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Abstract simplicity of locally compact Kac–Moody groups

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Abstract

In this paper, we establish that complete Kac–Moody groups over finite fields are abstractly simple. The proof makes essential use of Mathieu and Rousseau's construction of complete Kac–Moody groups over fields. This construction has the advantage that both real and imaginary root spaces of the Lie algebra lift to root subgroups over arbitrary fields. A key point in our proof is the fact, of independent interest, that both real and imaginary root subgroups are contracted by conjugation of positive powers of suitable Weyl group elements.

1. Introduction

Let $A = (A_{ij})_{1\leqslant i,j\leqslant n}$ be a generalised Cartan matrix and let $\mathfrak{G} = \mathfrak{G}_A$ denote the associated Kac–Moody–Tits functor of simply connected type, as defined by Tits [Tit87]. The value of $\mathfrak{G}$ over a field k is usually called a *minimal Kac–Moody group* of type A over k. This terminology is justified by the existence of larger groups associated with the same data, usually called *maximal* or *complete Kac–Moody groups*, and which are completions of $\mathfrak{G}(k)$ with respect to some suitable topology. One of them, introduced in [RR06], and which we will temporarily denote by $\hat{\mathfrak{G}}_A(k)$, is a totally disconnected topological group. It is, moreover, locally compact provided k is finite, and non-discrete (hence uncountable) as soon as A is not of finite type.

The question whether $\hat{\mathfrak{G}}_A(k)$ is (abstractly) simple for A indecomposable and k arbitrary is very natural and was explicitly addressed by Tits [Tit89]. Abstract simplicity results for $\hat{\mathfrak{G}}_A(k)$ over fields of characteristic 0 were first obtained in an unpublished note by Moody [Moo82]. Moody's proof has recently been generalised by Rousseau [Rou12, Théorème 6.19] who extended Moody's result to fields k of positive characteristic p that are not algebraic over $\mathbf{F}_p$. The abstract simplicity of $\hat{\mathfrak{G}}_A(k)$ when k is a finite field was shown in [CER08] in some important special cases, including groups of 2-spherical type over fields of order at least 4, as well as some other hyperbolic types under additional restrictions on the order of the ground field.

In this paper, we establish the abstract simplicity of complete Kac–Moody groups $\hat{\mathfrak{G}}_A(k)$ of indecomposable type over arbitrary finite fields, without any restriction. Our proof relies on an approach which is completely different from that used in [CER08].

Theorem A. *Let $\hat{\mathfrak{G}}_A(\mathbf{F}_q)$ be a complete Kac–Moody group over a finite field $\mathbf{F}_q$, with generalised Cartan matrix A. Assume that A is indecomposable of indefinite type. Then $\hat{\mathfrak{G}}_A(\mathbf{F}_q)$ is abstractly simple.*

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As it turns out, it does not matter which completion of $\mathfrak{G}_A(\mathbf{F}_q)$ we are considering; see Theorem B and Remark 1 below.

After completion of this work, I was informed by Bertrand Rémy that, in a recent joint work [CR13] with I. Capdeboscq, they obtained independently a special case of this theorem, namely the abstract simplicity over finite fields of order at least 4 and of characteristic p in the case where p is greater than $M = \max_{i \neq j} |A_{ij}|$. Their approach is similar to that used in [CER08].

Note that the topological simplicity of $\hat{\mathfrak{G}}_A(\mathbf{F}_q)$ (that is, the absence of non-trivial closed normal subgroups), which we will use in our proof of Theorem A, was previously established by Rémy when $q > 3$ (see [Rém04, Theorem 2.A.1]); the tiniest finite fields were later covered by Caprace and Rémy (see [CR09, Proposition 11]).

Note also that for incomplete groups, abstract simplicity fails in general since groups of affine type admit numerous congruence quotients. However, it has been shown by Caprace and Rémy [CR09] that $\mathfrak{G}_A(\mathbf{F}_q)$ is abstractly simple provided A is indecomposable, $q > n > 2$ and A is not of affine type. They also recently covered the rank 2 case for matrices A of the form $A = \left(\begin{smallmatrix} 2 & -m \\ -1 & 2 \end{smallmatrix}\right)$ with $m > 4$ (see [CR12, Theorem 2]).

As mentioned at the beginning of this introduction, different completions of $\mathfrak{G}(k)$ have been considered in the literature, and therefore all deserve the name 'complete Kac–Moody groups'. We now proceed to review them briefly.

Essentially three such completions have been constructed so far, from very different points of view. The first construction, due to Rémy and Ronan [RR06], is the one we considered above. It is the completion of the image of $\mathfrak{G}(k)$ in the automorphism group $\mathrm{Aut}(X_+)$ of its associated positive building X_+, where $\mathrm{Aut}(X_+)$ is equipped with the compact-open topology. For the rest of this paper, we will denote this group by $\mathfrak{G}^{rr}(k)$, so that $\hat{\mathfrak{G}}(k) = \mathfrak{G}^{rr}(k)$ in our previous notation. To avoid taking a quotient of $\mathfrak{G}(k)$, a variant of this group has also been considered by Caprace and Rémy [CR09, § 1.2]. This latter group, here denoted $\mathfrak{G}^{crr}(k)$, contains $\mathfrak{G}(k)$ as a dense subgroup and admits $\mathfrak{G}^{rr}(k)$ as a quotient.

The second construction, due to Carbone and Garland [CG03], associates to a regular dominant integral weight λ the completion, here denoted by $\mathfrak{G}^{cg\lambda}(k)$, of $\mathfrak{G}(k)$ for the so-called weight topology.

The third construction, of which we will make essential use, was first introduced by Mathieu [Mat88] and further developed by Rousseau [Rou12]. It is more algebraic and closer in spirit to the construction of $\mathfrak{G}$. In fact, one gets a group functor over the category of $\mathbf{Z}$-algebras, which we will subsequently denote by $\mathfrak{G}^{pma}$. As noted in [Rou12, 3.20], this functor is a generalisation of the complete Kac–Moody group over $\mathbf{C}$ constructed by Kumar [Kum02, § 6.1.6]. Note that in this case the closure $\overline{\mathfrak{G}(k)}$ of $\mathfrak{G}(k)$ in $\mathfrak{G}^{pma}(k)$ need not be the whole of $\mathfrak{G}^{pma}(k)$. However, $\overline{\mathfrak{G}(k)} = \mathfrak{G}^{pma}(k)$ as soon as the characteristic of k is zero or greater than the maximum M (in absolute value) of the non-diagonal entries of A (see [Rou12, Proposition 6.11]).

These three constructions are strongly related, and hopefully equivalent. In particular, they all possess refined Tits systems whose associated building is the positive building X_+ of $\mathfrak{G}(k)$ (with possibly different apartment systems). Moreover, there are natural continuous group homomorphisms $\overline{\mathfrak{G}(k)} \to \mathfrak{G}^{cg\lambda}(k)$ and $\mathfrak{G}^{cg\lambda}(k) \to \mathfrak{G}^{crr}(k)$ extending the identity on $\mathfrak{G}(k)$ (see [Rou12, 6.3]). Their composition $\phi\colon \overline{\mathfrak{G}(k)} \to \mathfrak{G}^{cg\lambda}(k) \to \mathfrak{G}^{crr}(k)$ is an isomorphism of topological groups in many cases (see [Rou12, Théorème 6.12]) and conjecturally in all cases.

If G is either $\mathfrak{G}^{pma}(k)$ or $\overline{\mathfrak{G}(k)}$ or $\mathfrak{G}^{cg\lambda}(k)$ or $\mathfrak{G}^{crr}(k)$, we let $Z'(G)$ denote the kernel of the G-action on X_+. As mentioned in Remark 1 below, Theorem A immediately implies the abstract simplicity of $G/Z'(G)$ whenever G is one of $\overline{\mathfrak{G}(k)}$ or $\mathfrak{G}^{cg\lambda}(k)$ or $\mathfrak{G}^{crr}(k)$ (and k is finite). As pointed out to me by Pierre-Emmanuel Caprace, our arguments in fact also imply the abstract simplicity of $\mathfrak{G}^{pma}(k)/Z'(\mathfrak{G}^{pma}(k))$, even when $\overline{\mathfrak{G}(k)} \neq \mathfrak{G}^{pma}(k)$.

Theorem B. *Assume that the generalised Cartan matrix A is indecomposable of indefinite type. Then $\mathfrak{G}_A^{pma}(\mathbf{F}_q)/Z'(\mathfrak{G}_A^{pma}(\mathbf{F}_q))$ is abstractly simple over any finite field $\mathbf{F}_q$.*

Note that even the topological simplicity of $\mathfrak{G}_A^{pma}(\mathbf{F}_q)/Z'(\mathfrak{G}_A^{pma}(\mathbf{F}_q))$ was not previously known in full generality (see [Rou12, Lemme 6.14 and Proposition 6.16] for known results).

While the Rémy–Ronan construction is more geometric in nature, the Mathieu–Rousseau construction is purely algebraic and hence *a priori* more suitable for establishing algebraic properties of complete Kac–Moody groups. The present paper is a good illustration of this idea, and we hope it provides a strong motivation for studying these 'algebraic completions' further.

Remark 1. When the field k is finite, the several group homomorphisms $\overline{\mathfrak{G}(k)} \to \mathfrak{G}^{cg\lambda}(k) \to \mathfrak{G}^{crr}(k) \to \mathfrak{G}^{rr}(k) \leqslant \mathrm{Aut}(X_+)$ are all surjective (see [Rou12, 6.3]), and if G is either $\overline{\mathfrak{G}(k)}$ or $\mathfrak{G}^{cg\lambda}(k)$ or $\mathfrak{G}^{crr}(k)$, the effective quotient of G by the kernel $Z'(G)$ of its action on X_+ coincides with $\mathfrak{G}^{rr}(k)$. If, moreover, the characteristic p of k is greater than the maximum M (in absolute value) of the non-diagonal entries of A, one has $\overline{\mathfrak{G}(k)} = \mathfrak{G}^{pma}(k)$, and thus in that case there is only one simple group $G/Z'(G)$. Hence Theorem B is a consequence of Theorem A when $p > M$. If $p \leqslant M$, it is possible that the effective quotient of $\mathfrak{G}^{pma}(k)$ inside $\mathrm{Aut}(X_+)$ properly contains $\mathfrak{G}^{rr}(k)$ (see Corollary F below). When this happens, Theorem B thus asserts the abstract simplicity of a different group than the one considered in Theorem A.

Finally, we notice that, although we assumed the Kac–Moody groups to be of simply connected type to simplify the notation, the results remain valid for an arbitrary Kac–Moody root datum (see Remark 6.3).

In the proof of Theorems A and B, we establish other results of independent interest, which we now proceed to describe.

Let k be an arbitrary field. Fix a realisation of the generalised Cartan matrix $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ as in [Kac90, § 1.1]. Let $Q = \sum_{i=1}^n \mathbf{Z}\alpha_i$ be the associated root lattice, where $\alpha_1, \dots, \alpha_n$ are the simple roots. Let also Δ (respectively, $\Delta_\pm$) be the set of roots (respectively, positive/negative roots), so that $\Delta = \Delta_+ \sqcup \Delta_-$. Write also Δ^{re} and Δ^{im} (respectively, Δ_+^{re} and Δ_+^{im}) for the set of (positive) real and imaginary roots.

Recall that a subset Ψ of Δ is *closed* if $\alpha + \beta \in \Psi$ whenever $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Delta$. For a closed subset Ψ of Δ_+, define the subgroup scheme $\mathfrak{U}_\Psi^{ma}$ of $\mathfrak{G}^{pma}$ as in [Rou12, 3.1]. Set $\mathfrak{U}^{ma+} = \mathfrak{U}_{\Delta_+}^{ma}$. One can then define *root groups* $\mathfrak{U}_{(\alpha)}^{ma}$ in $\mathfrak{U}^{ma+}$ by setting $\mathfrak{U}_{(\alpha)}^{ma} := \mathfrak{U}_{\{\alpha\}}^{ma}$ for $\alpha \in \Delta_+^{\mathrm{re}}$ and $\mathfrak{U}_{(\alpha)}^{ma} := \mathfrak{U}_{\mathbf{N}^*\alpha}^{ma}$ for $\alpha \in \Delta_+^{\mathrm{im}}$, where $\mathbf{N}^* = \mathbf{N}\backslash\{0\}$.

We also let $\mathfrak{B}^+$, $\mathfrak{U}^+$, $\mathfrak{N}$ and $\mathfrak{T}$ denote, as in [Rou12, 1.6], the sub-functors of $\mathfrak{G} = \mathfrak{G}_A$ such that over k, $(\mathfrak{B}^+(k) = \mathfrak{U}^+(k) \rtimes \mathfrak{T}(k), \mathfrak{N}(k))$ is the canonical positive BN-pair attached to $\mathfrak{G}(k)$, and $\mathfrak{N}(k)/\mathfrak{T}(k) \cong W$, where $W = W(A)$ is the Coxeter group attached to A. We fix once for all a section $W \cong \mathfrak{N}(k)/\mathfrak{T}(k) \to \mathfrak{N}(k) : w \mapsto \overline{w}$. Note that $\mathfrak{N}$ can be viewed as a sub-functor of G^{pma} (see [Rou12, 3.12, Remarque 1]).

Finally, given a topological group H and an element $a \in H$, we define the *contraction group* $\mathrm{con}^H(a)$, or simply $\mathrm{con}(a)$, as the set of elements $g \in H$ such that $a^n g a^{-n} \xrightarrow{n\to\infty} 1$. Note then that for any $a \in \overline{\mathfrak{G}(k)} \subseteq \mathfrak{G}^{pma}(k)$, one has $\varphi(\mathrm{con}^{\mathfrak{G}^{pma}(k)}(a) \cap \overline{\mathfrak{G}(k)}) \subseteq \mathrm{con}^{\mathfrak{G}^{rr}(k)}(\varphi(a))$, where we denote by φ the composition $\overline{\mathfrak{G}(k)} \xrightarrow{\phi} \mathfrak{G}^{crr}(k) \to \mathfrak{G}^{rr}(k)$.

Theorem C. *Let k be an arbitrary field.*

(i) *Let $\omega \in W$ and let $\Psi \subseteq \Delta_+$ be a closed set of positive roots such that $\omega\Psi \subseteq \Delta_+$. Then $\overline{\omega}\mathfrak{U}^{ma}_{\Psi}\overline{\omega}^{-1} = \mathfrak{U}^{ma}_{\omega\Psi}$.*

(ii) *Let $\omega \in W$ and $\alpha \in \Delta_+$ be such that $\omega^l\alpha$ is positive and different from α for all positive integers l. Then $\mathfrak{U}^{ma}_{(\alpha)} \subseteq \mathrm{con}^{\mathfrak{G}^{pma}(k)}(\overline{\omega})$. In particular, $\varphi(\mathfrak{U}^{ma}_{(\alpha)} \cap \overline{\mathfrak{G}(k)}) \subseteq \mathrm{con}^{\mathfrak{G}^{rr}(k)}(\overline{\omega})$.*

(iii) *Assume that A is of indefinite type. Then there exists some $\omega \in W$ such that $\mathfrak{U}^{ma}_{(\alpha)} \subseteq \mathrm{con}^{\mathfrak{G}^{pma}(k)}(\overline{\omega}) \cup \mathrm{con}^{\mathfrak{G}^{pma}(k)}(\overline{\omega}^{-1})$ for all $\alpha \in \Delta_+$. Hence root groups (associated to both real and imaginary roots) are contracted.*

The proof of Theorem C can be found at the end of Section 4. The idea behind proving Theorem A once Theorem C is established is as follows. We let $a \in \mathfrak{N}(\mathbf{F}_q)$ be such that $\mathfrak{U}^{ma}_{(\alpha)}(\mathbf{F}_q) \subseteq \mathrm{con}^{\mathfrak{G}^{pma}(\mathbf{F}_q)}(a) \cup \mathrm{con}^{\mathfrak{G}^{pma}(\mathbf{F}_q)}(a^{-1})$ for all $\alpha \in \Delta_+$, as in Theorem C(iii). We deduce that $\mathfrak{U}^{rr+}(\mathbf{F}_q)$ is contained in the subgroup generated by the closures of $\mathrm{con}^{\mathfrak{G}^{rr}(\mathbf{F}_q)}(a^{\pm 1})$. Now, as the topological simplicity of $\mathfrak{G}^{rr}(\mathbf{F}_q)$ is known, it suffices to consider a dense normal subgroup K of $\mathfrak{G}^{rr}(\mathbf{F}_q)$. We can then conclude by invoking the following result of Caprace, Reid and Willis [CRW13, Theorem 1.1].

Theorem 1.1. *Let G be a totally disconnected locally compact group and let K be a dense normal subgroup of G. Then K contains the closure in G of $\mathrm{con}(g)$ for any $g \in G$.*

The proof of Theorem B follows the exact same lines, except that in this case the topological simplicity of the group is not known in full generality, and we need one more argument to establish it.

We also point out that Theorem C has another application, concerning the existence of non-closed contraction groups in complete Kac–Moody groups of non-affine type. Recall that in simple algebraic groups over local fields, contraction groups are always closed (in fact they either are trivial or coincide with the unipotent radical of some parabolic subgroup). In particular, they are closed in a complete Kac–Moody group G over a finite field as soon as the defining generalised Cartan matrix A is of non-twisted affine type. It has been shown in [BRR08] that, on the other hand, if A is indecomposable non-spherical, non-affine and of size at least 3, then the contraction group $\mathrm{con}(a)$ of some element $a \in G$ must be non-closed. The following result shows that this also holds when A is indecomposable non-spherical, non-affine and of size 2.

Theorem D. *Let A denote an $n \times n$ generalised Cartan matrix of indecomposable indefinite type, let $W = W(A)$ be the associated Weyl group, and let $w = s_1 \dots s_n$ denote the Coxeter element of W. Let also G be one the complete Kac–Moody groups $\mathfrak{G}^{rr}_A(\mathbf{F}_q)$ or $\mathfrak{G}^{pma}_A(\mathbf{F}_q)$ of simply connected type. Then the contraction group $\mathrm{con}^G(w)$ is not closed in G, unless perhaps if $G = \mathfrak{G}^{rr}_A(\mathbf{F}_q)$ and $\mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(\mathbf{F}_q) \cap \overline{\mathfrak{G}(\mathbf{F}_q)}$ is contained in the kernel of φ.*

Here is a final application of our results concerning isomorphism classes of Kac–Moody groups and their completions. While over infinite fields, it is known that two minimal Kac–Moody groups can be isomorphic only if their ground fields are isomorphic and their underlying generalised Cartan matrices coincide up to a row–column permutation (see [Cap09, Theorem A]), this fails to be true over finite fields. Indeed, over a given finite field, two minimal Kac–Moody groups associated with two different generalised Cartan matrices of size 2 can be isomorphic, as noticed in [Cap09, Lemma 4.3]. The following result shows, however, that the corresponding Mathieu–Rousseau completions should not be expected to be isomorphic as topological groups.

Theorem E. *Let $k = \mathbf{F}_q$ be a finite field with* $\operatorname{char} k \neq 2$*. Then there exist minimal Kac–Moody groups $G_1 = \mathfrak{G}_{A_1}(\mathbf{F}_q)$ and $G_2 = \mathfrak{G}_{A_2}(\mathbf{F}_q)$ over $\mathbf{F}_q$ associated to 2×2 generalised Cartan matrices A_1, A_2, such that G_1 and G_2 are isomorphic as abstract groups, but their Mathieu–Rousseau completions $\mathfrak{G}^{pma}_{A_1}(\mathbf{F}_q)$ and $\mathfrak{G}^{pma}_{A_2}(\mathbf{F}_q)$ are not isomorphic as topological groups.*

This surprising result provides, in particular, the first known families of examples over arbitrary finite fields (of characteristic at least 3) of minimal Kac–Moody groups that are not dense in their Mathieu–Rousseau completion (up to now, the only known such family was given over $\mathbf{F}_2$ in [Rou12, 6.10]).

Corollary F. *Let $k = \mathbf{F}_q$ be a finite field with* $\operatorname{char} k \neq 2$*. Let $A = \left(\begin{smallmatrix} 2 & -m \\ -n & 2 \end{smallmatrix}\right)$ be a generalised Cartan matrix with $m, n > 2$ and assume that $m \equiv n \equiv 2 \pmod{q-1}$. Then the minimal Kac–Moody group $\mathfrak{G}_A(\mathbf{F}_q)$ of simply connected type is not dense in its Mathieu–Rousseau completion $\mathfrak{G}^{pma}_A(\mathbf{F}_q)$.*

The proof of these statements will be given in § 5.

The paper is organised as follows. We first fix some notation and provide an outline of the construction of Mathieu–Rousseau completions in § 2. We next prove some preliminary results about the Coxeter group W and the set of roots Δ in § 3. We then use these results to prove a more precise version of Theorem C in § 4. We establish its consequences in § 5, and we conclude the proof of Theorems A and B in § 6.

2. Preliminaries

2.1 Notations

Throughout this paper, we write $\mathbf{N}^*$ (respectively, $\mathbf{Z}^*$) for the set of non-zero natural numbers (respectively, non-zero integers).

For the rest of this paper, k denotes an arbitrary field and $A = (a_{ij})_{i,j \in I}$ denotes a generalised Cartan matrix indexed by $I = \{1, \ldots, n\}$. We fix a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A as in [Kac90, § 1.1]. We then retain all notation from the introduction. In particular, Δ is the corresponding set of roots and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ (respectively, $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$) the set of simple roots (respectively, coroots). For $\alpha \in \Delta$, we denote by $\operatorname{ht}(\alpha)$ its height.

Recall the definition of the Tits functor $\mathfrak{G} = \mathfrak{G}_A$ and of its sub-functors $\mathfrak{B}^+$, $\mathfrak{U}^+$, $\mathfrak{N}$ and $\mathfrak{T}$. Again, $\mathfrak{G}^{rr}(k)$ denotes the Rémy–Ronan completion of $\mathfrak{G}(k)$ and $\mathfrak{U}^{rr+}(k)$ the completion in $\mathfrak{G}^{rr}(k)$ of $\mathfrak{U}^+(k)$, so that $(\mathfrak{U}^{rr+}(k) \rtimes \mathfrak{T}(k), \mathfrak{N}(k))$ is a BN-pair for $\mathfrak{G}^{rr}(k)$ (see [CR09, Proposition 1]). We will give more details about the Mathieu–Rousseau completion $\mathfrak{G}^{pma}(k)$ of $\mathfrak{G}(k)$ in § 2.2 below.

As before, $W = W(A) \cong \mathfrak{N}(k)/\mathfrak{T}(k)$ is the Coxeter group associated to A, with generating set $S = \{s_1, \ldots, s_n\}$ such that $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all $i, j \in I$, and we fix a section $W \cong \mathfrak{N}(k)/\mathfrak{T}(k) \to \mathfrak{N}(k) : w \mapsto \overline{w}$.

Finally, to avoid cumbersome notation, we will write $\operatorname{con}(a)$ for both contraction groups $\operatorname{con}^{\mathfrak{G}^{pma}(k)}(a)$ and $\operatorname{con}^{\mathfrak{G}^{rr}(k)}(a)$, as k is fixed and as it will be always clear in which group we are working.

2.2 The Mathieu–Rousseau completion

We now outline the construction of the Mathieu–Rousseau completion of $\mathfrak{G}$ and give its basic properties, as it will play an important role in what follows. The general reference for this section is [Rou12].

Some notation. Let $\Lambda^\vee$ be the free $\mathbf{Z}$-module generated by $\Pi^\vee$, and let Λ be its $\mathbf{Z}$-dual, which we view as a $\mathbf{Z}$-form of the dual $\mathfrak{h}^*$. In particular, Λ contains Π. Then, as we are considering a Tits functor $\mathfrak{G}_A$ of simply connected type, the torus $\mathfrak{T}(k) = \mathfrak{T}_\Lambda(k) = \mathrm{Hom}_{gr}(\Lambda, k^\times)$ is generated by $\{r^h \mid r \in k^\times,\ h \in \Pi^\vee\}$, where

$$r^h\colon\ \Lambda \to k^\times : \lambda \mapsto r^{\langle \lambda, h\rangle}.$$

Let $\mathfrak{g}$ denote the Kac–Moody algebra of $\mathfrak{G}$ with root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, and let $e_1, \dots, e_n$ and $f_1, \dots, f_n$ be the corresponding Chevalley generators, so that $\mathfrak{g}_{\alpha_i} = \mathbf{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbf{C}f_i$ for all $i \in I$. Let also $\mathcal{U}$ denote the $\mathbf{Z}$-form of the enveloping algebra $\mathcal{U}_{\mathbf{C}}(\mathfrak{g})$ of $\mathfrak{g}$ introduced by Tits (see [Rou12, § 2]): this is a $\mathbf{Z}$-bialgebra graded by $Q := \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ and containing the elements $e_i^{(l)} := e_i^l/l!$ and $f_i^{(l)} := f_i^l/l!$ ($l \in \mathbf{N}$, $i \in I$). We write $\mathcal{U}_\alpha$ for the weight space corresponding to $\alpha \in Q$. The W-action on Δ induces a W-action on $\mathcal{U}_{\mathbf{C}}(\mathfrak{g})$ with s_i ($i \in I$) acting as

$$s_i^* = \exp(\mathrm{ad}\, e_i) \exp(\mathrm{ad}\, f_i) \exp(\mathrm{ad}\, e_i) \in \mathrm{Aut}(\mathcal{U}_{\mathbf{C}}(\mathfrak{g})).$$

This W-action preserves $\mathcal{U}$, and given $\alpha \in \Delta^{\mathrm{re}}$ such that $\alpha = w\alpha_i$ for some $w \in W$ and $i \in I$, the element $e_\alpha = w^* e_i$ is well defined (up to a choice of sign) and is a $\mathbf{Z}$-basis for $\mathfrak{g}_{\alpha\mathbf{Z}} := \mathfrak{g}_\alpha \cap \mathcal{U}$. In particular, we may choose $e_{-\alpha_i} := f_i$ as a basis for $\mathfrak{g}_{-\alpha_i\mathbf{Z}}$. For a ring R, we also set $\mathfrak{g}_{\alpha R} := \mathfrak{g}_{\alpha\mathbf{Z}} \otimes_{\mathbf{Z}} R$.

For a closed set $\Psi \subseteq \Delta$, we define the $\mathbf{Z}$-subalgebra $\mathcal{U}(\Psi)$ of $\mathcal{U}$ generated by all $\mathcal{U}^\alpha := \mathcal{U}_{\mathbf{C}}(\oplus_{n\geqslant 1} \mathfrak{g}_{n\alpha}) \cap \mathcal{U}$ for $\alpha \in \Psi$. If in addition $\Psi \subseteq w(\Delta_+)$ for some $w \in W$, we may also define the completion $\widehat{\mathcal{U}}_R(\Psi)$ of $\mathcal{U}(\Psi)$ over any ring R as

$$\widehat{\mathcal{U}}_R(\Psi) = \prod_{\alpha \in w.Q_+} (\mathcal{U}(\Psi)_\alpha \otimes_{\mathbf{Z}} R),$$

where $Q_+ := \bigoplus_{i\in I} \mathbf{N}\alpha_i$ and $\mathcal{U}(\Psi)_\alpha = \mathcal{U}(\Psi) \cap \mathcal{U}_\alpha$.

Pro-unipotent groups. The first step in the construction of $\mathfrak{G}^{pma}$ is to define for each closed set $\Psi \subseteq \Delta_+$ of positive roots the affine group scheme $\mathfrak{U}^{ma}_\Psi$ (which we view as a group functor) whose algebra is the restricted dual

$$\mathbf{Z}[\mathfrak{U}^{ma}_\Psi] := \bigoplus_{\alpha \in \mathbf{N}\Psi} \mathcal{U}(\Psi)^*_\alpha$$

of $\mathcal{U}(\Psi)$. In other words,

$$\mathfrak{U}^{ma}_\Psi(R) = \mathrm{Hom}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}[\mathfrak{U}^{ma}_\Psi], R) \quad \text{for any ring } R.$$

One can then define *root groups* $\mathfrak{U}^{ma}_{(\alpha)}$ in $\mathfrak{U}^{ma+} := \mathfrak{U}^{ma}_{\Delta_+}$ by setting $\mathfrak{U}^{ma}_{(\alpha)} = \mathfrak{U}_\alpha := \mathfrak{U}^{ma}_{\{\alpha\}}$ for $\alpha \in \Delta^{\mathrm{re}}_+$ and $\mathfrak{U}^{ma}_{(\alpha)} := \mathfrak{U}^{ma}_{\mathbf{N}^*\alpha}$ for $\alpha \in \Delta^{\mathrm{im}}_+$. For $\alpha \in \Delta^{\mathrm{re}}_+$, one can similarly define the root group $\mathfrak{U}_{-\alpha} = \mathfrak{U}^{ma}_{\{-\alpha\}}$ as above, with Ψ replaced by $\{-\alpha\}$. In other words, for each $\alpha \in \Delta^{\mathrm{re}}$, the real root group $\mathfrak{U}_\alpha$ is isomorphic to the additive group scheme $\mathbb{G}_a$ by

$$x_\alpha\colon\ \mathbb{G}_a(R) \xrightarrow{\sim} \mathfrak{U}_\alpha(R) : r \mapsto \exp(re_\alpha)$$

for any ring R (see Proposition 2.1 below). Note that, identifying $\{\mathfrak{U}_\alpha(R) \mid \alpha \in \Delta^{\mathrm{re}}\}$ with the root group datum of $\mathfrak{G}(R)$, the element $\overline{s_i} \in \mathfrak{N}(R)$ lifting $s_i \in W$ may then be chosen as

$$\overline{s_i} = x_{\alpha_i}(1) x_{-\alpha_i}(1) x_{\alpha_i}(1) = \exp(e_i)\exp(f_i)\exp(e_i).$$

The group functor $\mathfrak{U}^{ma}_\Psi$ admits a nice description in terms of root groups, which we now briefly review. For each $x \in \mathfrak{g}_{\alpha\mathbf{Z}}$, $\alpha \in \Delta_+$, G. Rousseau makes a choice of an *exponential sequence*, namely of a sequence $(x^{[n]})_{n\in\mathbf{N}}$ where $x^{[0]}=1$, $x^{[1]}=x$, and $x^{[n]}\in\mathcal{U}_{n\alpha}$ is such that $x^{[n]}-x^n/n!$ has filtration less than n in $\mathcal{U}_{\mathbf{C}}(\mathfrak{g})$ for each $n\in\mathbf{N}$, and which satisfies some additional compatibility condition with the bialgebra structure on $\mathcal{U}$ (see [Rou12, Propositions 2.4 and 2.7]). Such an exponential sequence for x is unique up to modifying each $x^{[n]}$, $n\geqslant 2$, by an element of $\mathfrak{g}_{n\alpha\mathbf{Z}}$. For a ring R and an element $\lambda\in R$, one can then define the *twisted exponential*

$$[\exp]\lambda x := \sum_{n\geqslant 0}\lambda^n x^{[n]}\in\widehat{\mathcal{U}}_R(\Delta_+).$$

Note that for α a real root, one can take the usual exponential. For each $\alpha\in\Delta_+$, let $\mathcal{B}_\alpha$ be a $\mathbf{Z}$-basis of $\mathfrak{g}_{\alpha\mathbf{Z}}$. For $\alpha\in\Delta^{\mathrm{re}}$, we choose $\mathcal{B}_\alpha=\{e_\alpha\}$. Finally, for a closed subset $\Psi\subseteq\Delta_+$, set $\mathcal{B}_\Psi=\bigcup_{\alpha\in\Psi}\mathcal{B}_\alpha$. Here is the description of $\mathfrak{U}^{ma}_\Psi$.

Proposition 2.1 [Rou12, Proposition 3.2]. *Let $\Psi\subseteq\Delta_+$ be closed and let R be a ring. Then $\mathfrak{U}^{ma}_\Psi(R)$ can be identified with the multiplicative subgroup of $\widehat{\mathcal{U}}_R(\Psi)$ consisting of the products*

$$\prod_{x\in\mathcal{B}_\Psi}[\exp]\lambda_x x$$

for $\lambda_x\in R$, where the product is taken in any (arbitrary) chosen order on $\mathcal{B}_\Psi$. The expression for an element of $\mathfrak{U}^{ma}_\Psi(R)$ in the form of such a product is unique.

A consequence of this proposition which we will use later on is the following lemma (see [Rou12, Lemme 3.3]).

Lemma 2.2. *Let $\Psi'\subseteq\Psi\subseteq\Delta_+$ be closed subsets of roots. Then $\mathfrak{U}^{ma}_{\Psi'}$ is a closed subgroup of $\mathfrak{U}^{ma}_\Psi$. Moreover, if $\Psi\backslash\Psi'$ is closed as well, then there is a unique decomposition $\mathfrak{U}^{ma}_\Psi=\mathfrak{U}^{ma}_{\Psi'}\cdot\mathfrak{U}^{ma}_{\Psi\backslash\Psi'}$.*

Minimal parabolics. The next step in the construction of $\mathfrak{G}^{pma}$ is to define, for each $i\in I$, the *minimal parabolic subgroup* $\mathfrak{B}^{ma+}_i$ of type i as the semi-direct product of $\mathfrak{U}^{ma}_{\Delta_+\backslash\{\alpha_i\}}$ with the unique connected affine algebraic group $\mathfrak{A}^\Lambda_i$ associated to the Kac–Moody root datum $(\{1\},(2),\Lambda,\{\alpha_i\},\{\alpha_i^\vee\})$ (see [Spr98, Theorem 10.1.1]). Note that $\mathfrak{A}^\Lambda_i$ contains $\mathfrak{T}$, $\mathfrak{U}_{\alpha_i}$ and $\mathfrak{U}_{-\alpha_i}$ as closed subgroups and is generated by them. To define this semi-direct product, it is thus sufficient to describe for each ring R conjugation actions of $\mathfrak{T}(R)=\mathrm{Hom}_{gr}(\Lambda,R^\times)$ and $\mathfrak{U}_\alpha(R)=\{\exp(re_\alpha)\mid r\in R\}$ on $\mathfrak{U}^{ma}_{\Delta_+\backslash\{\alpha_i\}}$, for $\alpha\in\{\pm\alpha_i\}$. For $t\in\mathfrak{T}(R)$, this is defined using Proposition 2.1 by

$$\mathrm{Int}(t)\cdot[\exp]\lambda x=[\exp]t(\gamma)\lambda x\quad\text{if } x\in\mathfrak{g}_{\gamma R}.$$

For $\alpha\in\{\pm\alpha_i\}$, we set

$$\mathrm{Int}(\exp(e_\alpha))(z)=\sum_{m\geqslant 0}(\mathrm{ad}(e_\alpha)^m/m!)(z)$$

for all $z\in\mathfrak{U}^{ma}_{\Delta_+\backslash\{\alpha_i\}}(R)$, where $\mathfrak{U}^{ma}_{\Delta_+\backslash\{\alpha_i\}}(R)$ is viewed as a subset of either $\widehat{\mathcal{U}}_R(\Delta_+)$ or $\widehat{\mathcal{U}}_R(s_i(\Delta_+))$, depending on whether $\alpha=\alpha_i$ or $\alpha=-\alpha_i$.

The following lemma will be crucial for us.

Lemma 2.3. *For any $\alpha\in\Delta_+$ and any $w\in W$ such that $w\alpha\in\Delta_+$,*

$$\overline{w}\mathfrak{U}^{ma}_{(\alpha)}\overline{w}^{\,-1}=\mathfrak{U}^{ma}_{(w\alpha)}.$$

Proof. For α a real root, this is [Rou12, 3.11]. In any case, this amounts to showing that, whenever $s_i\in W$ is such that $s_i(\alpha)\in\Delta_+$,

$$\overline{s_i}\cdot([\exp]x)\cdot\overline{s_i}^{\,-1}=[\exp](s_i^*x)$$

for any homogenous $x \in \oplus_{n\geqslant 1}\mathfrak{g}_{n\alpha R}$, with R an arbitrary ring. This last statement readily follows from the definition of the semi-direct product defining $\mathfrak{B}_i^{ma+}$. □

The group scheme $\mathfrak{G}^{pma}$. The *Mathieu–Rousseau completion* $\mathfrak{G}^{pma}$ of $\mathfrak{G}$ is then defined as some amalgamated product of the minimal parabolics $\mathfrak{B}_i^{ma+}$, $i \in I$ (see [Rou12, 3.6]). Over the field k, the identification of $\{\mathfrak{U}_\alpha(k) \mid \alpha \in \Delta^{\mathrm{re}}\}$ with the root group datum of $\mathfrak{G}(k)$ (as well as the identification of the tori $\mathfrak{T}(k)$ of $\mathfrak{G}(k)$ and $\mathfrak{G}^{pma}(k)$) induces an injection of $\mathfrak{G}(k)$ in $\mathfrak{G}^{pma}(k)$ (see [Rou12, Proposition 3.13]). The Borel subgroup $\mathfrak{B}^{ma+}(k) = \mathfrak{T}(k) \ltimes \mathfrak{U}^{ma+}(k)$ and $\mathfrak{N}(k)$ form a BN-pair for $\mathfrak{G}^{pma}(k)$ with associated building the positive building of $\mathfrak{G}(k)$ (see [Rou12, 3.16]).

The topology on $\mathfrak{G}^{pma}(k)$ is given as follows. For each $n \in \mathbf{N}$, set $\mathfrak{U}_n^{ma} := \mathfrak{U}_{\Psi(n)}^{ma}$, where $\Psi(n) = \{\alpha \in \Delta^+ \mid \mathrm{ht}(\alpha) \geqslant n\}$.

Lemma 2.4 [Rou12, 6.3.6]. *$\mathfrak{G}^{pma}(k)$ is a complete (Hausdorff) topological group with basis of neighbourhoods of the identity the subgroups $\mathfrak{U}_n^{ma}(k)$, $n \in \mathbf{N}$.*

Comparison with the Rémy–Ronan completion. Recall from the introduction the continuous homomorphism $\varphi\colon \overline{\mathfrak{G}(k)} \to \mathfrak{G}^{rr}(k)$, where $\overline{\mathfrak{G}(k)}$ denotes the closure of $\mathfrak{G}(k)$ in $\mathfrak{G}^{pma}(k)$. Write also $\overline{\mathfrak{U}^+(k)}$ for the closure of $\mathfrak{U}^+(k)$ in $\mathfrak{U}^{ma+}(k)$.

Lemma 2.5 [Rou12, 6.3.5]. *Assume that the field k is finite. Then the restriction of φ to $\overline{\mathfrak{U}^+(k)}$ is surjective onto $\mathfrak{U}^{rr+}(k)$.*

3. Coxeter groups and root systems

In this section, we prepare the ground for the proof of Theorem A by establishing several results which concern the Coxeter group W and the set of roots Δ. Basics on these two topics are covered in [AB08, chs. 1–3] and [Kac90, chs 1–5], respectively.

Throughout this section, we let $\Sigma = \Sigma(W, S)$ denote the Coxeter complex of W. Also, we let C_0 be the fundamental chamber of Σ. Finally, with the exception of Lemma 3.1 below where no particular assumption on W is made, we will always assume that W is infinite irreducible. Note that this is equivalent to saying that A is indecomposable of non-finite type.

Lemma 3.1. *Let $w = s_1 \ldots s_n$ be a Coxeter element of W. Let $A = A_1 + A_2$ be the unique decomposition of A as a sum of matrices A_1, A_2 such that A_1 (respectively, A_2) is an upper (respectively, lower) triangular matrix with 1s on the diagonal. Then the matrix of w in the basis $\{\alpha_1, \ldots, \alpha_n\}$ of simple roots is $-A_1^{-1}A_2 = I_n - A_1^{-1}A$.*

Proof. For a certain property P of two integer variables i, j (e.g. $\mathrm{P}(i,j) \equiv j \leqslant i$), we introduce the Kronecker symbol $\delta_{\mathrm{P}(i,j)}$ taking value 1 if $\mathrm{P}(i,j)$ is satisfied and 0 otherwise.

Let $B = (b_{ij})$ denote the matrix of w in the basis $\{\alpha_1, \ldots, \alpha_n\}$. Thus, b_{ij} is the coefficient of α_i in the expression for $s_1 \ldots s_n\alpha_j$ as a linear combination of the simple roots, which we will write as $[s_1 \ldots s_n\alpha_j]_i$. Thus $b_{ij} = [s_1 \ldots s_n\alpha_j]_i = [s_i \ldots s_n\alpha_j]_i$. Note that

$$s_{i+1} \ldots s_n\alpha_j = \sum_{k=i+1}^{n} [s_{i+1} \ldots s_n\alpha_j]_k\alpha_k + \delta_{j\leqslant i}\alpha_j = \sum_{k=i+1}^{n} b_{kj}\alpha_k + \delta_{j\leqslant i}\alpha_j.$$

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Therefore

$$b_{ij} = \left[s_i\left(\sum_{k=i+1}^{n} b_{kj}\alpha_k + \delta_{j\leqslant i}\alpha_j\right)\right]_i = -\sum_{k=i+1}^{n} a_{ik}b_{kj} - \delta_{j\leqslant i}a_{ij} + \delta_{i=j}$$
$$= \left(-\sum_{k=1}^{n}(A_1)_{ik}b_{kj} + b_{ij}\right) + (\delta_{j>i}a_{ij} - a_{ij}) + \delta_{i=j}$$
$$= -\sum_{k=1}^{n}(A_1)_{ik}b_{kj} + b_{ij} - a_{ij} + \sum_{k=1}^{n}(A_1)_{ik}(I_n)_{kj}.$$

Thus $A = -A_1B + A_1$, so that $B = -A_1^{-1}A_2$, as desired. □

For $\omega \in W$ and $\alpha \in \Delta_+$, define the function $f_\alpha^\omega \colon \mathbf{Z} \to \{\pm 1\} : k \mapsto \operatorname{sign}(\omega^k\alpha)$, where $\operatorname{sign}(\Delta_\pm) = \pm 1$.

Lemma 3.2. *Let $\omega \in W$ be such that $\ell(\omega^l) = |l|\ell(\omega)$ for all $l \in \mathbf{Z}$. Then f_α^ω is monotonic for all $\alpha \in \Delta_+$.*

Proof. Let $\omega \in W$ be such that $\ell(\omega^l) = |l|\ell(\omega)$ for all $l \in \mathbf{Z}$ and let $\omega = t_1t_2\dots t_k$ be a reduced expression for ω, where $t_j \in S$ for all $j \in \{1,\dots,k\}$. Let $\alpha \in \Delta_+$ and assume that f_α^ω is not constant. Then α is a real root because $W.\Delta_+^{\mathrm{im}} = \Delta_+^{\mathrm{im}}$. Let $k_\alpha \in \mathbf{Z}^*$ be minimal (in absolute value) so that $f_\alpha^\omega(k_\alpha) = -1$. We deal with the case when $k_\alpha > 0$; the same proof applies for $k_\alpha < 0$ by replacing ω with its inverse. We have to show that $\omega^l\alpha \in \Delta_-$ if and only if $l \geqslant k_\alpha$.

Let $\beta := \omega^{k_\alpha - 1}\alpha$. Thus $\beta \in \Delta_+^{\mathrm{re}}$ and $\omega\beta \in \Delta_-^{\mathrm{re}}$. It follows that there is some $i \in \{1,\dots,k\}$ such that $\beta = t_kt_{k-1}\dots t_{i+1}\alpha_{t_i}$. In other words, β is one of the n positive roots whose wall $\partial\beta$ in the Coxeter complex Σ of W separates the fundamental chamber C_0 from $\omega^{-1}C_0$. We want to show that $\omega^l\beta \in \Delta_-$ if and only if $l \geqslant 1$.

Assume first for a contradiction that there is some $l \geqslant 1$ such that $\omega^{l+1}\beta \in \Delta_+$, that is, $\omega^{l+1}\beta$ contains C_0. Since $\omega^{l+1}\beta$ contains $\omega^{l+1}C_0$ but not ω^lC_0, its wall $\omega^{l+1}\partial\beta$ separates ω^lC_0 from $\omega^{l+1}C_0$ and C_0. In particular, any gallery from C_0 to $\omega^{l+1}C_0$ going through ω^lC_0 cannot be minimal. This contradicts the assumption that $\ell(\omega^l) = |l|\ell(\omega)$ for all $l \in \mathbf{Z}$ since this implies that the product of $l+1$ copies of $t_1\dots t_k$ is a reduced expression for ω^{l+1}.

Assume next for a contradiction that there is some $l \geqslant 1$ such that $\omega^{-l}\beta \in \Delta_-$. Then as before, $\omega^{-l}\partial\beta$ separates $\omega^{-l}C_0$ from $\omega^{-l-1}C_0$ and C_0. Again, this implies that any gallery from C_0 to $\omega^{-l-1}C_0$ going through $\omega^{-l}C_0$ cannot be minimal, yielding the desired contradiction. □

Corollary 3.3. *Let $w = s_1\dots s_n$ be a Coxeter element of W. Then f_α^w is monotonic for all $\alpha \in \Delta_+$.*

Proof. As $\ell(w^l) = |l|\ell(w)$ for all $l \in \mathbf{Z}$ by the main result of [Spe09], this readily follows from Lemma 3.2. □

Lemma 3.4. *Assume that A is of indefinite type. Let $w = s_1\dots s_n$ be a Coxeter element of W, and let $\alpha \in \Delta_+$. Then $w^l\alpha \neq \alpha$ for all non-zero integers l.*

Proof. Assume for a contradiction that $w^k\alpha = \alpha$ for some $k \in \mathbf{N}^*$. It then follows from Corollary 3.3 that $w^i\alpha \in \Delta_+$ for all $i \in \{0,\dots,k-1\}$. Viewing w as an automorphism of the root lattice, we get that

$$(w - \mathrm{Id})(w^{k-1} + \cdots + w + \mathrm{Id})\alpha = 0.$$

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Moreover, $\beta := (w^{k-1} + \cdots + w + \mathrm{Id})\alpha$ is a sum of positive roots, and hence can be viewed as a non-zero vector of $\mathbf{R}^n$ with non-negative entries. Recall from Lemma 3.1 that w is represented by the matrix $-A_1^{-1}A_2$. Thus, multiplying the above equality by $-A_1$, we get that $A\beta = 0$. Since A is indecomposable of indefinite type, this gives the desired contradiction by [Kac90, Theorem 4.3]. □

Lemma 3.5. *Let $\omega \in W$ and $\alpha \in \Delta_+$ be such that $\omega^l\alpha \neq \alpha$ for all positive integers l. Then $|\mathrm{ht}(\omega^l\alpha)|$ goes to infinity as l goes to infinity.*

Proof. If $|\mathrm{ht}(\omega^l\alpha)|$ were bounded as l goes to infinity, the set of roots $\{\omega^l\alpha \mid l \in \mathbf{N}\}$ would be finite, and so there would exist an $l \in \mathbf{N}^*$ such that $\omega^l\alpha = \alpha$, a contradiction. □

Lemma 3.6. *Assume that A is of indefinite type. Let $w = s_1 \dots s_n$ be a Coxeter element of W, and let $\alpha \in \Delta_+$. Then there exists some $\epsilon \in \{\pm\}$ such that $w^{\epsilon l}\alpha \in \Delta_+$ for all $l \in \mathbf{N}$. Moreover, $\mathrm{ht}(w^{\epsilon l}\alpha)$ goes to infinity as l goes to infinity.*

Proof. The existence of ϵ readily follows from Corollary 3.3, while the second statement is a consequence of Lemmas 3.4 and 3.5. □

4. Contraction groups

In this section, we make use of the results proven so far to establish, under suitable hypotheses, that the subgroups $\mathfrak{U}^{ma+}(k)$ of $\mathfrak{G}^{pma}(k)$ and $\mathfrak{U}^{rr+}(k)$ of $\mathfrak{G}^{rr}(k)$ are contracted. Throughout this section, W is assumed to be infinite irreducible, and we fix some Coxeter element $w = s_1 \dots s_n$ of W.

Lemma 4.1. *Let $\Psi_1 \subseteq \Psi_2 \subseteq \cdots \subseteq \Delta_+$ be an increasing sequence of closed subsets of Δ_+ and set $\Psi = \bigcup_{i=1}^{\infty} \Psi_i$. Then the corresponding increasing union of subgroups $\bigcup_{i=1}^{\infty} \mathfrak{U}^{ma}_{\Psi_i}(k)$ is dense in $\mathfrak{U}^{ma}_{\Psi}(k)$.*

Proof. This readily follows from Proposition 2.1. □

Proposition 4.2. *Let $\Psi \subseteq \Delta_+$ be closed. Let $\omega \in W$ be such that $\omega\Psi \subseteq \Delta_+$. Then $\overline{\omega}\mathfrak{U}^{ma}_{\Psi}\overline{\omega}^{-1} = \mathfrak{U}^{ma}_{\omega\Psi}$.*

Proof. For a positive root $\alpha \in \Delta_+$, consider the root group $\mathfrak{U}^{ma}_{(\alpha)}$ as in Section 2.2. Let also Ψ and ω be as in the statement of the lemma. By Lemma 2.3, we know that

$$\overline{\omega}\langle \mathfrak{U}^{ma}_{(\alpha)} \mid \alpha \in \Psi\rangle\overline{\omega}^{-1} = \langle \mathfrak{U}^{ma}_{(\omega\alpha)} \mid \alpha \in \Psi\rangle.$$

Passing to the closures, it follows from Lemma 4.1 that $\overline{\omega}\mathfrak{U}^{ma}_{\Psi}\overline{\omega}^{-1} = \mathfrak{U}^{ma}_{\omega\Psi}$, as desired. □

Lemma 4.3. *Let $\Psi \subseteq \Delta_+$ be the set of positive roots α such that $w^l\alpha \in \Delta_+$ for all $l \in \mathbf{N}$. Then both Ψ and $\Delta_+\backslash\Psi$ are closed. In particular, one has a unique decomposition $\mathfrak{U}^{ma+} = \mathfrak{U}^{ma}_{\Psi} \cdot \mathfrak{U}^{ma}_{\Delta_+\backslash\Psi}$.*

Proof. Clearly, Ψ is closed. Let $\alpha, \beta \in \Delta_+\backslash\Psi$ be such that $\alpha + \beta \in \Delta$. Thus there exist some positive integers l_1, l_2 such that $w^{l_1}\alpha \in \Delta_-$ and $w^{l_2}\beta \in \Delta_-$. Then $w^l(\alpha + \beta) \in \Delta_-$ for all $l \geqslant \max\{l_1, l_2\}$ by Corollary 3.3 and hence $\alpha + \beta \in \Delta_+\backslash\Psi$. Thus $\Delta_+\backslash\Psi$ is closed, as desired. The second statement follows from Lemma 2.2. □

Remark 4.4. Let $\Psi \subseteq \Delta_+$ be as in Lemma 4.3. Put an arbitrary order on Δ_+. This yields enumerations $\Psi = \{\beta_1, \beta_2, \dots\}$ and $\Delta_+\backslash\Psi = \{\alpha_1, \alpha_2, \dots\}$. For each $i \in \mathbf{N}^*$, we let Ψ_i (respectively, Φ_i) denote the closure in Δ_+ of $\{\beta_1, \dots, \beta_i\}$ (respectively, of $\{\alpha_1, \dots, \alpha_i\}$). It follows from Lemma 4.3 that $\Psi = \bigcup_{i=1}^{\infty} \Psi_i$ and that $\Delta_+\backslash\Psi = \bigcup_{i=1}^{\infty} \Phi_i$.

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Lemma 4.5. *Fix* $i \in \mathbf{N}^*$, *and let* $\Psi_i, \Phi_i \subseteq \Delta_+$ *be as in Remark 4.4. Assume that A is of indefinite type. Then there exists a sequence of positive integers* $(n_l)_{l\in\mathbf{N}}$ *going to infinity as* l *goes to infinity, such that* $\overline{w}^l \mathfrak{U}^{ma}_{\Psi_i} \overline{w}^{-l} \subseteq \mathfrak{U}^{ma}_{n_l}$ *and* $\overline{w}^{-l} \mathfrak{U}^{ma}_{\Phi_i} \overline{w}^{l} \subseteq \mathfrak{U}^{ma}_{n_l}$ *for all* $l \in \mathbf{N}$.

Proof. Let $\alpha_j, \beta_j \in \Delta_+$ be as in Remark 4.4. By Lemma 3.6 together with Corollary 3.3, one can find for each $j \in \{1, \dots, i\}$ sequences of positive integers $(m^j_l)_{l\in\mathbf{N}}$ and $(n^j_l)_{l\in\mathbf{N}}$ going to infinity as l goes to infinity, such that $\mathrm{ht}(w^{-l}\alpha_j) \geqslant m^j_l$ and $\mathrm{ht}(w^l\beta_j) \geqslant n^j_l$ for all $l \in \mathbf{N}$. For each $l \in \mathbf{N}$, set $n_l = \min\{m^l_j, n^l_j \mid 1 \leqslant j \leqslant i\}$. Then the sequence $(n_l)_{l\in\mathbf{N}}$ goes to infinity as l goes to infinity. Moreover, $\mathrm{ht}(\alpha) \geqslant n_l$ for all $\alpha \in w^{-l}\Phi_i$ and $\mathrm{ht}(\beta) \geqslant n_l$ for all $\beta \in w^l\Psi_i$. The conclusion then follows from Proposition 4.2. □

Theorem 4.6. *Let* $a = \overline{w} \in \mathfrak{G}(k) \subseteq \mathfrak{G}^{pma}(k)$, *and let* Ψ, Ψ_i, Φ_i *be as in Remark 4.4. Assume that A is of indefinite type. Then the following hold.*

(i) $\mathfrak{U}^{ma}_{\Psi_i}(k) \subseteq \mathrm{con}(a)$ *and* $\mathfrak{U}^{ma}_{\Phi_i}(k) \subseteq \mathrm{con}(a^{-1})$ *for all* $i \in \mathbf{N}^*$.

(ii) $\mathfrak{U}^{ma}_{\Psi}(k) \subseteq \overline{\mathrm{con}(a)}$ *and* $\mathfrak{U}^{ma}_{\Delta_+\setminus\Psi}(k) \subseteq \overline{\mathrm{con}(a^{-1})}$.

(iii) $\mathfrak{U}^{ma+}(k) \subseteq \overline{\langle \overline{\mathrm{con}(a)} \cup \overline{\mathrm{con}(a^{-1})} \rangle}$.

Proof. The first statement follows from Lemma 4.5. The second statement is a consequence of the first together with Lemma 4.1. The third statement follows from the second together with Lemma 4.3. □

Recall from Lemma 2.5 that $\varphi(\overline{\mathfrak{U}^+(k)}) = \mathfrak{U}^{rr+}(k)$ whenever k is finite.

Lemma 4.7. *Let* $a = \overline{w} \in \mathfrak{G}(k) \subseteq \mathfrak{G}^{rr}(k)$. *Assume that A is of indefinite type and that* $\varphi(\overline{\mathfrak{U}^+(k)}) = \mathfrak{U}^{rr+}(k)$ *(e.g. k finite). Then* $\mathfrak{U}^{rr+}(k) \subseteq \overline{\langle \overline{\mathrm{con}(a)} \cup \overline{\mathrm{con}(a^{-1})} \rangle}$.

Proof. We know from Theorem 4.6(iii) that $\overline{\mathfrak{U}^+(k)} \subseteq \overline{\langle \overline{\mathrm{con}(a)} \cup \overline{\mathrm{con}(a^{-1})} \rangle}$. Applying φ then yields the desired inclusion since $\varphi(\overline{\mathrm{con}(a^{\pm 1})}) \subseteq \overline{\mathrm{con}(\varphi(a)^{\pm 1})} = \overline{\mathrm{con}(a^{\pm 1})}$ by continuity of φ. □

Proof of Theorem C. The first statement is Proposition 4.2 and the third follows from Theorem 4.6(i). The second statement is a consequence of the first together with Lemmas 2.4 and 3.5. □

5. Consequences of Theorem C

Before we give the proof of Theorems A and B in the next section, we examine the consequences, stated in the introduction, of Theorem C. More precisely, we will make use of the following lemma. Recall from [Wil12, § 3] the definition of the *nub* of an automorphism α of a totally disconnected locally compact group G. It possesses many equivalent definitions (see [Wil12, Theorem 4.12]) and, given an element $a \in G$ (viewed as a conjugation automorphism), it can be characterised as $\mathrm{nub}(a) = \overline{\mathrm{con}(a)} \cap \overline{\mathrm{con}(a^{-1})}$ (see [Wil12, Remark 3.3(b) and (d)]).

Lemma 5.1. *Let* $G = \mathfrak{G}^{pma}_A(\mathbf{F}_q)$ *be a complete Kac–Moody group of simply connected type over a finite field* $\mathbf{F}_q$, *with indecomposable generalised Cartan matrix A of indefinite type. Let* $U^{\mathrm{im}+} = \mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(\mathbf{F}_q)$ *denote its positive imaginary subgroup, let* $w \in W = W(A)$ *denote a Coxeter element of W, and set* $a := \overline{w} \in \mathfrak{N}(\mathbf{F}_q)$. *Then*

$$U^{\mathrm{im}+} \subseteq \mathrm{nub}(a) = \overline{\mathrm{con}(a)} \cap \overline{\mathrm{con}(a^{-1})}.$$

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Proof. Notice that Lemma 4.5 remains valid if one replaces Ψ by its (closed) subset Δ_+^{im} and w by w^{-1}. As in the proof of the second statement of Theorem 4.6, Lemma 4.1 then allows us to conclude. □

To establish Theorem D, we need one more technical lemma regarding contraction groups, whose proof is an adaptation of [Wan84, the proof of Proposition 2.1].

LEMMA 5.2. *Let G be a locally compact group, let a be an element of G, and let Q be a compact subset of G such that $Q \subseteq \mathrm{con}(a)$. Then Q is uniformly contracted by a, that is, for every open neighbourhood U of the identity one has $a^nQa^{-n} \subset U$ for all large enough n.*

Proof. Fix an open neighbourhood U of the identity, and let V be a compact neighbourhood of the identity such that $V^2 \subset U$. By hypothesis, for all $x \in Q$ there exists an N_x such that $a^nxa^{-n} \in V$ for all $n \geqslant N_x$. In other words,

$$Q \subset \bigcup_{N\geqslant 0} \bigcap_{n\geqslant N} a^{-n}Va^n.$$

Note that the sets $C_N = \bigcap_{n\geqslant N} a^{-n}Va^n$ form an ascending chain of compact sets. It follows from Baire's theorem that $Q \cap C_N$ has non-empty interior in Q for a large enough N.

By compactness of Q, one then finds a finite subset F of Q such that

$$Q \subset F \cdot C_N.$$

Since F is finite and contained in $\mathrm{con}(a)$, we know that $a^nFa^{-n} \subset V$ for all large enough n. Moreover, by construction, $a^nC_Na^{-n} \subset V$ for $n \geqslant N$, and hence

$$a^nQa^{-n} = (a^nFa^{-n}) \cdot (a^nC_Na^{-n}) \subset V^2 \subset U$$

for all large enough n, as desired. □

Proof of Theorem D. Let A denote an $n \times n$ generalised Cartan matrix of indecomposable indefinite type, let $W = W(A)$ be the associated Weyl group, and let $w = s_1 \dots s_n$ denote a Coxeter element of W. Set $a := \overline{w} \in \mathfrak{N}(\mathbf{F}_q)$. It then follows from Lemma 5.1 that

$$U_{\mathrm{im}}^{ma+} := \mathfrak{U}^{ma}_{\Delta_+^{\mathrm{im}}}(\mathbf{F}_q) \subseteq \overline{\mathrm{con}(a)} \quad \text{in } \mathfrak{G}_A^{pma}(\mathbf{F}_q)$$

and that

$$U_{\mathrm{im}}^{rr+} := \varphi(\mathfrak{U}^{ma}_{\Delta_+^{\mathrm{im}}}(\mathbf{F}_q) \cap \overline{\mathfrak{U}^+(\mathbf{F}_q)}) \subseteq \overline{\mathrm{con}(a)} \quad \text{in } \mathfrak{G}_A^{rr}(\mathbf{F}_q).$$

Since U_{im}^{ma+} is closed in $\mathfrak{U}^{ma+}(\mathbf{F}_q)$ which is compact (see [Rou12, 6.3]), both the groups U_{im}^{ma+} and U_{im}^{rr+} are compact. Moreover, they are normalised by a by Proposition 4.2. Hence they cannot be contracted by a because of Lemma 5.2, since by assumption U_{im}^{rr+} is non-trivial. In particular, $\mathrm{con}(a) \neq \overline{\mathrm{con}(a)}$ and hence $\mathrm{con}(a)$ cannot be closed.

Note that one could also directly use the fact that $\mathrm{con}(a)$ is closed if and only if $\mathrm{nub}(a) = \{1\}$ (see [Wil12, Remark 3.3(b)]) together with Lemma 5.1. We preferred, however, to present a more elementary proof as well, as Lemma 5.2 will be used anyway in the proof of Theorem E below. □

To prove Theorem E, we need two additional technical lemmas. The first lemma is an adaptation of [Cap09, Lemma 4.3].

LEMMA 5.3. *Let k be a finite field of order q, and consider the generalised Cartan matrices $A = \left(\begin{smallmatrix} 2 & -m \\ -n & 2 \end{smallmatrix}\right)$ and $A' = \left(\begin{smallmatrix} 2 & -m' \\ -n' & 2 \end{smallmatrix}\right)$ such that $m, m', n, n' \geqslant 2$. Assume, moreover, that $m \equiv m' \pmod{q-1}$ and $n \equiv n' \pmod{q-1}$. Then the minimal Kac–Moody groups $\mathfrak{G}_A(k)$ and $\mathfrak{G}_{A'}(k)$ of simply connected type are isomorphic.*

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Proof. As the Weyl groups of A and A' are isomorphic (to the infinite dihedral group), one can identify the corresponding sets of real roots. Moreover, as noted in the proof of [Cap09, Lemma 4.3], the commutation relations between root groups corresponding to prenilpotent pairs of roots are trivial in $\mathfrak{G}_A(k)$ (respectively, $\mathfrak{G}_{A'}(k)$). In particular, one can identify the Steinberg functors of $\mathfrak{G}_A$ and $\mathfrak{G}_{A'}$.

Recall from § 2.2 (and in the notation of that section) that the torus $\mathfrak{T}_\Lambda(k)$ of $\mathfrak{G}_A(k)$ is generated by $\{r^{\alpha_i^\vee} \mid r \in k^\times, i = 1, 2\}$, and similarly for the torus $\mathfrak{T}_{\Lambda'}(k)$ of $\mathfrak{G}_{A'}(k)$. This yields identifications of $\mathfrak{T}_\Lambda(k)$ and $\mathfrak{T}_{\Lambda'}(k)$. As $r^m = r^{m'}$ and $r^n = r^{n'}$ for all $r \in k$, it then follows from the above identifications that $\mathfrak{G}_A(k)$ and $\mathfrak{G}_{A'}(k)$ admit the same Steinberg type presentation (see [Tit87, § 3.6]), as desired. □

Lemma 5.4. *Let k be an arbitrary field with $\operatorname{char} k \neq 2$. Let $G = \mathfrak{G}^{pma}(k)$ be the Mathieu–Rousseau completion associated with a 2×2 generalised Cartan matrix $A = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$ for some $m, n > 2$. Then the imaginary subgroup $U^{\mathrm{im}+} = \mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(k)$ of G is not contained in the kernel $Z'(G)$ of the action of G on its associated building.*

Proof. Let p denote the characteristic of k. Thus $p = 0$ or $p \geqslant 3$. Assume for a contradiction that $U^{\mathrm{im}+}$ is contained in $Z'(G)$.

Note first that $U^{\mathrm{im}+} = \bigcap_{w \in W} \overline{w}\mathfrak{U}^{ma+}(k)\overline{w}^{\,-1}$. Indeed, as Δ^{im}_+ is W-stable, the inclusion $\subseteq$ is clear from Proposition 2.1 and Lemma 2.3. Assume now for a contradiction that there is some $u \in \bigcap_{w \in W} \overline{w}\mathfrak{U}^{ma+}(k)\overline{w}^{\,-1}$ that is not in $U^{\mathrm{im}+}$. Write u as a product $u = \prod_{x \in \mathcal{B}_{\Delta_+}} [\exp]\lambda_x x$ as in Proposition 2.1, and let Φ_u be the set of positive real roots β such that $\lambda_x \neq 0$ for $x \in \mathcal{B}_\beta$. Thus Φ_u is non-empty. Choose $\beta \in \Phi_u$ and $v \in W$ such that $-v\beta$ is a simple root and $v\beta' \in \Delta_+$ for all $\beta' \in \Phi_u \backslash \{\beta\}$. Then by Lemma 2.3, the element $\overline{v}$ conjugates u outside $\mathfrak{U}^{ma+}(k)$, yielding the desired contradiction.

As $Z'(G) = Z(G).(Z'(G) \cap \mathfrak{U}^{ma+}(k))$ and as $Z'(G) \cap \mathfrak{U}^{ma+}(k)$ is normal in G by [Rou12, Proposition 6.4], where $Z(G)$ denotes the centre of G, we deduce that $U^{\mathrm{im}+} = Z'(G) \cap \mathfrak{U}^{ma+}(k)$ is normal in G.

Recall the notation from § 2.2. In particular, e_1, e_2 and f_1, f_2 denote the Chevalley generators of the Kac–Moody algebra $\mathfrak{g}$ with generalised Cartan matrix A, and α_1, α_2 (respectively, $\alpha_1^\vee$, $\alpha_2^\vee$) are the corresponding simple roots (respectively, coroots). Recall also the definition of the $\mathbf{Z}$-form $\mathcal{U}$ of $\mathcal{U}_{\mathbf{C}}(\mathfrak{g})$, as well as the Lie algebras $\mathfrak{g}_{\mathbf{Z}} = \mathfrak{g} \cap \mathcal{U}$ and $\mathfrak{g}_k = \mathfrak{g}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$. Finally, for each real root $\gamma \in \Delta^{\mathrm{re}}$, choose as before a $\mathbf{Z}$-basis element e_γ of $\mathfrak{g}_{\gamma\mathbf{Z}}$.

We will show that there exist an imaginary root $\delta \in \Delta^{\mathrm{im}}_+$, a simple root α_i, and a non-zero element $x \in \mathfrak{g}_{\delta k}$ such that $\delta - \alpha_i \in \Delta^{\mathrm{re}}_+$ and $\operatorname{ad}(f_i)x$ is non-zero in $\mathfrak{g}_k$. Recalling from § 2.2 the definition of the semi-direct product $\mathfrak{B}^{ma+}_i = \mathfrak{A}^\Lambda_i \ltimes \mathfrak{U}^{ma}_{\Delta_+ \backslash \{\alpha_i\}}$, this will imply that the root group $\mathfrak{U}_{-\alpha_i}(k)$ conjugates the imaginary root group $\mathfrak{U}^{ma}_{(\delta)}(k)$ outside $U^{\mathrm{im}+}$, so that $U^{\mathrm{im}+}$ cannot be normal in G, yielding the desired contradiction.

Assume first that m is not a multiple of p. As $m, n \geqslant 3$, we know that $\delta := \alpha_1 + \alpha_2$ is an imaginary root (see [Kac90, Lemma 5.3]) and that $x := [e_1, e_2] \in \mathfrak{g}_{\mathbf{Z}}$ is non-zero. Moreover, $\operatorname{ad}(f_1)x = -me_2$ is non-zero in $\mathfrak{g}_k$ since m is not a multiple of p, as desired.

Assume next that m is a multiple of p. Set $\gamma := s_1(\alpha_2) = \alpha_2 + m\alpha_1 \in \Delta^{\mathrm{re}}_+$. Then again $\delta := \alpha_2 + \gamma \in \Delta^{\mathrm{im}}_+$ since $\langle \delta, \alpha_1^\vee \rangle = 0$ and $\langle \delta, \alpha_2^\vee \rangle = 4 - mn < 0$. Set $x := [e_2, e_\gamma] \in \mathfrak{g}_{\mathbf{Z}}$. Note that $\operatorname{ad}(f_2)e_\gamma = 0$ since $\gamma - \alpha_2 = m\alpha_1 \notin \Delta$. As $p \neq 2$, we deduce that $\operatorname{ad}(f_2)x = (2 - mn)e_\gamma$ is non-zero in $\mathfrak{g}_k$, as desired. □

Proof of Theorem E. It follows from Lemma 5.3 that the minimal Kac–Moody group $G_1 = \mathfrak{G}_{A_1}(\mathbf{F}_q)$ over $\mathbf{F}_q$ of simply connected type with generalised Cartan matrix $A_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ (hence

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of affine type) is isomorphic to any minimal Kac–Moody group $G_2 = \mathfrak{G}_{A_2}(\mathbf{F}_q)$ over $\mathbf{F}_q$ of simply connected type with generalised Cartan matrix $A_2 = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$ for some $m, n > 2$ (hence of indefinite type) with $m \equiv n \equiv 2 \pmod{q-1}$. We fix such a group G_2.

For $i = 1, 2$ set $\widehat{G}_i := \mathfrak{G}^{pma}_{A_i}(\mathbf{F}_q)$ and let Z'_i denote the kernel of the action of $\widehat{G}_i$ on its associated building. Assume for a contradiction that there is an isomorphism $\psi\colon \widehat{G}_1 \to \widehat{G}_2$ of topological groups. As noticed in [Rou12, Remarque 6.20(4)], the quotient $\widehat{G}_1/Z'_1$ is a simple algebraic group over the local field $\mathbf{F}_q(\!(t)\!)$. In particular, all the contraction groups of $\widehat{G}_1/Z'_1$ are closed. Moreover, $\psi(Z'_1)$ is the unique maximal proper normal subgroup of $\widehat{G}_2$, and it is compact. It follows that $\psi(Z'_1) = Z'_2$, for otherwise, by Tits' lemma (see [AB08, Lemma 6.61]), the group $\widehat{G}_2$ would be compact, a contradiction. Hence ψ induces an isomorphism of topological groups between $\widehat{G}_1/Z'_1$ and $\widehat{G}_2/Z'_2$, so that, in particular, all contraction groups of $\widehat{G}_2/Z'_2$ are closed. Let $\pi\colon \widehat{G}_2 \to \widehat{G}_2/Z'_2$ denote the canonical projection, and let a be any element of $\widehat{G}_2$. Then

$$\pi(\overline{\mathrm{con}(a)}) \subseteq \overline{\pi(\mathrm{con}(a))} \subseteq \overline{\mathrm{con}(\pi(a))} = \mathrm{con}(\pi(a)).$$

It follows from Lemma 5.1 that the subgroup $U^{\mathrm{im}+} := \mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(\mathbf{F}_q)$ of $\mathfrak{U}^{ma}_{\Delta_+}(\mathbf{F}_q)$ in $\widehat{G}_2$ is such that

$$\pi(U^{\mathrm{im}+}) \subseteq \pi(\overline{\mathrm{con}(a)}) \subseteq \mathrm{con}(\pi(a))$$

for a suitably chosen $a \in \widehat{G}_2$ normalising $U^{\mathrm{im}+}$. Thus Lemma 5.2 implies that $\pi(U^{\mathrm{im}+}) = \{1\}$, that is, $U^{\mathrm{im}+} \subseteq Z'_2$. This contradicts Lemma 5.4, as desired. □

Proof of Corollary F. Let $k = \mathbf{F}_q$ with $\mathrm{char}\, k \neq 2$, and let the generalised Cartan matrix A be as in the statement of Corollary F. As we saw in the proof of Theorem E above, the minimal Kac–Moody group $G_2 = \mathfrak{G}_{A_2}(\mathbf{F}_q)$ (where we set $A_2 = A$) is then isomorphic to the minimal Kac–Moody group $G_1 = \mathfrak{G}_{A_1}(\mathbf{F}_q)$ over $\mathbf{F}_q$ of simply connected type with generalised Cartan matrix $A_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, whereas the quotients $\widehat{G}_1/Z'_1$ and $\widehat{G}_2/Z'_2$ cannot be isomorphic as topological groups, where as before $\widehat{G}_i := \mathfrak{G}^{pma}_{A_i}(\mathbf{F}_q)$ and Z'_i is the kernel of the action of $\widehat{G}_i$ on its associated building $(i = 1, 2)$.

Note that the isomorphism between G_1 and G_2 is the one provided by Lemma 5.3, and it maps the twin BN-pair of G_1 to that of G_2. In particular, the Rémy–Ronan completions $\mathfrak{G}^{rr}_{A_1}(\mathbf{F}_q)$ of G_1 and $\mathfrak{G}^{rr}_{A_2}(\mathbf{F}_q)$ of G_2 are isomorphic as topological groups.

Moreover, as $\mathrm{char}\, k > 2$, we know from [Rou12, Proposition 6.11] that G_1 is dense in $\widehat{G}_1$. Assume now for a contradiction that G_2 is dense in $\widehat{G}_2$. Then the surjective continuous homomorphisms $\varphi_i\colon \widehat{G}_i \to \mathfrak{G}^{rr}_{A_i}(\mathbf{F}_q)$ $(i = 1, 2)$ induce isomorphisms of topological groups

$$\widehat{G}_1/Z'_1 \cong \mathfrak{G}^{rr}_{A_1}(\mathbf{F}_q) \cong \mathfrak{G}^{rr}_{A_2}(\mathbf{F}_q) \cong \widehat{G}_2/Z'_2,$$

yielding the desired contradiction. □

6. Proof of Theorems A and B

We now let $k = \mathbf{F}_q$ be a finite field, A be an indecomposable generalised Cartan matrix of indefinite type, and G be one of the complete Kac–Moody groups $\mathfrak{G}^{rr}_A(\mathbf{F}_q)$ or $\mathfrak{G}^{pma}_A(\mathbf{F}_q)$. We also set $U^+ := \mathfrak{U}^{rr+}(\mathbf{F}_q)$ or $U^+ := \mathfrak{U}^{ma+}(\mathbf{F}_q)$ accordingly. Then G is a locally compact totally disconnected topological group, and U^+ is a compact open subgroup of G. Indeed, for $\mathfrak{G}^{rr}_A(\mathbf{F}_q)$, this follows from [CR09, Proposition 1]; $\mathfrak{G}^{pma}_A(\mathbf{F}_q)$ is locally compact because $\mathfrak{U}^{ma+}(\mathbf{F}_q)$ is compact open by [Rou12, 6.3], and it is totally disconnected because its filtration by the $\mathfrak{U}^{ma}_n(\mathbf{F}_q)$ is separated.

As mentioned in the introduction, we first need to establish the topological simplicity of $\mathfrak{G}_A^{pma}(\mathbf{F}_q)$ in full generality.

PROPOSITION 6.1. *Assume that the generalised Cartan matrix A is indecomposable of indefinite type. Then $\mathfrak{G}_A^{pma}(\mathbf{F}_q)/Z'(\mathfrak{G}_A^{pma}(\mathbf{F}_q))$ is topologically simple over any finite field $\mathbf{F}_q$.*

Proof. Set $G := \mathfrak{G}_A^{pma}(\mathbf{F}_q)$ and $Z' := Z'(\mathfrak{G}_A^{pma}(\mathbf{F}_q))$. It follows from [CM11, Corollary 3.1] that G possesses a closed cocompact normal subgroup H containing Z' and such that H/Z' is topologically simple. It thus remains to see that in fact $H = G$. Let $\pi\colon G \to G/H$ denote the canonical projection. Let also w be a Coxeter element of W, and set $a := \overline{w} \in \mathfrak{N}(\mathbf{F}_q) \subset G$. Since G/H is compact and totally disconnected, its contraction groups are trivial (see the introduction of [CRW13]). In particular,

$$\pi(\mathrm{con}(a^{\pm 1})) \subseteq \mathrm{con}(\pi(a^{\pm 1})) = \{1\},$$

and hence the closures of the contraction groups $\mathrm{con}(a)$ and $\mathrm{con}(a^{-1})$ are contained in $\ker \pi = H$. It follows from Theorem 4.6 that H contains $\mathfrak{U}^{ma+}(\mathbf{F}_q)$. But G normalises H and contains $\mathfrak{N}(\mathbf{F}_q)$, and hence H also contains all real root groups. Therefore $H = G$, as desired. □

We can now give a common proof for Theorems A and B.

THEOREM 6.2. *Assume that the generalised Cartan matrix A is indecomposable of indefinite type, and let G be one of the complete Kac–Moody groups $\mathfrak{G}_A^{rr}(\mathbf{F}_q)$ or $\mathfrak{G}_A^{pma}(\mathbf{F}_q)$. Then $G/Z'(G)$ is abstractly simple.*

Proof. Set $U^+ := \mathfrak{U}^{rr+}(\mathbf{F}_q)$ or $U^+ := \mathfrak{U}^{ma+}(\mathbf{F}_q)$ so that $U^+ \leqslant G$. Let K be a non-trivial normal subgroup of $G/Z'(G)$. Since $G/Z'(G)$ is topologically simple (see [CR09, Proposition 11] for $\mathfrak{G}_A^{rr}(\mathbf{F}_q)$ and Proposition 6.1 for $\mathfrak{G}_A^{pma}(\mathbf{F}_q)$), K must be dense in G. Since G is locally compact and totally disconnected, it then follows from Theorem 4.6 and Lemma 4.7, together with Theorem 1.1, that K contains U^+. Since U^+ is open, K is open as well, and hence closed in G. Therefore $K = G$, as desired. □

Remark 6.3. We remark that, although we made the assumption that the Kac–Moody group $\mathfrak{G}(k)$ is of simply connected type (to get simplified statements), this of course does not have any impact on the simplicity results, and one might as well consider an arbitrary Kac–Moody root datum $\mathcal{D}$ and the Kac–Moody group $\mathfrak{G}_{\mathcal{D}}(k)$. The essential difference is that, in general, $\mathfrak{G}_{\mathcal{D}}(k)$ is no longer generated by its root subgroups, and one then has to consider a sub-quotient of $\mathfrak{G}^{pma}(k)$ (or else $\mathfrak{G}^{rr}(k)$). More precisely, let G be either $\mathfrak{G}^{pma}(k)$ or $\mathfrak{G}^{rr}(k)$, let U^+ be the corresponding subgroup $\mathfrak{U}^{ma+}(k)$ or $\mathfrak{U}^{rr+}(k)$, and let $G_{(1)}$ be the subgroup of G generated by U^+ and by all root groups of $\mathfrak{G}(k)$. Then $G_{(1)}$ is normal in G (and $G = \mathfrak{T}(k).G_{(1)}$), and what we have proved is the abstract simplicity of $G_{(1)}/(Z'(G) \cap G_{(1)})$.

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