A REMARK ON BOUNDEDNESS OF BLOCH FUNCTIONS

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Two consequences of a theorem of Dahlberg are derived. Let $f$ be a holomorphic function in the unit disk $D$ of the complex plane, and let $E$ be an $F_\sigma$ subset of the unit circle $T$. Suppose that $\lim_{r \to 1^-} |f(rw)| \leq M$, $w \in T \setminus E$, for some constant $M$.

Then $f$ is bounded in either of the two cases:

(i) if $f$ is in the Bloch space and $E$ is of zero measure with respect to the Hausdorff measure associated with the function $\psi(t) = t \log(2\pi e^t/t)$,

(ii) if $f$ is integrable with respect to the planar Lebesgue measure on $D$ and $E$ is of zero measure with respect to the Hausdorff measure associated with the function $\psi(t) = t \log(2\pi e^t/t)$.

Let $D$ be the unit disk of the complex plane, and let $T$ be the unit circle. For a function $\varphi$ satisfying the usual conditions, let $\Lambda_\varphi$ be the Hausdorff measure on $T$ corresponding to $\varphi$. Let $\phi_1(t) = \log(2\pi e^t/t)$, $\phi_2(t) = \log(\phi_1(t))$, $0 < t \leq 2\pi$, and let $\psi_j(t) = t\phi_j(t)$, $j = 1, 2$.

The following theorem is a modification of a theorem of Dahlberg [2, Theorem 4], and can be proved in a very similar manner.

**Theorem.** (Dahlberg). Let $j = 1$ or $j = 2$. Let $E \subset T$ be an $F_\sigma$ set with $\Lambda_\psi_j(E) = 0$. Let $u$ be a sub-harmonic function on $D$. Suppose that there are constants $C$ and $M$ such that

$$u(z) \leq C \phi_j(1 - |z|)$$

for all $z \in D$, and

$$\lim_{r \to 1^-} u(rw) \leq M$$

if $w \in T \setminus E$. Then the function $u$ is bounded above in $D$.

Let $B$ denote the Bloch space, that is, the space of those holomorphic functions $f$ on $D$ which satisfy the condition

$$\sup_{z \in D} \left(1 - |z|^2\right) |f'(z)| < +\infty.$$

Since for any $f \in B$ the function $u = \log |f|$ is sub-harmonic on $D$ and satisfies (A) with $j = 2$ and some constant $C$, we have the following corollary, which is a generalisation of [1, Theorem 2].

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**Corollary 1.** Let $f \in B$. Suppose that there are: a constant $M$ and an $F_\sigma$ set $E \subset \mathbb{T}$ with $\Lambda_{\psi_2}(E) = 0$ such that

$$\lim_{r \to 1-} |f(rw)| \leq M$$

for $w \in \mathbb{T} \setminus E$. Then $f$ is bounded in $D$.

Now, let us observe that if $f$ is a holomorphic function in $D$ and

$$\iint_{x^2+y^2<1} |f(x + iy)| \, dx \, dy < +\infty,$$

then by the mean value property we have

$$|f(z)| \leq \frac{1}{\pi(1-|z|)^2} \iint_{|z+iy-z|<1-|z|} f(x+iy) \, dx \, dy \leq \frac{1}{\pi(1-|z|)^2} \iint_{x^2+y^2<1} |f(x+iy)| \, dx \, dy.$$ 

Therefore the subharmonic function $u = \log |f|$ satisfies (A) with $j = 1$ and some constant $C$. Thus we have

**Corollary 2.** Let $f$ be a holomorphic function in $D$ integrable with respect to the planar Lebesgue measure on $D$. Suppose that there are: a constant $M$ and an $F_\sigma$ set $E \subset \mathbb{T}$ with $\Lambda_{\psi_1}(E) = 0$ such that

$$\lim_{r \to 1-} |f(rw)| \leq M$$

for $w \in \mathbb{T} \setminus E$. Then $f$ is bounded in $D$.

Taking $E = \emptyset$ in Corollary 2 we obtain the affirmative answer to a question asked in [1, p.37]. For a finite set $E$, Corollary 2 proves a conjecture from [3].

**References**

