1$ be any arrow. Then the *cyclic derivative* of p with respect to a is

$$\partial_a p := \sum_{a_i = a} a_{i-1} \cdots a_1 a_n \cdots a_{i+1}.$$

Extending ∂_a by linearity and continuity, we obtain a map $\mathbb{K}\langle\langle Q\rangle\rangle_{\mathrm{cyc}} \to \mathbb{K}\langle\langle Q\rangle\rangle$. For an ice quiver with potential (Q, F, W), we define the *frozen Jacobian algebra*

$$\mathcal{J}(Q, F, W) = \mathbb{K}\langle\langle Q\rangle\rangle / \overline{\langle \partial_a W : a \in Q_1 \setminus F_1 \rangle}.$$

Here the closure is taken in the J-adic topology on $\mathbb{K}\langle\langle Q\rangle\rangle$, where J is the ideal generated by arrows.

Definition 2.4. Let D be a Postnikov diagram in the disc. We associate to D an ice quiver (Q_D, F_D) as follows. The vertices of Q_D are the alternating regions of D, as in Definition 2.1. When the closures of two different alternating regions v_1 and v_2 meet in a crossing point between strands of D, or at one of the boundary marked points, we draw an arrow between v_1 and v_2, oriented in a way consistent with these strands, as shown in Figure 1.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

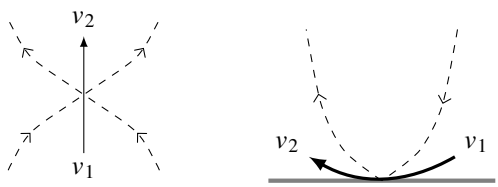


Figure 1. *Arrows associated to strand crossings.*

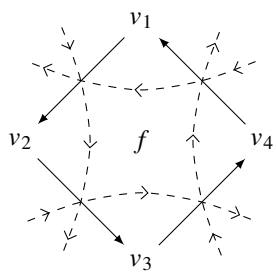


Figure 2. *A fundamental cycle around an oriented region.*

The vertices of F_D are the boundary regions, and the arrows of F_D are those corresponding to boundary marked points, as on the right in Figure 1.

Each clockwise or anticlockwise region f of D determines a *fundamental cycle* C_f in Q, by following the arrows corresponding to the crossing points in the boundary of the region, as shown in Figure 2 for a quadrilateral anticlockwise region.

Thus we may define a potential

$$W_D = \sum_{f \text{ anticlockwise}} C_f - \sum_{f \text{ clockwise}} C_f$$

on Q_D, and we call $A_D := \mathcal{J}(Q_D, F_D, W_D)$ the *dimer algebra* of D.

Remark 2.5. While it is straightforward to describe the defining relations of A_D directly without using a potential (see, e.g., [11, Def. 2.11]), results on frozen Jacobian algebras will play a crucial role for us in Section 3, making it important that A_D has this structure.

Proposition 2.6 (compare to [2, Rem. 3.3]). *The quiver Q_D is strongly connected: that is, any two vertices are connected by a directed path.*

Proof. Since the regions of D cover the disc, the underlying graph of Q_D is connected. Moreover, every arrow a of Q_D is contained in at least one fundamental cycle—exactly one if a is frozen, and exactly two otherwise. Thus every arrow has a path from its head to its tail, so Q_D is even strongly connected. □

Remark 2.7. Having now defined Postnikov diagrams and their associated quivers and algebras more precisely, we may provide a few more details on the relationship to positroid varieties discussed in Section 1.2.

Label the boundary marked points of Σ by $\{1, \ldots, n\}$, in the clockwise direction. These labels can be transferred first to the strands, by giving a strand the label of its source, and then to the alternating regions, by applying the label of a strand to every alternating region on its left. By the definition of k, this process labels each alternating region v of D by a *k-label* $I_v \in \binom{[n]}{k}$, and it turns out [39, Thm. 6.6] that the collection $\mathcal{C}_D$ of these labels is a maximal weakly separated collection in the positroid $\mathcal{P}_D$. We note here that $\mathcal{P}_D = \binom{[n]}{k}$ (i.e. $\mathcal{P}_D$ is the uniform positroid) if and only if D is a (k, n)-diagram.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

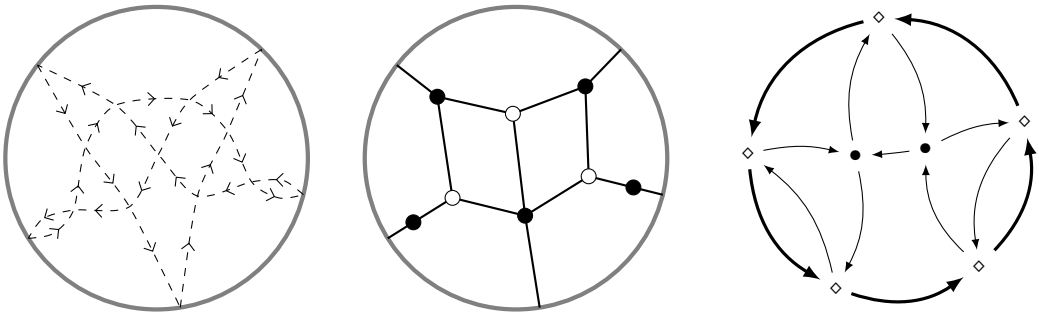


Figure 3. *A Postnikov diagram (left), its corresponding dimer model (centre) and the ice quiver of its dimer algebra (right). The frozen vertices of this ice quiver are shown as white diamonds, and the frozen arrows are drawn in bold.*

The cluster algebra $\mathscr{A}_D$ has initial cluster variables x_v indexed by the vertices of Q_D. By construction, these vertices are the alternating regions of D, so each v has an attached k-label $I_v \in \binom{[n]}{k}$ and Plücker coordinate Δ_{I_v}. The assignment $x_v \mapsto \Delta_{I_v}$ induces Galashin and Lam's isomorphism $\mathscr{A}_D \to \mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$ [18]. (Strictly speaking, Galashin and Lam use the complementary set of $(n-k)$-labels obtained by labelling regions to the right of each strand, so they put a cluster structure on the coordinate ring of an isomorphic positroid variety in Gr^n_{n-k}. Our isomorphism is obtained from theirs using the natural isomorphism $\mathbb{C}[\mathrm{Gr}^n_{n-k}] \xrightarrow{\sim} \mathbb{C}[\mathrm{Gr}^n_k]$ with $\Delta_I \mapsto \Delta_{I^c}$.)

We note that we could have obtained a different set of k-labels, and hence a different specialisation map $\mathscr{A}_D \to \mathbb{C}(\widehat{\Pi}^\circ(\mathcal{P}_D))$, by instead giving the strands of D the labels of their targets. This can produce a different cluster algebra structure (in the sense that different functions are cluster variables) on $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$, isomorphic to the same abstract cluster algebra $\mathscr{A}_D$. While this paper is mostly concerned with categorifying $\mathscr{A}_D$, and so is insensitive to such labelling conventions, they will reappear in Section 7 when we use results of the author with Çanakçı and King [11] to relate our categories more directly with the coordinate rings $\mathbb{C}[\widehat{\Pi}^\circ(\mathcal{P}_D)]$. In particular, it will turn out to be the source labelling convention that is most compatible with our categorification and its relationship to Jensen–King–Su's Grassmannian cluster category $\mathrm{CM}(C)$ [28].

Remark 2.8. Given a Postnikov diagram D in the disc Σ, we can construct from D a bipartite graph G as follows. The vertices of G are given by the anticlockwise and clockwise regions of D, and two such vertices are connected by an edge if the the closures of the regions meet at a crossing of strands in the interior of Σ. The graph G is connected if and only if D is connected. Adding to G the data of half-edges, which connect the vertex corresponding to a region to any boundary marked points in its closure, we obtain a dimer model in Σ in the sense of [44, Ex. 2.5]. This construction appears in [41, §14], where the dimer model is called a plabic graph. The dimer algebra A_D is precisely the dimer algebra of (Σ, G), as defined in [44, Ex. 2.12]. An example of a Postnikov diagram, its corresponding dimer model and the ice quiver of its dimer algebra is shown in Figure 3.

Given the data of a dimer model G in the disc, or indeed in any oriented surface Σ with or without boundary, it is always possible to produce a collection of strands, dividing Σ into alternating, clockwise and anticlockwise regions, from which G may be recovered by the above rules. These strands are usually called the *zig-zag paths* or flows of G [3, §3.3]. Note that in general these strands can either travel between points on the boundary of Σ, as in Definition 2.1, or be closed curves in the interior. In the case that Σ is a disc, these strands will always satisfy conditions (P0)–(P2) from Definition 2.1. Requiring that they also satisfy (P3) and (P4)—which in particular rules out any closed curves in the interior—places an extra condition on the dimer model G, the analogue of Broomhead's geometric consistency [3, Prop. 3.12]. Since these conditions, particularly (P4), will be important for us, we prefer to start from the data of a Postnikov diagram.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

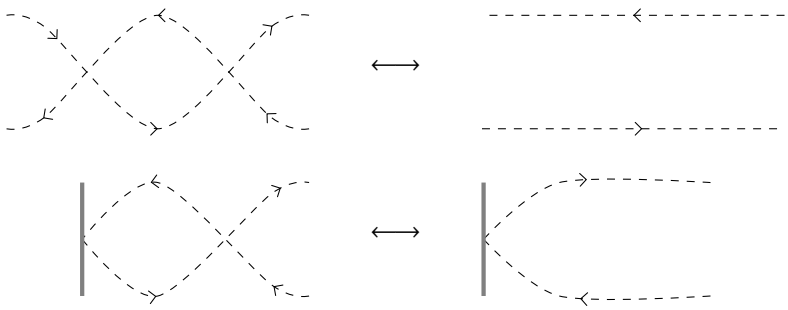


Figure 4. *Twisting (right-to-left) and untwisting (left-to-right) moves for a Postnikov diagram, in the interior (above) and at the boundary (below). The reflections of these figures in a horizontal line also show twisting and untwisting moves.*

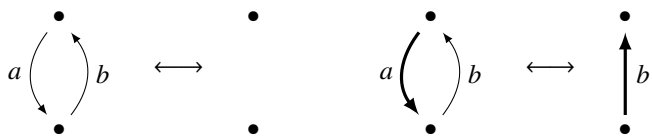


Figure 5. *The effect on the quiver of applying a twisting or untwisting move in the interior (left) or at the boundary (right). Bold arrows are frozen.*

Postnikov diagrams are typically considered up to isotopy fixing the boundary, which does not affect the construction of the dimer algebra at all, and certain twisting and untwisting moves shown in Figure 4. These moves do affect the construction of the dimer algebra, in the sense that two Postnikov diagrams related by such a move determine different ice quivers with potential, as shown in Figure 5 (compare to [44, Ex. 4.5]). However, the frozen Jacobian algebras of these two quivers with potential—that is, the dimer algebras of the two Postnikov diagrams—are isomorphic in a natural way. These moves also do not affect the k-labels attached to the quiver vertices.

Proposition 2.9 (compare to [2, Lem. 12.1]). *Let D and D' be Postnikov diagrams such that D' is obtained from D via an untwisting move. Then the dimer algebras A_D and $A_{D'}$ are isomorphic.*

Proof. First we consider the case that the untwisting move happens in the interior, as in the upper part of Figure 4. Then $Q_{D'}$ is obtained from Q_D by removing two arrows a and b, which form a fundamental cycle, as in the left-hand side of Figure 5. Since these arrows are unfrozen, they are each contained in a second fundamental cycle, and we write these cycles as ap and bq for some paths p and q, respectively. Since these paths do not contain the arrows a or b, they are also paths in $Q_{D'}$. Moreover, the relation $\partial_a W_D = 0$ implies that $b = p$ in A_D, and the relation $\partial_b W_D = 0$ that $a = q$ in A_D. Define a map $\varphi\colon \mathbb{K}\langle\langle Q_D\rangle\rangle \to \mathbb{K}\langle\langle Q_{D'}\rangle\rangle$, fixing all vertex idempotents (the two quivers having the same vertex set) and all arrows different from a and b, and with $\varphi(a) = q$ and $\varphi(b) = p$. This map induces a homomorphism $A_D \to A_{D'}$ inverse to that induced by the map $\mathbb{K}\langle\langle Q_{D'}\rangle\rangle \to \mathbb{K}\langle\langle Q_D\rangle\rangle$ arising from the inclusion map between arrow sets.

On the other hand, if the untwisting move takes place at the boundary, then $Q_{D'}$ is obtained from Q_D by removing a frozen arrow a lying in a fundamental 2-cycle, and $F_{D'}$ is obtained from F_D by replacing a by the other arrow b in this 2-cycle. As before, since b is unfrozen it lies in a second fundamental cycle bq for some path q not involving a, and it follows from the relation $\partial_b W_D = 0$ that $a = q$ in A_D. Thus defining φ as in the first case except for $\varphi(b) = b$ provides an isomorphism of dimer algebras in an analogous way. □

We call a Postnikov diagram *reduced* if no untwisting moves can be applied to it; any Postnikov diagram D is equivalent under untwisting moves to a reduced diagram D', unique up to isotopy fixing

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

the boundary. The reader is warned that this does not correspond to Postnikov's definition of reducedness for plabic graphs [41, §12] but rather is compatible with terminology for quivers with potential: the ice quiver with potential $(Q_{D'}, F_{D'}, W_{D'})$ is the reduction of (Q_D, F_D, W_D), in the sense of [44, §3]. Indeed, the above isomorphisms of dimer algebras in the proof of Proposition 2.9 are local versions of the reduction isomorphism given at the end of the proof of [44, Thm. 3.6].

Proposition 2.10. *If D is reduced, the quiver Q_D has no 2-cycles.*

Proof. A 2-cycle in Q_D containing at least one unfrozen arrow corresponds to a possible untwisting move (see Figures 4 and 5) and so does not occur when D is reduced. A 2-cycle consisting only of frozen arrows would correspond to a pair of boundary regions meeting at two points on the boundary of the disc, which must then be the only boundary regions. But D has at least 3 boundary regions, so this is also impossible. □

Before moving on to the homological part of the paper, we discuss one additional feature of the dimer algebra A_D of a connected Postnikov diagram D. For each vertex v of Q_D, choose a path $t_v : v \to v$ representing a fundamental cycle. Writing $t = \sum_{v \in Q_0} t_v \in A_D$, it follows from connectedness of D that t does not depend on the choices of paths t_v. Moreover, t is a central element of the algebra. Thus A_D has the structure of an algebra over $Z = \mathbb{K}[[t]]$; the abuse of notation in identifying the abstract generator of Z with the element $t \in A_D$, thus making the Z-action clear, is justified by the following result, which is also fundamental to the arguments in Section 3. The proof is close to that of Baur–King–Marsh [2] for the case that D is a (k, n)-diagram, and we refer to their paper when the arguments apply without change to our more general setting.

Proposition 2.11. *The dimer algebra A_D of a connected Postnikov diagram D is thin, meaning that for any vertices v and w, there is some $p \in e_w A_D e_v$ such that*

$$e_w A_D e_v = Zp,$$

and moreover that Zp is a free Z-module of rank 1. Moreover, there is a unique such p expressible as the image of a path from v to w under the projection $\mathbb{K}\langle\langle Q_D \rangle\rangle \to A_D$.

Proof. As in [2, §4], we may weight the arrows of $Q = Q_D$ by elements of the set $\{1, \dots, n\}$, labelling the boundary marked points of Σ. An arrow a of Q is crossed by two strands of D, say that starting at marked point i from right to left and that starting at j from left to right. We then weight a by the (indicator function of) the cyclic interval $[i, j-1]$ (compare to [2, Defn. 4.1]). A path in Q is weighted by the sum wt_p of weights of its arrows, and its total weight is defined to be $\sum_{i=1}^{n} \mathrm{wt}_p(i)$, which is always at least 1 if the path has nonzero length.

The proof of [2, Cor. 4.4], stated for (k, n)-Postnikov diagrams, remains valid in our more general setting to show that every fundamental cycle has constant weight $w(i) = 1$ for all $1 \le i \le n$. If $p_+ = p_-$ is one of the defining relations of A_D, then there is an arrow $a \in Q_1$ such that both ap_+ and ap_- are fundamental cycles, from which it follows that the weights of p_+ and p_- agree. It follows that the weight, and hence the total weight, descends to a grading of A.

Now let $v, w \in Q_0$. Since Q is strongly connected as in Proposition 2.6, there is some path from v to w in Q, and we choose p to be such a path with minimal total weight. If q is any other path from v to w, then [2, Prop. 9.3] applies to show that there is a path $r : v \to w$ and nonnegative integers N_p and N_q such that

$$p = t^{N_p} r, \qquad q = t^{N_q} r$$

in A_D. As before, this proposition is stated only in the case that D is a (k, n)-Postnikov diagram, but its proof is still valid under our weaker assumptions—the key property of D here is (P4).

Since the total weight of t is nonzero, and p has minimal total weight among paths from v to w, we must have $N_p = 0$ and $p = r$ in A_D. Thus $q = t^{N_q} p$, and we see that $e_w A e_v$ is generated over Z by p.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

It is moreover freely generated since each element of $\{t^N p : N \geq 0\}$ has a different total weight, which implies that these elements are linearly independent in A_D. □

Definition 2.12. We call a path $p\colon v \to w$ in Q_D a *minimal path* if $e_w A_D e_v = Zp$. Proposition 2.11 then states that there is a minimal path between any pair of vertices in Q_D and that any two such paths define the same element of A_D. We write p_{wv} for the class in A_D of any minimal path from v to w.

Thinness of A_D, as in Proposition 2.11, is the analogue of algebraic consistency for dimer models on the torus, as described by Broomhead [3, Def. 5.12] (the analogue of geometric consistency, compare to [3, Prop. 3.12], being conditions (P3) and (P4) in the definition of a Postnikov diagram). The following observation about minimal paths is straightforward but useful.

Lemma 2.13. *Let D be a connected Postnikov diagram, and let $p\colon u \to v$ and $q\colon v \to w$ be paths in Q_D. If the composition qp is a minimal path, then both p and q are minimal paths.*

Proof. By Proposition 2.11, there are $m, n \geq 0$ such that $p = t^m p_{vu}$ and $q = t^n p_{wv}$ in A_D. Thus $qp = t^{m+n} p_{wv} p_{vu} = t^{m+n+\delta} p_{wu}$ for some $\delta \geq 0$. But qp is minimal, so by Proposition 2.11 again we must have $m + n + \delta = 0$, and so in particular $m = n = 0$. Hence both p and q are minimal paths. □

3. The Calabi–Yau property

Let $A = A_D$ be the dimer algebra of a connected Postnikov diagram D. In this section, we show that this algebra is bimodule internally 3-Calabi–Yau, in the sense of [43], with respect to the idempotent corresponding to the boundary regions of D, thus proving Theorem 2. Since A is presented as a frozen Jacobian algebra, we will prove this statement via [43, Thm. 5.6], which tells us that it is enough to check the exactness of a certain complex of A-bimodules.

Throughout the proof, we will write

$$\mathrm{H}_v = \{a \in Q_1 : ha = v\}, \qquad \mathrm{T}_v = \{a \in Q_1 : ta = v\}$$

for the set of arrows with head, respectively tail, at $v \in Q_0$, and

$$\mathrm{H}_v^{\mathrm{m}} = \mathrm{H}_v \cap Q_1^{\mathrm{m}}, \qquad \mathrm{T}_v^{\mathrm{m}} = \mathrm{T}_v \cap Q_1^{\mathrm{m}}$$

for their intersections with the set Q_1^{m} of unfrozen arrows.

As well as using the fact that A is thin, as in Proposition 2.11, we will also rely on the following gradability property of A, which allows us to deduce exactness of the relevant complex of bimodules from the exactness of certain complexes of (one-sided) A-modules.

Proposition 3.1. *The dimer algebra A_D of a connected Postnikov diagram D admits a grading in which all arrows have positive degrees.*

Proof. The algebra A_D may be graded by total weights of paths, as discussed in the proof of Proposition 2.11. As already mentioned in that proof, each arrow has positive total weight. □

Most of the arguments in this section will depend on the fact that A is thin, as in Proposition 2.11, so that each piece $e_w A e_v$ is freely generated over $Z = \mathbb{K}[[t]]$ by a minimal path. Before introducing the complexes of A-modules whose exactness implies the required 3-Calabi–Yau property of A, we give one more lemma concerning these paths.

Lemma 3.2. *Let $v, w \in Q_0$ with v mutable, and consider a minimal path $p_{wv}\colon v \to w$. Then there is some $a \in \mathrm{H}_v$ such that $p_{wv} a = p_{w,ta}$ in A.*

Proof. We first observe that if $p\colon u \to w$ is a minimal path in Q passing through the vertex v, then p includes our desired arrow as follows. Write $p = p_2 a p_1$ for some $a \in \mathrm{H}_v$. Then by Lemma 2.13, each subpath of p is also minimal, so in A we have $p_2 = p_{wv}$ and $p_2 a = p_{w,ta}$, and hence a is our desired arrow.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

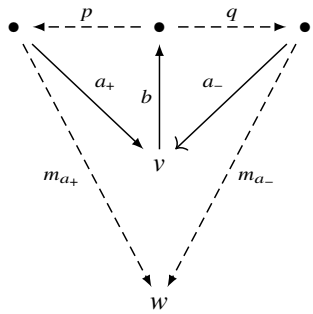


Figure 6. *Fundamental cycles a_+pb and a_-qb together with minimal paths $m_{a_\pm}\colon ta_\pm \to w$ such that $p' = m_{a_+}p$ and $q' = m_{a_-}q$ form a digon containing v. Solid arrows represent arrows in Q, whereas dashed arrows represent paths.*

Now for each $a \in \mathrm{H}_v$, choose a minimal path $m_a\colon ta \to w$. If any of these paths passes through v, then we find our desired arrow as above, so we assume the contrary. Since v is mutable, each $b \in \mathrm{T}_v$ is involved in two fundamental cycles a_+pb and a_-qb with $a_\pm \in \mathrm{H}_v$, and we can write $p' = m_{a_+}p$ and $q' = m_{a_-}q$. By our assumption on the m_a, neither p' nor q' passes through v.

Using again that v is mutable, the union of fundamental cycles containing v is a disc with v in the interior, so there must be some b for which the paths p' and q' are two sides of a digon containing v (compare to the argument in [3, Lem. 6.17]). From now on, we assume that we are in this situation, shown in Figure 6.

Observe that $p_{wv}a_+ = t^\delta p_{w,ta_+}$ for some $\delta \geq 0$ by Proposition 2.11. If $\delta = 0$, then a_+ was our desired arrow, so assume $\delta > 0$. In a similar way, we find $m, n \geq 0$ such that $p_{w,ta_+}p = t^m p_{w,hb}$ and $p_{w,hb}b = t^n p_{wv}$. Since pba_+ is a fundamental cycle, we must have

$$tp_{w,ta_+} = p_{w,ta_+}pba_+ = t^m p_{w,hb}ba_+ = t^{m+n}p_{wv}a_+ = t^{m+n+\delta}p_{w,ta_+}.$$

By Proposition 2.11 again, we must have $m + n + \delta = 1$, so we conclude from positivity of δ that $m = n = 0$ (and $\delta = 1$). In particular, this means $p' = m_{a_+}p$ is a minimal path, representing $p_{w,hb}$. Repeating the argument with a_- and q replacing a_+ and p, we see that either a_- is our desired arrow or $q' = m_{a_-}q$ is also a minimal path representing $p_{w,hb}$.

In the latter case, p' and q' are paths from hb to w defining the same element of A, namely $p_{w,hb}$, and bounding a digon containing v. It is then a consequence of [3, Lem. 6.17] that there is a path $r\colon hb \to w$ passing through v and also defining $p_{w,hb}$ in A. In particular, r is a minimal path passing through v, so we may find our desired arrow as in the first paragraph of the proof. □

We now turn to the main part of the argument. In [43, §5], it is explained how an ice quiver with potential (Q, F, W) determines a complex of projective bimodules for the associated frozen Jacobian algebra $A = \mathcal{J}(Q, F, W)$; we will denote this complex by $\mathbf{P}(A)$, although strictly it depends on the presentation of A determined by (Q, F, W). For the convenience of the reader, we repeat its definition here, following [42, §4].

Let $S = A/\overline{J}$, where J is the the ideal generated by arrows. As a left A-module, S is the direct sum of the vertex simple modules and has a basis consisting of the vertex idempotents e_v. For the remainder of this section, we write $\otimes = \otimes_S$.

Introduce formal symbols ρ_a for each $a \in Q_1$ and ω_v for each $v \in Q_0$, and define S-bimodule structures on the vector spaces

$$\mathbb{K}Q_0 = \bigoplus_{v\in Q_0} \mathbb{K}e_v, \qquad \mathbb{K}Q_1 = \bigoplus_{a\in Q_1} \mathbb{K}a, \qquad \mathbb{K}Q_2^{\mathrm{m}} = \bigoplus_{a\in Q_1^{\mathrm{m}}} \mathbb{K}\rho_a, \qquad \mathbb{K}Q_3^{\mathrm{m}} = \bigoplus_{v\in Q_0^{\mathrm{m}}} \mathbb{K}\omega_v,$$

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

via the formulae

$$e_v \cdot e_v \cdot e_v = e_v, \qquad e_{ha} \cdot a \cdot e_{ta} = a, \qquad e_{ta} \cdot \rho_a \cdot e_{ha} = \rho_a, \qquad e_v \cdot \omega_v \cdot e_v = \omega_v.$$

We recall here that $Q_0^{\mathrm{m}} = Q_0 \setminus F_0$ and $Q_1^{\mathrm{m}} = Q_1 \setminus F_1$ are the sets of mutable vertices and unfrozen arrows, respectively.

Since $\mathbb{K}Q_0$ is naturally isomorphic to S as a bimodule, there is a natural isomorphism

$$A \otimes \mathbb{K}Q_0 \otimes A \xrightarrow{\sim} A \otimes A,$$

which we can compose with the multiplication map for A to obtain a map $\mu_0 \colon A \otimes \mathbb{K}Q_0 \otimes A \to A$. Define $\mu_1 \colon A \otimes \mathbb{K}Q_1 \otimes A \to A \otimes \mathbb{K}Q_0 \otimes A$ by

$$\mu_1(x \otimes a \otimes y) = x \otimes e_{ha} \otimes ay - xa \otimes e_{ta} \otimes y.$$

For any path $p = a_m \cdots a_1$ in Q, we may define

$$\Delta_a(p) = \sum_{a_i = a} a_m \cdots a_{i+1} \otimes a_i \otimes a_{i-1} \cdots a_1,$$

and extend by linearity and continuity to obtain a map $\Delta_a \colon \mathbb{K}\langle\langle Q \rangle\rangle \to A \otimes \mathbb{K}Q_1 \otimes A$. We then define $\mu_2 \colon A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes A \to A \otimes \mathbb{K}Q_1 \otimes A$ by

$$\mu_2(x \otimes \rho_a \otimes y) = \sum_{b \in Q_1} x \Delta_b(\partial_a W) y.$$

Finally, define $\mu_3 \colon A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes A \to A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes A$ by

$$\mu_3(x \otimes \omega_v \otimes y) = \sum_{a \in \mathrm{T}_v} x \otimes \rho_a \otimes ay - \sum_{b \in \mathrm{H}_v} xb \otimes \rho_b \otimes y.$$

As $\mathrm{T}_v \cup \mathrm{H}_v \subseteq Q_1^{\mathrm{m}}$ for any $v \in Q_0^{\mathrm{m}}$, this map has the claimed codomain.

Definition 3.3. For an ice quiver with potential (Q, F, W), with frozen Jacobian algebra $A = \mathcal{J}(Q, F, W)$, let $\mathbf{P}(A)$ be the complex of A-bimodules with nonzero terms

$$A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes A \xrightarrow{\mu_3} A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes A \xrightarrow{\mu_2} A \otimes \mathbb{K}Q_1 \otimes A \xrightarrow{\mu_1} A \otimes \mathbb{K}Q_0 \otimes A$$

and $A \otimes \mathbb{K}Q_0 \otimes A$ in degree 0.

Using this complex, we can give a sufficient condition for the frozen Jacobian algebra to be bimodule internally 3-Calabi–Yau [43, Defn. 2.4] with respect to the idempotent determined by the vertices of F. This condition will suffice as a definition for the purposes of the present paper.

Theorem 3.4 [43, Thm. 5.7]. *If A is a frozen Jacobian algebra such that*

$$0 \longrightarrow \mathbf{P}(A) \xrightarrow{\mu_0} A \longrightarrow 0 \tag{3.1}$$

is exact, then A is bimodule internally 3*-Calabi–Yau with respect to the idempotent $e = \sum_{v \in F_0} e_v$.*

Remark 3.5. By standard results on presentations of algebras, as in Butler–King [9], we need only check exactness at the two left-most nonzero terms of $\mathbf{P}(A)$ (in degrees -3 and -2), since the rest of the complex in equation (3.1) is the standard projective bimodule presentation of an algebra defined by a quiver with relations.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

Proposition 3.6. *For $A = A_D$ the dimer algebra of a connected Postnikov diagram, the complex in equation (3.1) is exact in degree d if and only if the complex*

$$0 \longrightarrow \mathbf{P}(A) \otimes_A S \xrightarrow{\mu_0 \otimes S} S \longrightarrow 0 \tag{3.2}$$

is exact in degree d, if and only if the complex

$$0 \longrightarrow \mathbf{P}(A) \otimes_A S_v \xrightarrow{\mu_0 \otimes S_v} S_v \longrightarrow 0 \tag{3.3}$$

is exact in degree d for any $v \in Q_0$, where S_v denotes the simple left A-module at v.

Proof. Exactness of equation (3.1) and equation (3.2) are equivalent because of the existence of a grading as in Proposition 3.1, by an argument that is essentially due to Broomhead [3, Prop. 7.5]. The reader can find an explanation of how this argument extends to the case of frozen Jacobian algebras in [42, §4]. The second equivalence is simply the observation that the complex in equation (3.2) is the direct sum of the complexes in equation (3.3). □

It will be useful to introduce some notation for elements of terms of $\mathbf{P}(A) \otimes_A S_v$, following [42, §4]. First note that we have isomorphisms

$$\begin{aligned} A \otimes \mathbb{K}Q_1 \otimes A \otimes_A S_v &\cong \bigoplus_{b \in \mathrm{T}_v} Ae_{hb}, \\ A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes A \otimes_A S_v &\cong \bigoplus_{a \in \mathrm{H}_v^{\mathrm{m}}} Ae_{ta}, \\ A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes A \otimes_A S_v &\cong \begin{cases} Ae_v, & v \in Q_0^{\mathrm{m}}, \\ 0, & v \in F_0. \end{cases} \end{aligned} \tag{3.4}$$

In the first two cases, the right-hand sides are of the form $\bigoplus_{a \in S} Ae_{\delta a}$, where S is a set of arrows and $\delta \colon S \to Q_0$. The map $Ae_{\delta a} \to \bigoplus_{a \in S} Ae_{\delta a}$ including the domain as the summand indexed by a will be denoted by $x \mapsto x \otimes [a]$; this helps us to distinguish these various inclusions when δ is not injective. As a consequence, a general element of the direct sum is

$$x = \sum_{a \in S} x_a \otimes [a] \tag{3.5}$$

with $x_a \in Ae_{\delta a}$.

The most complicated map in the complex $\mathbf{P}(A)$ is μ_2, so we will spell out $\mu_2 \otimes_A S_v$ explicitly. Using the isomorphisms from equation (3.4) and our notation for elements of the direct sums, we have

$$(\mu_2 \otimes_A S_v)(x) = \sum_{b \in \mathrm{T}_v} \left(\sum_{a \in \mathrm{H}_v^{\mathrm{m}}} x_a \partial_b^r \partial_a W \right) \otimes [b],$$

where ∂_b^r, called the *right derivative* with respect to b, is defined on paths by

$$\partial_b^r (a_k \cdots a_1) = \begin{cases} a_k \cdots a_2, & a_1 = b, \\ 0, & a_1 \neq b \end{cases} \tag{3.6}$$

and extended linearly and continuously. Similarly, there is a *left derivative*, defined on paths by

$$\partial_b^l (a_k \cdots a_1) = \begin{cases} a_{k-1} \cdots a_1, & a_k = b, \\ 0, & a_k \neq b. \end{cases} \tag{3.7}$$

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

Given a Postnikov diagram D and two arrows a and b of Q_D, we may observe that

$$\partial_b^r \partial_a W = \partial_a^l \partial_b W. \tag{3.8}$$

Indeed, there are at most two fundamental cycles in Q_D containing a, so the part of W consisting of terms containing both a and b is of the form $apb - aqb$ for some p and q, which are either paths, not containing the arrows a or b, or are zero. Thus one can directly calculate each side of equation (3.8), which both result in $p - q$.

We are now ready to prove our main theorem.

Theorem 3.7. *For $A = A_D$ the dimer algebra of a connected Postnikov diagram D, the complex in equation (3.1) is exact, and hence A is bimodule internally 3-Calabi–Yau with respect to the idempotent determined by the boundary vertices.*

Proof. As already noted in Remark 3.5, the complex in equation (3.1) can only fail to be exact in degrees -2 and -3, so, by Proposition 3.6, we need only check that the complex in equation (3.3) is exact at the terms $A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes S_v$ and $A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes S_v$ for each $v \in Q_0$.

We first deal with exactness at $A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes S_v$. This term is zero unless $v \in Q_0^{\mathrm{m}}$, so we may additionally assume this. By the third isomorphism in equation (3.4), we then have $A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes S_v \cong Ae_v$, so let $x \in Ae_v$. We calculate

$$(\mu_3 \otimes S_v)(x) = \sum_{a \in \mathrm{H}_v^{\mathrm{m}}} xa \otimes [a],$$

using the notation in (3.5) for elements of $A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes S_v \cong \bigoplus_{a \in \mathrm{H}_v^{\mathrm{m}}} Ae_{ta}$. Thus $x \in \ker(\mu_3 \otimes S_v)$ if and only if $xa = 0$ for all $a \in \mathrm{H}_v^{\mathrm{m}}$. Note that since v is mutable, $\mathrm{H}_v^{\mathrm{m}} = \mathrm{H}_v \neq \varnothing$.

Let $w \in Q_0$, so $e_w x \in e_w A e_v$. Since A is thin by Proposition 2.11, we have $e_w A e_v = Z p_{wv}$, so we may write $e_w x = \sum_{n \in \mathbb{N}} z_n t^n \cdot p_{wv}$ for some sequence of scalars $z_n \in \mathbb{K}$. Using thinness again, for any $a \in \mathrm{H}_v^{\mathrm{m}}$, there is some $\delta_a \in \mathbb{N}$ such that $p_{wv} a = t^{\delta_a} p_{w,ta}$. It then follows that

$$e_w xa = \sum_{n \in \mathbb{N}} z_n t^n \cdot p_{wv} a = \sum_{n \in \mathbb{N}} z_n t^{n+\delta_a} \cdot p_{w,ta}.$$

Thus if $xa = 0$, we have $z_n = 0$ for all n, and so $e_w x = 0$. Since w was chosen arbitrarily, we conclude that $x = 0$, and so the kernel of $\mu_3 \otimes S_v$ is trivial. This establishes exactness of equation (3.3) at the term $A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes S_v$.

We now move to the term $A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes S_v$. Using the relevant isomorphisms in equation (3.4) and the notation of equation (3.5), a general element of this term is of the form $\phi = \sum_{a \in \mathrm{H}_v^{\mathrm{m}}} x_a \otimes [a]$ for $x \in Ae_{ta}$, and its image under $\mu_2 \otimes S_v$ is

$$(\mu_2 \otimes S_v)(\phi) = \sum_{b \in \mathrm{T}_v} \left(\sum_{a \in \mathrm{H}_v^{\mathrm{m}}} x_a \partial_b^r \partial_a W \right) \otimes [b] = \sum_{b \in \mathrm{T}_v} \left(\sum_{a \in \mathrm{H}_v^{\mathrm{m}}} x_a \partial_a^l \partial_b W \right) \otimes [b],$$

where the second equality follows from equation (3.8). The reader is warned that since $b \in \mathrm{T}_v$ may be frozen, the derivative $\partial_b W$ is not necessarily zero in A.

Now assume that $(\mu_2 \otimes S_v)(\phi) = 0$ or, equivalently by the above calculation, that

$$\sum_{a \in \mathrm{H}_v^{\mathrm{m}}} x_a \partial_a^l \partial_b W = 0 \tag{3.9}$$

for all $b \in \mathrm{T}_v$. Picking $w \in Q_0$ and multiplying by e_w on the left, we obtain elements $e_w x_a \in e_w A e_{ta}$ for each $a \in \mathrm{H}_v^{\mathrm{m}}$.

As in the first part of the proof, we will now use thinness of A to write the elements $e_w x_a$ as the product of a power series in t with $p_{w,ta}$, but in order to keep the notation clean in the subsequent

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

argument, we will do this in a slightly unusual way. First, note that, having fixed the vertex w, any path p determines $\delta_p \in \mathbb{N}$ with the property that $p_{w,hp}p = t^{\delta_p} p_{w,tp}$. Since this equality is formulated in the algebra A, the quantity δ_p depends only on the class of p in the algebra.

Using this notation, we can write

$$e_w x_a = t^{\delta_a} \sum_{n\in\mathbb{Z}} z_n^{w,a} t^n \cdot p_{w,ta} \tag{3.10}$$

for some scalars $z_n^{w,a} \in \mathbb{K}$. Since the right-hand side of this expression may only involve nonnegative powers of t, we have $z_n^{w,a} = 0$ whenever $n < -\delta_a$.

The key step in the argument is to show, under our assumption that $(\mu_2 \otimes S_v)(\phi) = 0$, that the coefficient $z_n^{w,a}$ depends only on n and w, and not on $a \in \mathrm{H}_v^{\mathrm{m}}$. We consider the configuration

$$\begin{tikzcd}
 & v \arrow[dl, "a_-"'] \arrow[d, "b"] & \\
\bullet \arrow[r, dashed, "p"] \arrow[dr, dotted, "e_w x_+"] & \bullet \arrow[l, dashed, "q"'] \arrow[d, dotted, "e_w x_-"'] & \bullet \arrow[ul, "a_+"'] \arrow[dl, dotted, "e_w x_+"] \\
 & w &
\end{tikzcd} \tag{3.11}$$

Here solid arrows represent arrows in Q_1, dashed arrows represent paths in Q and dotted arrows represent linear combinations of paths. Our assumption is that a_+ and a_- are unfrozen arrows, with head at v as depicted, which appear in the two fundamental cycles a_+pb and a_-qb containing the arrow b, also unfrozen, with tail at v. As indicated by the notation in equation (3.11), we will usually omit the letter a when a_+ or a_- appears in subscripts or superscripts in the following argument—for example, this means $x_+ := x_{a_+}$. Our aim is to show that $z_n^{w,+} = z_n^{w,-}$ for all $n \in \mathbb{Z}$; it will then follow from the way the quiver Q is embedded in the disc that $z_n^{w,a}$ is independent of $a \in \mathrm{H}_v^{\mathrm{m}}$, since if $a, a' \in \mathrm{H}_v^{\mathrm{m}}$, we can find sequences $a_0, \dots, a_k$ and $b_1 \cdots b_k$ such that $a_0 = a$, $a_k = a'$, and for all $1 \le i \le k$ the triple of arrows (a_{i-1}, b_i, a_i) has the same configuration as (a_-, b, a_+) in equation (3.11) (or its mirror image), so that $z_n^{w,a_{i-1}} = z_n^{w,a_i}$.

So consider the configuration shown in equation (3.11). We see from the figure that $\partial_b W = a_+p - a_-q$. While it can happen that $a_+ = a_-$ (if and only if v is a bivalent vertex of Q), in this case there is nothing to prove, so we may assume $a_+ \neq a_-$. Then for $a \in \mathrm{H}_v^{\mathrm{m}}$, we have

$$\partial_a^l \partial_b W = \begin{cases} p, & a = a_+, \\ -q, & a = a_-, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{a\in\mathrm{H}_v^{\mathrm{m}}} x_a \partial_a^l \partial_b W = x_+p - x_-q.$$

Since $\phi = \sum_{a\in\mathrm{H}_v^{\mathrm{m}}} x_a \otimes [a]$ is in the kernel of $\mu_2 \otimes S_v$, this quantity is zero by equation (3.9), so $x_+p = x_-q$. It follows that

$$e_w x_+ p = t^{\delta_+} \sum_{n\in\mathbb{Z}} z_n^{w,+} t^n \cdot p_{w,ta_+} p = t^{\delta_+ + \delta_p} \sum_{n\in\mathbb{Z}} z_n^{w,+} t^n \cdot p_{w,hb},$$
$$e_w x_- q = t^{\delta_-} \sum_{n\in\mathbb{Z}} z_n^{w,-} t^n \cdot p_{w,ta_-} q = t^{\delta_- + \delta_q} \sum_{n\in\mathbb{Z}} z_n^{w,-} t^n \cdot p_{w,hb},$$

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

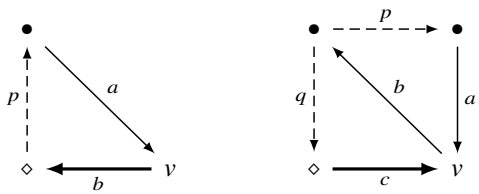
must also be equal. Since $a_+p = a_-q$ in A (because of the relation $\partial_b W$), we must have $\delta_+ + \delta_p = \delta_- + \delta_q$, so we can conclude, for any $n \in \mathbb{Z}$, that $z_n^{w,+} = z_n^{w,-}$, and further that $z_n^{w,a}$ is independent of $a \in \mathrm{H}_v^{\mathrm{m}}$, as required. From now on, we abbreviate $z_n^w := z_n^{w,a}$, and for any $a \in \mathrm{H}_v^{\mathrm{m}}$, we have

$$e_w x_a = t^{\delta_a} \sum_{n \in \mathbb{Z}} z_n^w t^n \cdot p_{w,ta}. \tag{3.12}$$

We now return to our main purpose, to show that the complex in equation (3.1) is exact. We recall that we are assuming that $\phi = \sum_{a \in \mathrm{H}_v^{\mathrm{m}}} x_a \otimes [a]$ satisfies $(\mu_2 \otimes S_v)(\phi) = 0$ and wish to show that ϕ is in the image of $\mu_3 \otimes S_v$.

First we deal with the case that v is a frozen vertex, in which case $\mu_3 \otimes S_v = 0$, and we wish to conclude that $x_a = 0$ for all $a \in \mathrm{H}_v^{\mathrm{m}}$. This is equivalent to showing that the complex numbers z_n^w appearing in equation (3.12) are zero for all n and all w.

It follows from the construction of (Q_D, F_D) that, since v is frozen, it is incident with a frozen arrow. If $\mathrm{H}_v^{\mathrm{m}} = \varnothing$, then we have nothing to show, and otherwise there is an arrow $a \in \mathrm{H}_v^{\mathrm{m}}$ as in one of the following configurations



(3.13)

in which apb and cqb are fundamental cycles. As earlier in the paper, the bold arrows are frozen, and the dashed arrows represent paths. The vertices indicated by diamonds are frozen, like the vertex v, whereas the others may be mutable or frozen.

In the two configurations in equation (3.13), we have $\partial_b W = \pm ap$ and $\partial_b W = \pm(ap - cq)$, respectively. Thus in either case we calculate for $a' \in \mathrm{H}_v^{\mathrm{m}}$ that

$$\partial_{a'}^l \partial_b W = \begin{cases} \pm p, & a' = a, \\ 0, & \text{otherwise.} \end{cases}$$

Since $(\mu_2 \otimes S_v)(\phi) = 0$, we deduce from equation (3.9) that

$$x_a p = \pm \sum_{a' \in \mathrm{H}_v^{\mathrm{m}}} x_a \partial_{a'}^l \partial_b W = 0.$$

Multiplying on the left by e_w and using that A is thin, we see that

$$0 = e_w x_a p = t^{\delta_a} \sum_{n \in \mathbb{Z}} z_n^w t^n \cdot p_{w,ta} p = t^{\delta_a + \delta_p} \sum_{n \in \mathbb{Z}} z_n^w t^n \cdot p_{w,hb},$$

so $z_n^w = 0$ for all n and w, as required.

Now we treat the case that v is a mutable vertex; in this case, $\mathrm{H}_v^{\mathrm{m}} = \mathrm{H}_v$ and we use the latter to keep the notation simpler. In this case, we want to construct $y \in Ae_v$ such that

$$(\mu_3 \otimes S_v)(y) = \sum_{a \in \mathrm{H}_v} ya \otimes [a] = \phi.$$

Using the sequence of complex numbers z_n^w appearing in equation (3.12), we write

$$y_w = \sum_{n \in \mathbb{Z}} z_n^w t^n \cdot p_{wv}.$$

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

For this to be a well-defined element of $e_w A e_v$, we need $z_n^w = 0$ for all $n < 0$. Choosing any $a \in \mathrm{H}_v$, we have that $z_n^w = z_n^{w,a}$ is zero whenever $n < -\delta_a$, as in equation (3.10). But by Lemma 3.2, there is some $a \in \mathrm{H}_v$ such that $\delta_a = 0$, so $y_w \in e_w A e_v$ as required.

Now, letting $y = \sum_{w \in Q_0} y_w \in A e_v \cong A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes S_v$, we can calculate

$$\begin{aligned}(\mu_3 \otimes S_v)(y) = \sum_{w \in Q_0} \sum_{a \in \mathrm{H}_v} y_w a \otimes [a] &= \sum_{w \in Q_0} \sum_{a \in \mathrm{H}_v} \left(\sum_{n \in \mathbb{N}} z_n^w t^n \cdot p_{wv} a \right) \otimes [a] \\ &= \sum_{w \in Q_0} \sum_{a \in \mathrm{H}_v} \left(\sum_{n \in \mathbb{N}} z_n^w t^{n+\delta_a} \cdot p_{w,ta} \right) \otimes [a] \\ &= \sum_{w \in Q_0} \sum_{a \in \mathrm{H}_v} e_w x_a \otimes [a] = \sum_{a \in \mathrm{H}_v} x_a \otimes [a] = \phi,\end{aligned}$$

so we conclude that $\phi \in \mathrm{im}(\mu_3 \otimes S_v)$, as required. □

4. Categorification

Given a suitable algebra A, which is bimodule internally 3-Calabi–Yau with respect to an idempotent e, one can construct from A and e a Frobenius category having many desirable properties from the point of view of categorifying cluster algebras. This is done via the main result of [43] (specialised to the case that the Calabi–Yau dimension is 3), which we will recall after collecting the necessary algebraic definitions. Applied to the dimer algebra A_D of a connected Postnikov diagram D, this will allow us to show (Theorem 4.5) that the category $\mathrm{GP}(B_D)$ appearing in Theorem 1 has many of the properties required of an additive categorification of the cluster algebra $\mathscr{A}_D$. Further such properties will be proved in Section 6.

Definition 4.1. Let $\mathcal{E}$ be an exact $\mathbb{K}$-linear category, as in [8]. We do not recall the full definition of this structure here but simply note that any full extension-closed subcategory $\mathcal{E}$ of an abelian category $\mathcal{A}$ has a natural structure of an exact category, by taking the class of short exact sequences to be the short exact sequences of $\mathcal{A}$ with terms in $\mathcal{E}$.

We say $\mathcal{E}$ is a *Frobenius category* if it has enough projective objects and enough injective objects, and these two classes of objects coincide. The *stable category* $\underline{\mathcal{E}}$ is obtained from $\mathcal{E}$ by factoring out the ideal of morphisms that factor over a projective (equivalently injective) object and has a natural triangulated structure [22, §I.2]. We say that $\underline{\mathcal{E}}$ is 2-*Calabi–Yau*, and $\mathcal{E}$ is *stably* 2-*Calabi–Yau*, if $\underline{\mathcal{E}}$ is Hom-finite and there is a functorial isomorphism

$$\mathrm{Hom}_{\underline{\mathcal{E}}}(X, Y) = \mathrm{D}\,\mathrm{Hom}_{\underline{\mathcal{E}}}(Y, \Sigma^2 X)$$

for any two objects X and Y of $\underline{\mathcal{E}}$, where Σ is the shift functor in the triangulated structure of $\underline{\mathcal{E}}$ (computed by taking the cokernel of an injective envelope) and D is duality over the ground field $\mathbb{K}$.

A full subcategory $\mathcal{T} \subseteq \mathcal{E}$ is called *cluster-tilting* if it is functorially finite and

$$\mathcal{T} = \{X \in \mathcal{E} : \mathrm{Ext}^1_{\mathcal{E}}(X, T) = 0 \,\forall\, T \in \mathcal{T}\} = \{X \in \mathcal{E} : \mathrm{Ext}^1_{\mathcal{E}}(T, X) = 0 \,\forall\, T \in \mathcal{T}\}.$$

Note that cluster-tilting subcategories are necessarily additively closed: that is, closed under direct sums and summands. We can make the same definition in $\underline{\mathcal{E}}$, using the usual definition $\mathrm{Ext}^1_{\underline{\mathcal{E}}}(M, N) = \mathrm{Hom}_{\underline{\mathcal{E}}}(M, \Sigma N)$ for triangulated categories, and observe that a subcategory $\mathcal{T} \subseteq \mathcal{E}$ containing all projective objects is cluster-tilting if and only if its image in the stable category $\underline{\mathcal{E}}$ (i.e., the full subcategory of $\underline{\mathcal{E}}$ on the same set of objects as $\mathcal{T}$) is cluster-tilting. When $\mathcal{E}$ is stably 2-Calabi–Yau, the second equality in the definition of a cluster-tilting subcategory is automatic. An object $T \in \mathcal{E}$ is a *cluster-tilting object* if its additive closure $\mathrm{add}\, T$ is a cluster-tilting subcategory.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

An algebra B is called *Iwanaga–Gorenstein* if it is Noetherian and has finite injective dimension both as a left and as a right module over itself. In this case these two injective dimensions coincide [25], and we call this number the *Gorenstein dimension* of B. For an Iwanaga–Gorenstein algebra B, we define the category of *Gorenstein projective* B-modules to be the full subcategory

$$\mathrm{GP}(B) = \left\{X \in \operatorname{mod} B : \operatorname{Ext}^i_B(X, B) = 0 \text{ for all } i > 0\right\}$$

of the abelian category of finitely generated B-modules. This is a Frobenius category; see for example [7, §4], where this category is denoted by $\mathrm{MCM}(B)$ and its objects called maximal Cohen–Macaulay modules.

Remark 4.2. The reader is warned that when B is free and finitely generated over $Z = \mathbb{K}[[t]]$, as will be the case in our examples, one can define the (maximal) Cohen–Macaulay B-modules to be those B-modules that are free and finitely generated over Z, as in [28, §3]. While a module for the algebra C defining the Grassmannian cluster category is Gorenstein projective if and only if it is Cohen–Macaulay [28, Cor. 3.7], this will not be the case for B in general (see Example 4.8). To avoid having two nonequivalent definitions of Cohen–Macaulay B-modules, we prefer the terminology 'Gorenstein projective' for the homological condition. The reader is, however, further warned that Gorenstein projective modules are defined over an arbitrary ring (as in [23, Defn. 2.1] for example), but in a way that need not be equivalent to Definition 4.1 if the ring is not Iwanaga–Gorenstein.

Theorem 4.3 [43, Thm. 4.1, Thm. 4.10]. *Let A be an algebra and $e \in A$ an idempotent. If A is Noetherian, $\underline{A} = A/\langle e\rangle$ is finite-dimensional and A is bimodule internally 3-Calabi–Yau with respect to e, then*

(1) *$B = eAe$ is Iwanaga–Gorenstein with Gorenstein dimension at most 3,*
(2) *eA is a cluster-tilting object in the Frobenius category $\mathrm{GP}(B)$,*
(3) *the stable category $\underline{\mathrm{GP}}(B)$ is a 2-Calabi–Yau triangulated category, and*
(4) *the natural maps $A \to \operatorname{End}_B(eA)^{\mathrm{op}}$ and $\underline{A} \to \underline{\operatorname{End}}_B(eA)^{\mathrm{op}}$ are isomorphisms.*

We already saw in Theorem 3.7 that the dimer algebra A_D of a connected Postnikov diagram D is bimodule internally 3-Calabi–Yau with respect to its boundary idempotent. Checking the other conditions needed to apply Theorem 4.3 to A_D is comparatively straightforward, although we note that we once again rely heavily on Proposition 2.11.

Proposition 4.4. *Let D be a connected Postnikov diagram with dimer algebra A_D, and let e be the boundary idempotent of this algebra. Then A_D is Noetherian and $A_D/\langle e\rangle$ is finite-dimensional.*

Proof. Since A_D is thin, as in Proposition 2.11, it is finitely generated as a module over the commutative Noetherian ring $\mathbb{K}[[t]]$ and so is Noetherian.

By thinness again, A_D has a basis $\{t^n p_{wv} : v, w \in Q_0, n \in \mathbb{N}\}$. To see that $A_D/\langle e\rangle$ is finite-dimensional, we show that all but finitely many of these basis vectors are zero in the quotient. Picking any $v, w \in Q_0$, let c be any cycle at w passing through a boundary vertex, which exists since Q_D is strongly connected as in Proposition 2.6. Then since A_D is thin, we have $cp_{wv} = t^N p_{wv}$ for some $N \in \mathbb{N}$. By construction, this element is zero in the quotient $A_D/\langle e\rangle$, and thus $t^n p_{wv} = 0$ in this quotient algebra for all $n \geq N$. Running over all pairs $v, w \in Q_0$ completes the proof. □

Recall that we write $\mathscr{A}_D$ for the cluster algebra with frozen variables determined by the ice quiver (Q_D, F_D) of a Postnikov diagram D. Note that arrows between frozen vertices play no role in the construction of this cluster algebra. Since Q_D may contain 2-cycles, we are committing a small abuse of notation here; really, $\mathscr{A}_D$ is the cluster algebra determined by the quiver Q' of the unique reduced Postnikov diagram obtained from D by untwisting moves, as in Figure 4. Equivalently, Q' is obtained from Q_D by removing a maximal collection of 2-cycles incident with unfrozen vertices. Using the description via untwisting and Proposition 2.9, we see that Q' has a potential W' such that $\mathcal{J}(Q', F_D, W') \cong A_D$ is the dimer algebra of the original diagram D.

By Theorem 3.7 and Proposition 4.4, we may apply Theorem 4.3 to the dimer algebra A_D of a connected Postnikov diagram, to obtain the following.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

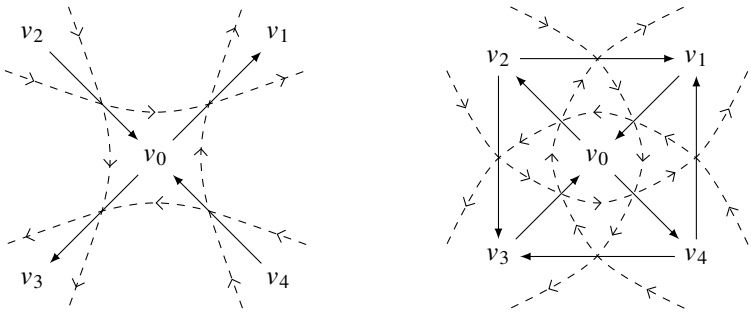


Figure 7. *A geometric exchange of a Postnikov diagram transforms the local configuration of a quadrilateral alternating region, shown on the left, to that shown on the right. The effect on the quiver with potential is a mutation [15, §5] at the vertex corresponding to the quadrilateral alternating region (compare to [49]).*

Theorem 4.5. *Let D be a connected Postnikov diagram with dimer algebra A_D. Let e be the boundary idempotent of this algebra, and write $B_D = eA_De$. Then*

(1) *B_D is Iwanaga–Gorenstein of Gorenstein dimension at most 3,*
(2) *$\mathrm{GP}(B_D)$ is a stably 2-Calabi–Yau Frobenius category,*
(3) *$T_D = eA_D \in \mathrm{GP}(B_D)$ is cluster-tilting, and*
(4) *$\mathrm{End}_{B_D}(T_D)^{\mathrm{op}} \cong A_D$.*

In particular, it follows from (4) that the Gabriel quiver of $\mathrm{End}_{B_D}(T_D)^{\mathrm{op}}$ is, up to arrows between frozen vertices, the quiver of a seed in the cluster algebra $\mathscr{A}_D$.

We note that if the full subquiver of Q_D on the mutable vertices is, after removing 2-cycles, mutation-equivalent to an acyclic quiver $\underline{Q}$, it follows from a result of Keller and Reiten [30, Thm. 2.1] that the stable category $\underline{\mathrm{GP}}(B_D)$ of our categorification is equivalent to the cluster category $\mathcal{C}_{\underline{Q}}$ [5]. This mutation-acyclicity assumption does not hold for most Postnikov diagrams, however. We conjecture that $\underline{\mathrm{GP}}(B_D)$ is always equivalent to Amiot's cluster category [1] for the quiver with potential obtained from (Q_D, F_D, W_D) by deleting the frozen vertices and all incident arrows from Q_D and removing terms passing through these vertices from W_D.

If D is a (k,n)-diagram, as in Definition 2.1, it follows from [2, Cor. 10.4] that the boundary algebra $B_D = eA_De$ does not depend on the choice of D within this class, up to isomorphism, and the category $\mathrm{GP}(B_D)$ is equivalent to Jensen–King–Su's categorification of the cluster algebra structure on the whole Grassmannian [28].

In general, we can consider two Postnikov diagrams D and D' related by geometric exchange as shown in Figure 7. Combining [41, Thm. 13.4] and [41, Thm. 17.1], we see that these diagrams determine the same positroid and hence the same positroid varieties. Moreover, the quivers Q_D and $Q_{D'}$ are related by a mutation, so the cluster algebras $\mathscr{A}_D$ and $\mathscr{A}_{Q_{D'}}$ are isomorphic. Thus we would like both D and D' to give rise to the same category of Gorenstein projective modules in Theorem 4.5. This is indeed the case, as follows.

Proposition 4.6. *Let D and D' be Postnikov diagrams related by geometric exchange. Then the boundary algebras B_D and $B_{D'}$ are isomorphic, and hence the categories $\mathrm{GP}(B_D)$ and $\mathrm{GP}(B_{D'})$ are equivalent.*

Proof. This can be proved exactly as in [2, Prop. 12.2]; while this proposition is stated only for (k,n)-diagrams, the proof does not use the extra assumption on the strand permutation. □

Corollary 4.7. *Given a Postnikov diagram D, the algebra B_D is determined up to isomorphism by the positroid $\mathcal{P}_D$.*

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

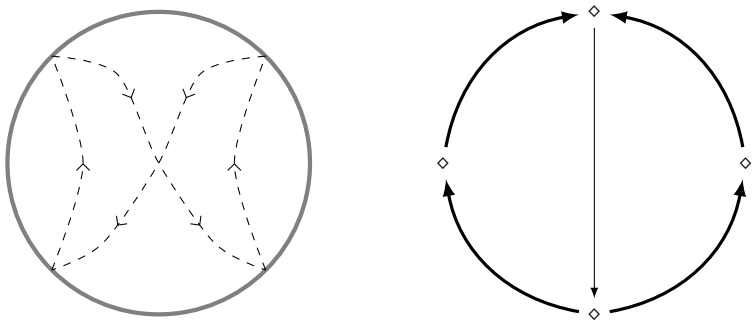


Figure 8. *A Postnikov diagram D of type $(2,4)$ giving rise to a boundary algebra B_D for which not all Cohen–Macaulay modules are Gorenstein projective. The right-hand figure shows the quiver Q_D, with frozen arrows marked in bold as usual, which is the Gabriel quiver of $A_D = B_D$.*

Proof. Again combining [41, Thm. 13.4] and [41, Thm. 17.1], any two diagrams determining the same positroid are connected by a sequence of geometric exchanges, so the result follows from Proposition 4.6. □

One consequence of the general results in Theorem 4.3 on bimodule internally 3-Calabi–Yau algebras is that the Gorenstein dimension of B_D is at most 3 for any connected Postnikov diagram D. In fact, this Gorenstein dimension never exceeds 2 by [11, Cor. 10.5]. When D is a (k,n)-diagram, B_D even has Gorenstein dimension 1, as in the proof of [28, Cor. 3.7]. Since 1 is also the Krull dimension of $Z = \mathbb{K}[[t]]$, in this case the category $\mathrm{GP}(B_D)$ coincides with the category of Cohen–Macaulay B_D-modules, meaning those B_D-modules free and finitely generated over Z. However, the Gorenstein dimension of B_D can be equal to 2, and the inclusion $\mathrm{GP}(B_D) \to \mathrm{CM}(B_D)$ can be proper, as the following example shows.

Example 4.8. Consider the Postnikov diagram D shown in Figure 8 and its dimer algebra A_D, the quiver of which is shown in the same figure. Since every alternating region of D is on the boundary (equivalently, every vertex of the quiver is frozen), we have $B_D = A_D$, which has finite global dimension by Theorem 3.7. In fact, since the left-most term of $\mathbf{P}(A_D)$ is zero in this case, again because all vertices are frozen, the global dimension of B_D is 2.

As is the case for any algebra of finite global dimension, B_D is Iwanaga–Gorenstein (of Gorenstein dimension equal to the global dimension, so in this case 2), and a B_D-module is Gorenstein projective if and only if it is projective. However, one can observe that the radical of the projective B_D-module at the lowest vertex (the head of the unfrozen arrow) is not projective, but it is free and finitely generated over Z: that is, Cohen–Macaulay, since it is a submodule of a projective B_D-module, which is Cohen–Macaulay by Proposition 2.11. Hence in this case the category of Gorenstein projective modules for B_D is a proper subcategory of the category of Cohen–Macaulay modules.

5. Loops and 2-cycles

Let $B = B_D$ be the boundary algebra of a connected Postnikov diagram D, and let $T_D = eA_D \in \mathrm{GP}(B)$ be the cluster-tilting object from Theorem 4.5. In this section, we show that the quiver of $\mathrm{End}_B(T)^{\mathrm{op}}$ has no loops or 2-cycles for any cluster-tilting object $T \in \mathrm{GP}(B)$ mutation equivalent to T_D—so far, we know this only for the relatively small number of such objects arising from the Postnikov diagrams related to D by sequences of geometric exchanges. This will allow us to complete the proof of Theorem 1 in the next section by giving us access to results of Fu and Keller [17]. The statement requires our nondegeneracy assumption that D has at least 3 strands (compare to [44, Prop. 5.17]).

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

For the remainder of the section, we fix a cluster-tilting object $T_0 \in \mathrm{GP}(B)$ and write $A_0 = \mathrm{End}_B(T_0)^{\mathrm{op}}$. We assume that

(1) A_0 is isomorphic to the frozen Jacobian algebra of an ice quiver with potential (Q, F, W) in such a way that the vertices of F correspond to projection onto indecomposable projective summands of T_0,
(2) the quiver Q has no loops or 2-cycles, and
(3) $A_0/\langle e_i \rangle$ is finite-dimensional whenever $i \in F_0$.

All of these assumptions hold if $T_0 = eA_D$ is the initial cluster-tilting object from Theorem 4.5—the absence of loops and 2-cycles in Q_D (after replacing D by a reduced diagram if necessary) follows from Proposition 2.10, and finite-dimensionality in (3) follows from Proposition 2.11 via an exactly analogous argument to that in the proof of Proposition 4.4. Our goal is to show that these properties are preserved by mutations and so hold in the entire mutation class of those cluster-tilting objects coming from Postnikov diagrams.

With this in mind, we additionally fix for the rest of the section a cluster-tilting object T obtained from T_0 by mutation at a nonprojective indecomposable summand and write $A = \mathrm{End}_B(T)^{\mathrm{op}}$. By [44, Thm. 5.14], the algebra A again satisfies (1), with the required ice quiver with potential being the mutation $\mu_k(Q, F, W) = (Q', F', W')$ at an appropriate vertex k. Thus Q' has no loops and no 2-cycles incident with k, but a priori it may have other 2-cycles. The algebra A satisfies (3), as the following lemma shows.

Lemma 5.1. *Let A_0 be the frozen Jacobian algebra of an ice quiver with potential (Q, F, W), and assume i is a frozen vertex such that $A_0/\langle e_i \rangle$ is finite-dimensional. If A is the frozen Jacobian algebra of $\mu_k(Q, F, W)$ for some mutable vertex k, then $A/\langle e_i \rangle$ is again finite-dimensional.*

Proof. Note that the quivers Q and $\mu_k Q$ have the same set of frozen vertices, so the statement makes sense. It is straightforward to check that $A_0/\langle e_i \rangle$ and $A/\langle e_i \rangle$ are, like A_0 and A, frozen Jacobian algebras of ice quivers with potential related by mutation at k. (The required ice quivers with potential are obtained from those of A_0 and A by removing the vertex i and all incident arrows from the quiver, and removing from the potential all terms given by cycles through i.) That mutations preserve finite dimensionality of frozen Jacobian algebras follows via a straightforward adaptation of the proof of [15, Cor. 6.6] for ordinary Jacobian algebras. □

Thus it remains to show that the quiver Q' has no 2-cycles. We start by showing that $A/\langle e_i \rangle$ has finite global dimension for any frozen vertex i.

Proposition 5.2. *Let i be a frozen vertex. Then $e_i A e_i \cong Z$.*

Proof. Since e_i corresponds to projection onto an indecomposable projective summand P_i of T, we have $e_i A e_i = \mathrm{End}_B(P_i)^{\mathrm{op}}$. Since P_i is an indecomposable summand of every cluster-tilting object of $\mathrm{GP}(B)$, including T_D, this endomorphism algebra is isomorphic to Z by Proposition 2.11 and Theorem 4.5. □

In the setting of Proposition 5.2, we thus have a recollement

$$\begin{tikzcd} \mathrm{mod}\, A/(e_i) \arrow[r, "\iota"] & \mathrm{mod}\, A \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "e_i"] & \mathrm{mod}\, Z \arrow[l, bend right, "\ell"'] \arrow[l, bend left] \end{tikzcd} \tag{5.1}$$

of abelian categories, in which ι is the natural inclusion, $e_i \colon X \mapsto e_i X$ and $\ell = Ae_i \otimes_Z -$ is left adjoint to e_i.

Lemma 5.3. *In the recollement in equation (5.1), the functor ι is a homological embedding, meaning it induces isomorphisms on all extension groups. As a consequence,* $\mathrm{gldim}\, A/\langle e_i \rangle \leq \mathrm{gldim}\, A \leq 3$.

Proof. We use a result of Psaroudakis [45, Prop. 4.15], implying that it is enough to check that the image, denoted by $\mathsf{F}(P)$ in [45], of the counit

$$\varepsilon \colon \ell(e_i P) = Ae_i \otimes_Z e_i P \to P$$

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

is projective for all projective $P \in \operatorname{mod} A$. We may write $P = \operatorname{Hom}_B(T, T')$ for $T' \in \operatorname{add} T \subseteq \operatorname{GP}(B) \subseteq \operatorname{CM}(B)$ so that $e_i P = \operatorname{Hom}_B(P_i, T')$ is a free Z-module. Thus $Ae_i \otimes_Z e_i P \cong (Ae_i)^n$, where n is the rank of the Z-module $e_i P$, is projective. We claim that the counit ε is injective so that its image is isomorphic to the projective module $(Ae_i)^n$.

To see this, observe that $Ae_i \otimes_Z e_i P \cong (Ae_i)^n$ and P are Cohen–Macaulay A-modules—that is, they are free and finitely generated over Z—so this property is inherited by the kernel K of ε. On the other hand, $e_i\varepsilon\colon e_i A e_i \otimes_Z e_i P \to e_i P$ is an isomorphism by Proposition 5.2, so $e_i K = 0$. Thus K is a finitely generated $A/\langle e_i \rangle$-module and hence finite-dimensional by Lemma 5.1. Since the only finite-dimensional Cohen–Macaulay A-module is 0, we have the result.

Having established that ι is a homological embedding, it is then immediate that $\operatorname{gldim} A/\langle e_i \rangle \leq \operatorname{gldim} A$. That $\operatorname{gldim} A \leq 3$ is a consequence of [43, Prop. 3.7], since A is the endomorphism algebra of the (2-)cluster-tilting object $T \in \operatorname{GP}(B)$, and $\operatorname{GP}(B) \subseteq \operatorname{mod}(B)$ is the category of third syzygy modules since B has Gorenstein dimension at most 3 by Theorem 4.5(1). □

Proposition 5.4. *Let $J = \mathcal{J}(Q, F, W)$ be the frozen Jacobian algebra of a reduced ice quiver with potential (Q, F, W). If J is finite-dimensional and* $\operatorname{gldim} J < \infty$*, then Q has no loops or* 2*-cycles.*

Proof. Since (Q, F, W) is reduced, Q is the Gabriel quiver of J [44, Rem. 3.4]. That Q has no loops then follows from the no-loops theorem [24, 35], without using that J is a frozen Jacobian algebra. However, this latter assumption means that for each $i \in Q_0$, there is an exact sequence

$$\bigoplus_{b \in \mathrm{H}_i^{\mathrm{m}}} J e_{tb} \to \bigoplus_{a \in \mathrm{T}_i} J e_{ha} \to J e_i \to S_i \to 0$$

for $S_i = J e_i / \operatorname{rad}(J) e_i$, the simple J-module supported at vertex i, and this sequence is the start of a projective resolution of S_i [6, Prop. 3.3(b)]. Since Q has no loops, $tb \neq i$ for all $b \in \mathrm{H}_i^{\mathrm{m}}$, so

$$\operatorname{Hom}_J\left(\bigoplus_{b \in \mathrm{H}_i^{\mathrm{m}}} J e_{tb}, S_i\right) = 0.$$

It follows that $\operatorname{Ext}_J^2(S_i, S_i)$, which is a subquotient of this space, is also zero. That Q has no 2-cycles then follows from [20, Prop. 3.11]. □

We now return to $A = \operatorname{End}_B(T)^{\mathrm{op}}$ for T obtained from T_0 by a mutation at a nonprojective indecomposable summand. Recall that $A \cong \mathcal{J}(Q', F', W')$, where (Q', F', W') is obtained by mutation from (Q, F, W) and is thus reduced by definition.

Proposition 5.5. *The quiver Q' has no* 2*-cycles.*

Proof. Let i be a frozen vertex and $e_i \in A$ the corresponding idempotent. As in the proof of Lemma 5.1, the algebra $A/\langle e_i \rangle$ is again a frozen Jacobian algebra. It is finite-dimensional by the same lemma and has finite global dimension by Lemma 5.3. Thus by Proposition 5.4, its quiver has no 2-cycles. Since this quiver is obtained from Q' by deleting the vertex i and all incident arrows, there are no 2-cycles in Q' incident with i. Since D has at least 3 boundary regions, Q' has at least three frozen vertices, and by repeating this process we rule out 2-cycles in Q' altogether. □

Remark 5.6. Our standing assumption that Postnikov diagrams have at least 3 strands is only used here to rule out loops and 2-cycles incident only with frozen vertices. It is not needed to rule out loops and 2-cycles incident with mutable vertices—indeed, if there are fewer than 3 strands, there are no mutable vertices anyway.

We sum up our arguments in the following theorem.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

Theorem 5.7. *Let D be a reduced and connected Postnikov diagram. Let B be the boundary algebra of D, and let $T_D \in \mathrm{GP}(B)$ be the initial cluster-tilting object from Theorem 4.5. If $T \in \mathrm{GP}(B)$ is a cluster-tilting object obtained from T_D by a sequence of mutations, then $\mathrm{End}_B(T)^{\mathrm{op}}$ is isomorphic to the frozen Jacobian algebra of an ice quiver with potential (Q, F, W) such that Q has no loops and no 2-cycles.*

Proof. The results of this section show that the desired properties of T are preserved by mutations. Since they hold in the case $T = T_D$ by Theorem 4.5—reducedness of D implying that Q_D has no loops or 2-cycles—we obtain the result by induction. □

6. Cluster structures and cluster characters

In this section, we recall definitions and results of Buan–Iyama–Reiten–Scott [5] and Fu–Keller [17] on cluster structures and cluster characters. For our applications here, we need to weaken some definitions slightly, since the conclusions of Theorem 5.7 hold only for one mutation class of cluster-tilting objects in the category $\mathrm{GP}(B_D)$ for a suitable Postnikov diagram D, whereas the original definition of a cluster structure requires these properties for all cluster-tilting objects. However, we will see in this section that (unsurprisingly) Fu–Keller's cluster character [17] still provides a bijection between the cluster-tilting objects in this mutation class and the clusters of $\mathscr{A}_D$, and between their indecomposable summands and the cluster variables of $\mathscr{A}_D$.

All categories in this section are assumed to be $\mathbb{K}$-linear for a field $\mathbb{K}$, which is of characteristic zero (as above) and now additionally algebraically closed. Note that the category $\mathrm{GP}(B_D)$ constructed above from a Postnikov diagram has this property for $\mathbb{K}$ the field over which the dimer algebra is defined.

Definition 6.1. Let $\mathcal{E}$ be a Krull–Schmidt category, and let $\mathcal{T} \subseteq \mathcal{E}$ be a full and additively closed subcategory. The quiver $Q(\mathcal{T})$ has vertices given by the set $\operatorname{indec}\mathcal{T}$ of isomorphism classes of indecomposable objects in $\mathcal{T}$, and $\dim(\mathrm{Rad}_{\mathcal{T}}(T,U)/\mathrm{Rad}^2_{\mathcal{T}}(T,U))$ arrows from U to T for each $T, U \in \operatorname{indec}\mathcal{T}$. Here $\mathrm{Rad}_{\mathcal{T}}(T,U)$ consists of nonisomorphisms $T \to U$, and $\mathrm{Rad}^2_{\mathcal{T}}(T,U)$ is the subspace consisting of those expressible as a composition $T \xrightarrow{f} V \xrightarrow{g} U$, where f is not a split monomorphism, g is not a split epimorphism and $V \in \mathcal{T}$. We treat $Q(\mathcal{T})$ as an ice quiver whose frozen vertices are those corresponding to indecomposable projective objects of $\mathcal{E}$, which lie in $\mathcal{T}$.

If $T \in \mathcal{E}$ is an object, we write $Q(T) := Q(\operatorname{add} T)$, noting that this is the ordinary quiver of the semi-perfect algebra $\mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$.

Definition 6.2 (compare to [5, §II.1], [17, Def. 2.4]). Let $\mathcal{E}$ be a Krull–Schmidt stably 2-Calabi–Yau Frobenius category, and let $\mathcal{T}_0 \subseteq \mathcal{E}$ be a cluster-tilting subcategory. We say that $(\mathcal{E}, \mathcal{T}_0)$ has a *cluster structure* if

(1) for each cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{E}$ and each nonprojective indecomposable $X \in \mathcal{T}$, there is a nonprojective indecomposable object X^*, unique up to isomorphism, such that the full and additively closed subcategory $\mathcal{T}' = \mu_X \mathcal{T}$, called the *mutation* of $\mathcal{T}$ at X and having indecomposable objects

$$\operatorname{indec}\mathcal{T}' = \operatorname{indec}\mathcal{T} \setminus \{X\} \cup \{X^*\},$$

is cluster-tilting;

(2) in the situation of (1), there are short exact sequences

$$0 \longrightarrow X^* \longrightarrow U^+ \longrightarrow X \longrightarrow 0$$
$$0 \longrightarrow X \longrightarrow U^- \longrightarrow X^* \longrightarrow 0$$

in which the nonzero maps are minimal $(\mathcal{T} \cap \mathcal{T}')$-approximations (on the left or the right as appropriate);

(3) for any cluster-tilting subcategory $\mathcal{T}$ in the mutation class of $\mathcal{T}_0$ (i.e., related to $\mathcal{T}_0$ by a sequence of mutations as in (1)), the quiver $Q(\mathcal{T})$ has no loops or 2-cycles; and

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

(4) for $\mathcal{T}$ as in (3) and $X \in \mathcal{T}$ nonprojective, the quiver $Q(\mu_X \mathcal{T})$ agrees, up to arrows between frozen vertices, with the Fomin–Zelevinsky mutation $\mu_X Q(\mathcal{T})$.

If T_0 is a cluster-tilting object, then the additive closure $\operatorname{add} T_0$ is a cluster-tilting subcategory, and we say that $(\mathcal{E}, T_0)$ has a cluster structure if $(\mathcal{E}, \operatorname{add} T_0)$ does. Cluster-tilting subcategories $\mathcal{T}$ in the mutation class of $\mathcal{T}_0$, and their objects, are said to be *reachable* from $\mathcal{T}_0$.

Under our assumptions on $\mathcal{E}$, conditions (1) and (2) of this definition are automatic by [5, Thm. II.1.10(a)] (see also [26, Thms. 5.1, 5.3]), but we include them in the definition anyway to aid comparison with [5].

Remark 6.3. In making Definition 6.2, we follow [17], weakening conditions (3) and (4) to restrict to a single mutation class of cluster-tilting subcategories. Thus $\mathcal{E}$ has a cluster structure, in the sense of [17, Def. 2.4], if and only if $\mathcal{E}$ has a cluster-tilting subcategory and $(\mathcal{E}, \mathcal{T})$ has a cluster structure for all such subcategories $\mathcal{T}$. In principle, one could weaken conditions (1) and (2) in a similar way, but this seems to be less useful since these conditions are known to hold as stated in wide generality. The original definition from [5] applies to a larger class of categories, but we will not need this generality here.

Lemma 6.4. *Assume that $(\mathcal{E}, \mathcal{T}_0)$ has a cluster structure. If $\mathcal{T}$ is reachable, and $X \in \mathcal{T}$ is indecomposable and nonprojective, then the object X^* as in Definition 6.2(1) satisfies*

$$\dim \operatorname{Ext}^1_{\mathcal{E}}(X, X^*) = \dim \operatorname{Ext}^1_{\mathcal{E}}(X^*, X) = 1,$$

and the exact sequences in Definition 6.2(2) are not split. Moreover, if we write $B(X, Y)$ for the number of arrows $X \to Y$ in $Q(\mathcal{T})$, the middle terms of the exact sequences in Definition 6.2(2) are

$$U^+ \cong \bigoplus_{Y \in \operatorname{indec} \mathcal{T}} N^{B(X,Y)}, \qquad U^- \cong \bigoplus_{Y \in \operatorname{indec} \mathcal{T}} N^{B(Y,X)}$$

Proof. This is proved in exactly the same way as [17, Lem. 2.2] (in particular, using the assumption that $\mathbb{K}$ is algebraically closed), noting that $Q(\mathcal{T})$ has no loops by Definition 6.2(3) and the assumption that $\mathcal{T}$ is reachable. □

Definition 6.5 (compare to [17, Def. 3.1]). Let $\mathcal{E}$ be a Krull–Schmidt stably 2-Calabi–Yau Frobenius category, and let R be a commutative ring. A *cluster character* is a map $\Phi\colon \operatorname{Ob}(\mathcal{E}) \to R$ such that

(1) if $X \cong X'$, then $\Phi(X) = \Phi(X')$,
(2) we have $\Phi(X \oplus Y) = \Phi(X)\Phi(Y)$ for all $X, Y \in \operatorname{Ob}(\mathcal{E})$, and
(3) if $X, Y \in \operatorname{Ob}(\mathcal{E})$ satisfy $\dim \operatorname{Ext}^1_{\mathcal{E}}(X, Y) = 1$ (and hence $\dim \operatorname{Ext}^1_{\mathcal{E}}(Y, X) = 1$ by the Calabi–Yau property) and

$$0 \longrightarrow X \longrightarrow E^+ \longrightarrow Y \longrightarrow 0$$
$$0 \longrightarrow Y \longrightarrow E^- \longrightarrow X \longrightarrow 0$$

are nonsplit sequences, then

$$\Phi(X)\Phi(Y) = \Phi(E^+) + \Phi(E^-).$$

Below, we will often employ a standard abuse of notation and write $\Phi\colon \mathcal{E} \to R$.

The next definition is due to Fu and Keller [17], following Palu [40] and Caldero–Chapoton [10] for triangulated categories, and describes a standard cluster character on a suitable Frobenius category with a cluster-tilting object.

Definition 6.6 (compare to [17, Rem. 3.5]). Let $\mathcal{E}$ be a Krull–Schmidt stably 2-Calabi–Yau Frobenius category. Assume $T \in \mathcal{E}$ is a cluster-tilting object such that $A = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ is a Noetherian algebra of

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

finite global dimension. The stable endomorphism algebra $\underline{A} = \underline{\mathrm{End}}_{\mathcal{E}}(T)^{\mathrm{op}}$ is finite-dimensional since $\mathcal{E}$ is stably 2-Calabi–Yau, and we view $\operatorname{mod} \underline{A}$ as a subcategory of $\operatorname{mod} A$ via the quotient map. Writing

$$F = \mathrm{Hom}_{\mathcal{E}}(T, -) \colon \mathcal{E} \to \operatorname{mod} A,$$
$$G = \mathrm{Ext}^1_{\mathcal{E}}(T, -) \colon \mathcal{E} \to \operatorname{mod} \underline{A},$$

we define, for each $X \in \mathcal{E}$,

$$\varphi^T(X) = x^{[FX]} \sum_d \chi(\mathrm{Gr}_d(GX)) x^{-[d]} \in \mathbb{C}[\mathrm{K}_0(\operatorname{per} A)]. \tag{6.1}$$

Here the sum is taken over possible dimension vectors d of $\underline{A}$-modules, $\mathrm{Gr}_e(GX)$ denotes the Grassmannian of submodules of GX of dimension vector d, and χ denotes the Euler characteristic. Moreover, $[M]$ denotes the class of an A-module M in the Grothendieck group $\mathrm{K}_0(\operatorname{per} A) = \mathrm{K}_0(\operatorname{mod} A)$, and $[d] = [N] \in \mathrm{K}_0(\operatorname{per} A)$ for any module N of dimension vector d, this class being independent of the choice of N as in the proof of [17, Prop. 3.2].

Remark 6.7. We may assume T is basic and choose a decomposition $T = \bigoplus_{i=1}^n T_i$ of T into indecomposable direct summands. This yields a basis (strictly speaking, a free generating set) $[P_i] = [FT_i]$ of $\mathrm{K}_0(\operatorname{per} A)$ consisting of the classes of indecomposable projective A-modules and allows us to write $\varphi^T(X)$ as a Laurent polynomial in the variables $x_i = x^{[P_i]}$. Since $GT_i = 0$, it follows immediately that $\varphi^T(T_i) = x_i$. This preference for the basis of indecomposable projectives is why we usually write $\mathrm{K}_0(\operatorname{per} A)$ instead of $\mathrm{K}_0(\operatorname{mod} A)$ for the lattice of exponents.

However, thinking instead in terms of $\mathrm{K}_0(\operatorname{mod} A)$, we may use the nondegenerate Euler pairing

$$\mathrm{K}_0(\operatorname{mod} A) \times \mathrm{K}_0(\operatorname{fd} A) \to \mathbb{Z} : ([M], [N]) \mapsto \langle M, N \rangle := \sum_{j \geq 0} \dim \mathrm{Ext}^j_A(M, N)$$

between the Grothendieck groups of finitely generated and finite-dimensional A-modules, well-defined since A has finite global dimension. This pairing exhibits the classes of simple modules $S_i = FT_i / \mathrm{Rad}_{\mathcal{E}}(T, T_i)$ as a basis of $\mathrm{K}_0(\operatorname{fd} A)$ dual to the basis of $\mathrm{K}_0(\operatorname{mod} A) = \mathrm{K}_0(\operatorname{per} A)$ given by the classes of indecomposable projectives $P_i = FT_i$. Thus we may write

$$\varphi^T(X) = \prod_{i=1}^n x_i^{\langle FX, S_i \rangle} \sum_d \chi(\mathrm{Gr}_d(GX)) \prod_{i=1}^n x_i^{-\langle d, S_i \rangle}.$$

Since d is a dimension vector for $\underline{A}$, we have $-\langle d, S_i \rangle = \langle S_i, d \rangle$ by the relative (or internal) Calabi–Yau property of A [31]. In this way, we recover the formula as stated in [17, Rem. 3.5].

Theorem 6.8 (compare to [17, Thm. 3.3]). *For $\mathcal{E}$ and T as in Definition 6.6, the function $\varphi^T : \mathcal{E} \to \mathbb{C}[\mathrm{K}_0(\operatorname{per} A)]$ is a cluster character on $\mathcal{E}$ in the sense of Definition 6.5.*

Proof. The argument given for [17, Thm. 3.3] applies equally well here; while Fu and Keller assume that $\mathcal{E}$ is Hom-finite, our assumption that $A = \mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ is Noetherian is sufficient for their proof to go through. (For example, it is still the case that any finitely generated A-module, such as a finite-dimensional $\underline{A}$-module, is finitely presented.) Indeed, our additional assumption that $\operatorname{gldim} A < \infty$ even allows some parts of the proof in [17] to be simplified. □

Definition 6.9 (compare to [17, Def. 5.1]). Let $\mathcal{E}$ be a Krull–Schmidt stably 2-Calabi–Yau Frobenius category with cluster-tilting object T, and let $\mathscr{A}_Q$ be the cluster algebra determined by an ice quiver Q. We say $(\mathcal{E}, T)$ is a *Frobenius 2-Calabi–Yau realisation* of $\mathscr{A}_Q$ if $(\mathcal{E}, T)$ has a cluster structure in the sense of Definition 6.2 and the quiver $Q(T)$ is isomorphic to Q as an ice quiver, up to arrows between frozen vertices.

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

Note that Definition 6.9 is slightly weaker than [17, Def. 5.1] since we have weakened the definition of a cluster structure. For example, we do not require that $Q(T')$ has no loops or 2-cycles if T' is a nonreachable cluster-tilting object. However, we still have the following important result.

Theorem 6.10 (compare to [17, Thm. 5.4]). *Let $\mathscr{A}_Q$ be a cluster algebra with Frobenius 2-Calabi–Yau realisation $(\mathcal{E}, T)$, and write $A = \mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$. The Grothendieck group $\mathrm{K}_0(\mathrm{per}\, A)$ has a basis given by the classes $[P_i]$, for $i \in Q_0$, of indecomposable projective A-modules, and so we may view $\mathscr{A}_Q$ as a subalgebra of the Laurent polynomial ring $\mathbb{C}[\mathrm{K}_0(\mathrm{per}\, A)]$ by identifying $x_i = x^{[P_i]}$ with the initial cluster variables. Then the cluster character φ^T induces bijections*

(1) *from the set of isomorphism classes of indecomposable projective objects in $\mathcal{E}$ to the set of frozen variables of $\mathscr{A}_Q$,*
(2) *from the set of isomorphism classes of reachable rigid indecomposable nonprojective objects of $(\mathcal{E}, T_0)$ to the set of nonfrozen cluster variables of $\mathscr{A}_Q$, and*
(3) *from the set of isomorphism classes of reachable cluster-tilting objects of $(\mathcal{E}, T_0)$ to the set of clusters of $\mathscr{A}_Q$.*

Proof. This is proved exactly as [17, Thm. 5.4], despite our weakening of the definition of a cluster structure. Indeed, the relevant maps are shown to be injective in [17, Lem. 4.2, Prop. 4.3] without assuming that $\mathcal{E}$ has a cluster structure in either sense. The proof of surjectivity (and that the maps are well-defined: that is, that they take values in the appropriate sets) uses an inductive argument. We have already seen in Remark 6.7 that φ^T takes the indecomposable summands of T to the initial cluster variables x_i of $\mathscr{A}_Q$ (and the indecomposable projective summands to the frozen variables, by the definition of the ice quiver structure on $Q(T) = Q$). The conclusion then follows by using compatibility between mutations of reachable cluster-tilting objects and mutations of clusters in $\mathscr{A}_Q$. For example, the quivers of endomorphism algebras transform correctly by parts (3) and (4) of Definition 6.2, and in the notation of Definition 6.2(2), we have

$$\varphi^T(X)\varphi^T(X^*) = \varphi^T(U^+) + \varphi^T(U^-)$$

by Lemma 6.4 and Theorem 6.8, corresponding to an exchange relation in $\mathscr{A}_Q$ by Lemma 6.4 again. The inductive nature of the argument means we do not require any properties of nonreachable cluster-tilting objects, since these do not appear. □

We close this section by applying our results to the category $\mathrm{GP}(B_D)$ obtained from a suitable connected Postnikov diagram D. The resulting theorem justifies our description of $\mathrm{GP}(B_D)$, with its distinguished mutation class of cluster-tilting objects reachable from T_D, as a categorification of the cluster algebra $\mathscr{A}_D$.

Theorem 6.11. *Let D be a connected Postnikov diagram, let B_D be its boundary algebra (defined over an algebraically closed field $\mathbb{K}$ of characteristic zero), and let $T_D \in \mathrm{GP}(B_D)$ be the cluster-tilting object from Theorem 4.5. Then $(\mathrm{GP}(B_D), T_D)$ is a Frobenius 2-Calabi–Yau realisation of the cluster algebra $\mathscr{A}_D$.*

In particular, the Fu–Keller cluster character φ^{T_D} provides a bijection between rigid indecomposable objects reachable from T_D and cluster variables of $\mathscr{A}_D$ (including a bijection between indecomposable projectives and frozen variables) and between cluster-tilting objects reachable from T_D and clusters of $\mathscr{A}_D$.

Proof. The category $\mathrm{GP}(B_D)$ is a stably 2-Calabi–Yau Frobenius category, and $T_D \in \mathrm{GP}(B_D)$ a cluster-tilting object, by Theorem 4.5. It is idempotent complete and admits a fully faithful embedding to a Krull–Schmidt category by [11, Prop. 3.6] (see Proposition 7.1 below), so it is Krull–Schmidt. Moreover, $(\mathrm{GP}(B_D), T_D)$ has a cluster structure as follows. Conditions (1) and (2) are automatic, as discussed in Definition 6.2. Condition (3) holds by Theorem 5.7, and as a consequence, condition (4) holds by [44, Thm. 5.15]. Since $Q(T_D)$ is isomorphic to Q_D by Theorem 4.5 again, $(\mathrm{GP}(B_D), T_D)$ is a Frobenius 2-Calabi–Yau realisation of $\mathscr{A}_D$, and the required bijections then follow from Theorem 6.10. □

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

7. Relationship to the Grassmannian cluster category

Fix a connected Postnikov diagram D of type (k,n), and abbreviate $A = A_D$, $B = B_D$ and $T = T_D$. To be able to use the results of Section 6, we take the ground field for these algebras to be algebraically closed.

Recall that D determines an ice quiver $Q = Q_D$ with cluster algebra $\mathscr{A} = \mathscr{A}_D$, a positroid $\mathcal{P} = \mathcal{P}_D$ with necklace $\mathcal{I} = \mathcal{I}_D$ and an open positroid variety $\Pi^\circ = \Pi^\circ(\mathcal{P}_D)$ with cone $\widehat{\Pi}^\circ$. In this final section, we explain the relationship between $\mathrm{GP}(B)$ and the cluster category $\mathrm{CM}(C)$ for the Grassmannian Gr_k^n introduced by Jensen, King and Su [28]. As a result, we will see that the composition of the Fu–Keller cluster character φ^T on $\mathrm{GP}(B)$ with Galashin–Lam's isomorphism $\mathscr{A} \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^\circ]$ (modified as in Remark 2.7) is a representation-theoretically natural map.

Proposition 7.1 [11, Prop. 3.6]. *Let $\mathrm{CM}(C)$ be the Grassmannian cluster category for the (k,n)-Grassmannian. Then there is a fully faithful functor $\rho\colon \mathrm{GP}(B) \to \mathrm{CM}(C)$ arising from an injective map $C \to B$ of Z-orders that becomes an isomorphism under tensor product with $Z[t^{-1}]$.*

Recall from Remark 2.7 that we can label each alternating region of D (or equivalently, each vertex $j \in Q_0$) by $I_j \in \binom{[n]}{k}$. Precisely, the alternating region corresponding to j lies to the left of k strands, and I_j consists of the labels of the start-points of these strands. This leads to the specialisation map $\mathbb{C}[x_j^{\pm 1} : j \in Q_0] \to \mathbb{C}(\widehat{\Pi}^\circ)$, taking each x_j to $\Delta_{I_j}|_{\widehat{\Pi}^\circ}$, which restricts to an isomorphism $\mathscr{A}_D \to \mathbb{C}[\widehat{\Pi}^\circ]$ [18]. We write $\Phi\colon \mathrm{GP}(B) \to \mathbb{C}[\widehat{\Pi}^\circ]$ for the composition of the Fu–Keller cluster character $\varphi^T : \mathrm{GP}(B) \to \mathbb{C}[\mathrm{K}_0(\operatorname{per} A)] = \mathbb{C}[x_j^{\pm 1} : j \in Q_0]$ with this specialisation map. Since the indecomposable summand $T_j = eAe_j$ of T satisfies $\varphi^T(T_j) = x_j$ for each $j \in Q_0$, we have $\Phi(T_j) = \Delta_{I_j}|_{\widehat{\Pi}^\circ}$.

For each $I \in \binom{[n]}{k}$, Jensen, King and Su explicitly describe an indecomposable rigid module $M_I \in \mathrm{CM}(C)$. These objects categorify the Plücker coordinates $\Delta_I \in \mathbb{C}[\widehat{\mathrm{Gr}}_k^n]$, all of which are cluster variables. Over the course of this section, we will show that our cluster character Φ is compatible with the embedding ρ. The next proposition is crucial for this.

Proposition 7.2 [27]. *The natural map*

$$\mathrm{Ext}^1_B(X,Y) \to \mathrm{Ext}^1_C(\rho X, \rho Y)$$

is an isomorphism whenever $X \in \mathrm{CM}(B)$ and $Y \in \mathrm{GP}(B)$. In particular, the essential image of ρ is extension-closed.

Proof. The map is well-defined since ρ is exact and injective since ρ is fully faithful; these conclusions hold even under the weaker assumption $Y \in \mathrm{CM}(B)$. For surjectivity, let $P \to X$ be a projective cover of X, and consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \rho Y \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & \rho P \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \rho Y \arrow[r] & E \arrow[r] & \rho X \arrow[r] & 0 \end{tikzcd} \tag{7.1}$$

in which the lower row is an extension in $\mathrm{CM}(C)$ and the right-hand square is a pull-back. Since ρ is fully faithful, to show that the the lower row comes from an extension of B-modules, it suffices to show that E is in the essential image of ρ.

Note that ρB is rigid, as follows. By [11, Prop. 8.2], we have $\rho B \cong \bigoplus_{I \in \mathcal{I}} M_I$. Since the elements of $\mathcal{I}$ are weakly separated [39, Lem. 4.5], the required rigidity follows from [28, Prop. 5.6]. As a consequence, if $Y = B$, then the upper row of equation (7.1) splits, since $P \in \mathrm{add}(B)$. It follows that E is a factor of $\rho B \oplus \rho P$, so it lies in the essential image of ρ by [12, §37, Ex. 10]; here we use the additional properties of ρ from Proposition 7.1 to see that $\rho B \subseteq C$ and $\rho B[t^{-1}] = C[t^{-1}]$. Thus

$$\mathrm{Ext}^1_B(X,B) \xrightarrow{\sim} \mathrm{Ext}^1_C(\rho X, \rho B). \tag{7.2}$$

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

Now, for general $Y \in \mathrm{GP}(B)$, the upper row of equation (7.1) is an element of

$$\mathrm{Ext}^1_C(\rho P, \rho Y) = \mathrm{D}\,\mathrm{Ext}^1_C(\rho Y, \rho P) = \mathrm{D}\,\mathrm{Ext}^1_B(Y, P) = 0,$$

where the first equality is the 2-Calabi–Yau property of $\mathrm{CM}(C)$, the second follows from equation (7.2) since $P \in \mathrm{add}\, B$, and the last uses that $P \in \mathrm{add}(B)$ and $Y \in \mathrm{GP}(B)$. Thus this upper row once again splits, and we conclude as before that E is in the essential image of ρ. □

Proposition 7.3 [11, Prop. 8.2]. *Let $j \in Q_0$, and let $T_j = eAe_j \in \mathrm{GP}(B)$ be the corresponding indecomposable summand of the initial cluster tilting object $T = eA$. Then $\rho T_j \cong M_{I_j}$.*

Over the course of the next several results, we will show that $\Phi(M) = \Delta_I|_{\widehat{\Pi}^\circ}$ whenever $\rho(M) \cong M_I$. We begin by recalling Jensen–King–Su's cluster character on $\mathrm{CM}(C)$ and relating it to our cluster character on $\mathrm{GP}(B)$.

Proposition 7.4 [28, Eq. 9.4]. *There is a cluster character $\Psi\colon \mathrm{CM}(C) \to \mathbb{C}[\widehat{\mathrm{Gr}}^n_k]$ such that $\Psi(M_I) = \Delta_I$ for every k-subset I.*

Proposition 7.5. *For any reachable rigid object $X \in \mathrm{GP}(B)$, we have*

$$\Phi(X) = \Psi(\rho X)|_{\widehat{\Pi}^\circ}.$$

Proof. First, observe that $X \mapsto \Psi(\rho X)|_{\widehat{\Pi}^\circ}$ defines a cluster character $\mathrm{GP}(B) \to \mathbb{C}[\widehat{\Pi}^\circ]$ since Ψ is a cluster character on $\mathrm{CM}(C)$; this uses both Proposition 7.1 and, to see that the restricted map still satisfies the multiplication formula from Definition 6.5(3), Proposition 7.2.

Thus it suffices to check the required equality on the indecomposable summands of a single reachable cluster-tilting object, which we may take to be the initial object T. For each $j \in Q_0$, the summand $T_j = eAe_j$ satisfies

$$\Psi(\rho T_j)|_{\widehat{\Pi}^\circ} = \Psi(M_{I_j})|_{\widehat{\Pi}^\circ} = \Delta_{I_j}|_{\widehat{\Pi}^\circ} = \Phi(T_j)$$

by Propositions 7.3 and 7.4. □

Theorem 7.6. *If $M \in \mathrm{GP}(B)$ satisfies $\rho M \cong M_I$ for some $I \in \binom{[n]}{k}$, then $\Phi(M) = \Delta_I|_{\widehat{\Pi}^\circ}$.*

Proof. By Proposition 7.4, we have $\Delta_I = \Psi(M_I) = \Psi(\rho M)$. Thus by Proposition 7.5, it suffices to check that M is reachable—it is rigid by Proposition 7.2 (see also the proof of [11, Lem. 10.4]).

Consider a geometric exchange at an alternating region j of D (corresponding to a mutation at the vertex j of Q), producing a new diagram D'. By considering the exchange sequences as in Definition 6.2(2), we may see that the mutation T'_j of the direct summand T_k of T satisfies $\rho(T'_j) = M_{I'_j}$. Since ρ is fully faithful, T'_j is also the unique such object in $\mathrm{GP}(B)$ up to isomorphism. Inductively, it follows that $M \in \mathrm{GP}(B)$ with $\rho(M) \cong M_I$ is reachable as long as I appears as a label in a Postnikov diagram related to D by a sequence of geometric exchanges. By [39, Thm. 6.6], such subsets I are precisely those appearing in some maximal weakly separated collection $\mathcal{C}$ with $\mathcal{I} \subseteq \mathcal{C} \subseteq \mathcal{P}$: that is, they are those $I \in \mathcal{P}$, which are weakly separated from every element of $\mathcal{I}$.

So assume $M \in \mathrm{GP}(B)$ has $\rho(M) \cong M_I$. Since $M \in \mathrm{CM}(B)$, it follows from [11, Prop. 8.6] that $I \in \mathcal{P}_D$. Since M is even Gorenstein projective, it follows from Proposition 7.2 that

$$0 = \mathrm{Ext}^1_B(M, T_j) = \mathrm{Ext}^1_C(M_I, M_{I_j})$$

for $T_j = eA_De_j$ with j a frozen vertex, these being precisely the indecomposable summands of B. By [28, Prop. 5.6], this means precisely that I and I_j are weakly separated. Since $\mathcal{I}$ is the set of labels I_j of frozen vertices j, we have the result. □

Remark 7.7. We do not yet know whether $\Phi(X) = \Psi(\rho X)|_{\widehat{\Pi}^\circ}$ for all $X \in \mathrm{GP}(B)$. However, we expect that this equality does hold, even for all $X \in \mathrm{CM}(B)$; note that the Fu–Keller cluster character φ^T, and

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press

hence Φ, may be defined on this larger category via the same formula in equation (6.1), although it is only a cluster character on the stably 2-Calabi–Yau Frobenius category $\mathrm{GP}(B)$. If this were the case, the conclusion of Theorem 7.6 would hold under the weaker assumption that $M \in \mathrm{CM}(B)$. The k-subsets I appearing in the resulting statement would then be all of the elements of the positroid $\mathcal{P}$, including those not weakly separated from the necklace, by [11, Prop. 8.6].

Acknowledgements. Some important realisations concerning the results of this paper were made during discussions with İlke Çanakçı and Alastair King concerning our work in [11], and we are grateful to them for the inspiration. We also thank Chris Fraser, Bernt Tore Jensen, Andrea Pasquali, Khrystyna Serhiyenko, Melissa Sherman-Bennett and Lauren Williams for useful conversations. Additional thanks are due to Bernt Tore Jensen for informing us of the statement and proof of Proposition 7.2.

Conflicts of Interest. None.

Funding statement. The author was supported by the EPSRC postdoctoral fellowship EP/T001771/1 while expanding and revising this paper. The author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Cluster Algebras and Representation Theory*, supported by EPSRC grant no. EP/R014604/1, where some of these revisions were undertaken.

References

[1] C. Amiot. "Cluster categories for algebras of global dimension 2 and quivers with potential". *Ann. Inst. Fourier (Grenoble)* **59**.6 (2009), pp. 2525–2590. arXiv:0805.1035 [math.RT].

[2] K. Baur, A. D. King, and B. R. Marsh. "Dimer models and cluster categories of Grassmannians". *Proc. Lond. Math. Soc.* **113**.2 (2016), pp. 213–260. arXiv:1309.6524 [math.RT].

[3] N. Broomhead. "Dimer models and Calabi–Yau algebras". *Mem. Amer. Math. Soc.* **215**.1011 (2012). arXiv:0901.4662 [math.AG].

[4] K. A. Brown, K. R. Goodearl, and M. Yakimov. "Poisson structures on affine spaces and flag varieties. I. Matrix affine Poisson space". *Adv. Math.* **206**.2 (2006), pp. 567–629. arXiv:math/0501109 [math.QA].

[5] A. B. Buan, O. Iyama, I. Reiten, and J. Scott. "Cluster structures for 2-Calabi–Yau categories and unipotent groups". *Compos. Math.* **145**.4 (2009), pp. 1035–1079. arXiv:math/0701557 [math.RT].

[6] A. B. Buan, O. Iyama, I. Reiten, and D. Smith. "Mutation of cluster-tilting objects and potentials". *Amer. J. Math.* **133**.4 (2011), pp. 835–887. arXiv:0804.3813 [math.RT].

[7] R.-O. Buchweitz. *Maximal Cohen–Macaulay modules and Tate cohomology*. Vol. 262. Mathematical Surveys and Monographs. With appendices and an introduction by Luchezar L. Avramov, Benjamin Briggs, Srikanth B. Iyengar and Janina C. Letz. American Mathematical Society, Providence, RI, 2021.

[8] T. Bühler. "Exact categories". *Expo. Math.* **28**.1 (2010), pp. 1–69. arXiv:0811.1480 [math.HO].

[9] M. C. R. Butler and A. King. "Minimal resolutions of algebras". *J. Algebra* **212**.1 (1999), pp. 323–362.

[10] P. Caldero and F. Chapoton. "Cluster algebras as Hall algebras of quiver representations". *Comment. Math. Helv.* 81.3 (2006), pp. 595–616. arXiv:math/0410187 [math.RT].

[11] İ. Çanakçı, A. King, and M. Pressland. *Perfect matchings, dimer partition functions and cluster characters*. 2021. arXiv:2106.15924v2 [math.RT].

[12] C. W. Curtis and **I**. Reiner. *Methods of representation theory with applications to finite groups and orders*. *Vol. I*. Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1981.

[13] B. Davison. "Consistency conditions for brane tilings". *J. Algebra* **338** (2011), pp. 1–23. arXiv:0812.4185 [math.AG].

[14] L. Demonet and X. Luo. "Ice quivers with potential associated with triangulations and Cohen–Macaulay modules over orders". *Trans. Amer. Math. Soc.* **368**.6 (2016), pp. 4257–4293. arXiv:1307.0676 [math.RT].

[15] H. Derksen, J. Weyman, and A. Zelevinsky. "Quivers with potentials and their representations. I. Mutations". *Selecta Math. (N.S.)* **14**.1 (2008), pp. 59–119. arXiv:0704.0649 [math.RA].

[16] S. Fomin and A. Zelevinsky. "Cluster algebras. I. Foundations". *J. Amer. Math. Soc.* **15**.2 (2002), pp. 497–529. arXiv:math/0104151 [math.RT].

[17] C. Fu and B. Keller. "On cluster algebras with coefficients and 2-Calabi-Yau categories". *Trans. Amer. Math. Soc.* **362**.2 (2010), pp. 859–895. arXiv:0710.3152 [math.RT].

[18] P. Galashin and T. Lam. *Positroid varieties and cluster algebras.* To appear in *Ann. Sci. École Norm. Sup.* 2019. arXiv:1906.03501v2 [math.CO].

[19] C. Geiß, B. Leclerc, and J. Schröer. "Partial flag varieties and preprojective algebras". *Ann. Inst. Fourier (Grenoble)* **58**.3 (2008), pp. 825–876. arXiv:math/0609138 [math.RT].

[20] C. Geiß, B. Leclerc, and J. Schröer. "Rigid modules over preprojective algebras". *Invent. Math.* **165**.3 (2006), pp. 589–632. arXiv:math/0503324 [math.RT].

[21] K. R. Goodearl and M. Yakimov. "Poisson structures on affine spaces and flag varieties. II". *Trans. Amer. Math. Soc.* **361**.11 (2009), pp. 5753–5780. arXiv:math/0509075 [math.QA].

[22] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*. London Math. Soc. Lecture Note Ser. **119**. Cambridge: Cambridge University Press, 1988.

[23] H. Holm. "Gorenstein homological dimensions". *J. Pure Appl. Algebra* **189**.1-3 (2004), pp. 167–193.

[24] K. Igusa. "Notes on the no loops conjecture". *J. Pure Appl. Algebra* **69**.2 (1990), pp. 161–176.

[25] Y. Iwanaga. "On rings with finite self-injective dimension. II". *Tsukuba J. Math.* **4**.1 (1980), pp. 107–113.

[26] O. Iyama and Y. Yoshino. "Mutation in triangulated categories and rigid Cohen–Macaulay modules". *Invent. Math.* **172**.1 (2008), pp. 117–168. arXiv:math/0607736 [math.RT].

[27] B. T. Jensen, A. D. King, and X. Su. In preparation.

[28] B. T. Jensen, A. D. King, and X. Su. "A categorification of Grassmannian cluster algebras". *Proc. Lond. Math. Soc.* **113**.2 (2016), pp. 185–212. arXiv:1309.7301 [math.RT].

[29] B. Keller. "Cluster algebras, quiver representations and triangulated categories". *Triangulated categories*. London Math. Soc. Lecture Note Ser. 375. Cambridge: Cambridge Univ. Press, 2010, pp. 76–160. arXiv:0807.1960 [math.RT].

[30] B. Keller and I. Reiten. "Acyclic Calabi–Yau categories". *Compos. Math.* **144**.5 (2008), pp. 1332–1348. arXiv:math/0610594 [math.RT]. Appendix by M. Van den Bergh.

[31] B. Keller and I. Reiten. "Cluster-tilted algebras are Gorenstein and stably Calabi–Yau". *Adv. Math.* **211**.1 (2007), pp. 123–151. arXiv:math/0512471 [math.RT].

[32] A. Knutson, T. Lam, and D. E. Speyer. "Positroid varieties: juggling and geometry". *Compos. Math.* **149**.10 (2013), pp. 1710–1752. arXiv:0903.3694 [math.AG].

[33] B. Leclerc. "Cluster structures on strata of flag varieties". *Adv. Math.* **300** (2016), pp. 190–228. arXiv:1402.4435 [math.RT].

[34] B. Leclerc and A. Zelevinsky. "Quasicommuting families of quantum Plücker coordinates". In: *Kirillov's seminar on representation theory*. Amer. Math. Soc. Transl. Ser. 2 **181**. Providence, RI: American Mathematical Society, 1998, pp. 85–108.

[35] H. Lenzing. "Nilpotente Elemente in Ringen von endlicher globaler Dimension". *Math. Z.* **108** (1969), pp. 313–324.

[36] G. Lusztig. "Total positivity in partial flag manifolds". *Represent. Theory* **2** (1998), pp. 70–78.

[37] G. Muller and D. E. Speyer. "Cluster algebras of Grassmannians are locally acyclic". *Proc. Amer. Math. Soc.* **144**.8 (2016), pp. 3267–3281. arXiv:1401.5137 [math.CO].

[38] G. Muller and D. E. Speyer. "The twist for positroid varieties". *Proc. Amer. Math. Soc.* **115**.5 (2017), pp. 1014–1071. arXiv:1606.08383 [math.CO].

[39] S. Oh, A. Postnikov, and D. E. Speyer. "Weak separation and plabic graphs". *Proc. Lond. Math. Soc.* **110**.3 (2015), pp. 721–754. arXiv:1109.4434 [math.CO].

[40] Y. Palu. "Cluster characters for 2-Calabi–Yau triangulated categories". *Ann. Inst. Fourier (Grenoble)* **58**.6 (2008), pp. 2221–2248. arXiv:math/0703540 [math.RT].

[41] A. Postnikov. *Total positivity, Grassmannians, and networks*. 2006. arXiv:math/0609764v1 [math.CO].

[42] M. Pressland. *A categorification of acyclic principal coefficient cluster algebras*. 2017. arXiv:1702.05352v3 [math.RT].

[43] M. Pressland. "Internally Calabi–Yau algebras and cluster-tilting objects". *Math. Z.* **287**.1-2 (2017), pp. 555–585. arXiv:1510.06224 [math.RT].

[44] M. Pressland. "Mutation of frozen Jacobian algebras". *J. Algebra* **546** (2020), pp. 236–273. arXiv:1810.01179 [math.RT].

[45] C. Psaroudakis. "Homological theory of recollements of abelian categories". *J. Algebra* **398** (2014), pp. 63–110.

[46] K. Rietsch. "Closure relations for totally nonnegative cells in G/P". *Math. Res. Lett.* **13**.5-6 (2006), pp. 775–786. arXiv:math/0509137 [math.AG].

[47] J. S. Scott. "Grassmannians and cluster algebras". *Proc. London Math. Soc. (3)* **92**.2 (2006), pp. 345–380. arXiv:math/0311148 [math.CO].

[48] K. Serhiyenko, M. Sherman-Bennett, and L. Williams. "Cluster structures in Schubert varieties in the Grassmannian". *Proc. Lond. Math. Soc. (3)* **119**.6 (2019), pp. 1694–1744. arXiv:1902.00807 [math.CO].

[49] J. Vitória. "Mutations vs. Seiberg duality". *J. Algebra* 321.3 (2009), pp. 816–828. arXiv:0709.3939 [math.RA].

https://doi.org/10.1017/fms.2022.52 Published online by Cambridge University Press