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REGULAR AND MERCERIAN GENERALIZED LOTOTSKY METHOD

BY

J. F. MILLER AND H. B. SKERRY

ABSTRACT. Necessary conditions and sufficient conditions are obtained for the generalized Lototsky summability matrix method (F, d_n) to be regular and Mercerian. In particular, a set of conditions equivalent to being regular and Mercerian is given for real $\{d_n\}$ and for complex $\{d_n\}$ eventually in any closed half-plane containing the origin.

1. **Introduction.** We are concerned with finding conditions under which the generalized Lototsky summability method (F, d_n) is both regular and Mercerian, i.e., is equivalent to the identity matrix method I. We will need some background material.

DEFINITION 1.1. The generalized Lototsky method (F, d_n) is defined by the triangular matrix $A=(a_{nk})$ which has $a_{00}=1$, $a_{0k}=0$ when $k>0$, and

$$\prod_{j=1}^{n} \frac{z+d_j}{1+d_j} = \sum_{k=0}^{n} a_{nk} z^k, \qquad n \geq 1. \tag{1.2}$$

Here $\{d_n\}_1^\infty$ is an arbitrary complex sequence with $d_n \neq -1$.

To facilitate the notation, we will use $(a+d_n)!$ for $\prod_{j=1}^{n}(a+d_j)$ and $C(0, R)$ for the circle of radius R centered at the origin.

Cauchy's formula, together with (1.2), yields

$$a_{nk} = \frac{1}{(1+d_n)!} \frac{1}{2\pi i} \int_{C(0,R)} \frac{(z+d_n)!}{z^{k+1}} dz. \tag{1.3}$$

If $A^{-1}=(b_{nk})$, an explicit formula for b_{nk} (see [3]) is

$$b_{nk} = (1+d_k)!\,(-1)^{n-k} \frac{1}{2\pi i} \int_{C(0,R)} \frac{z^n}{(z-d_{k+1})!} dz, \tag{1.4}$$

assuming d_j is interior to $C(0, R)$ for $j=1, \ldots, k+1$, and $d_0=0$.

A summability method is regular if it sums every convergent sequence to its natural limit, and is Mercerian if it sums the convergent sequences but no others. Necessary and sufficient conditions for a matrix $A=(a_{nk})$ to be regular

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are (see [1], p. 8)

(i) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$,

(ii) $\lim_{n\to 0} a_{nk} = 0$ for each $k \geq 0$,

(iii) $\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$.

A regular normal (lower triangular with non-zero diagonal elements) matrix A is Mercerian if and only if $\|A^{-1}\| < \infty$.

2. Regular and Mercerian (F, d_n) methods

THEOREM 2.1. *If (F, d_n) is regular and Mercerian, then*

(i) $(1+d_n)!$ *is bounded and bounded away from* 0,

(ii) $\sum_1^n d_j$ *is bounded,*

(iii) $\sum_1^n d_j^2$ *is bounded,*

(iv) $|d_n| < 1$ *for each* n.

Proof. (i) From the regularity condition (iii) above, $|a_{nn}| = 1/|1+d_n|! \leq \|A\| < \infty$, and, because A is Mercerian, $|b_{nn}| = |1+d_n|! \leq \|A^{-1}\| < \infty$. (ii) $a_{n,n-1} = \sum_1^n d_j/(1+d_n)!$, so $|\sum_1^n d_j| = |a_{n,n-1}| \cdot |1+d_n|! \leq \|A\| \cdot \|A^{-1}\|$. (iii) Because

$$a_{n,n-2} = \frac{1}{(1+d_n)!} \sum d_i d_j, \qquad 1 \leq i < j \leq n,$$

and $|a_{n,n-2}| \leq \|A\|$, it follows that

$$\left|\sum d_i d_j\right| \leq \|A\| \cdot |1+d_n|! = 0(1), \qquad 1 \leq i < j \leq n. \tag{2.2}$$

Now (see [3], (3.2)),

$$|b_{n,n-2}| = |1+d_{n-2}|! \cdot \left|\sum_1^{n-1} d_j^2 + \sum d_i d_j\right| \leq \|A^{-1}\|,$$

with $1 \leq i < j \leq n-1$ in the second sum, and this, together with (i) and (2.2) gives the result. (iv) Because A is regular and Mercerian, so is A^{-1}, whence A^{-1} has null columns. In particular, $b_{n0} = (-d_1)^n \to 0$, which implies $|d_1| < 1$. If N is an arbitrary positive integer greater than 1, form the sequence $\{d_n'\}_1^\infty$ from $\{d_n\}_1^\infty$ by interchanging d_1 and d_N, i.e., set

$$d_n' = \begin{cases} d_N, & n = 1 \\ d_1, & n = N \\ d_n, & 1 \neq n \neq N. \end{cases}$$

From (1.2) it is clear that the matrix A', defining the (F, d_n') method, agrees with A except for possibly the first $N-1$ rows after row 0, so the two matrices sum exactly the same sequences and are consistent. It follows that A' is also regular and Mercerian, and thus so is its inverse, so $b_{n0}' = (-d_1')^n = (-d_N)^n \to 0$ and $|d_N| < 1$.

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LEMMA 2.3. *Let* (i) $\sum_{j=1}^{\infty}|u_j(z)|^{k+1}$ *converge uniformly on* $C(0,R)$, *and, if*

$$v_j(z)=\sum_{\nu=1}^{k}(-1)^{\nu-1}\nu^{-1}[u_j(z)]^{\nu},$$

let (ii) $\sum_{j=1}^{n} v_j(z)$ *be bounded uniformly in n and z on* $C(0,R)$. *Suppose that on* $C(0,R)$, *for each n we have* $u_n(z)\neq -1$, $|u_n(z)|\leq 1$, *and* u_n *is continuous. Then* $(1+u_n(z))!$ *is bounded and bounded away from zero uniformly in n and z on* $C(0,R)$.

Proof (This is essentially problem 5, p. 294, of [2].) Write

$$\begin{aligned} |1+u_n(z)|! &= \left|\exp\left\{\sum_1^n \operatorname{Log}(1+u_j(z))\right\}\right| \\ &= \exp\left\{\operatorname{Re}\sum_1^n \operatorname{Log}(1+u_j(z))\right\} \end{aligned} \tag{2.4}$$

to see that the problem reduces to showing that the last exponent is bounded. The first hypothesis implies that $|u_j(z)|\leq 1/2$ on $C(0,R)$ for $j>N>0$. But if $|w|\leq 1/2$, we have

$$\left|\operatorname{Log}(1+w)-\sum_{\nu=1}^{k}(-1)^{\nu-1}\frac{w^{\nu}}{\nu}\right|\leq |w|^{k+1},$$

so for $j>N$ it follows that $|\operatorname{Log}(1+u_j(z))-v_j(z)|\leq |u_j(z)|^{k+1}$, whence

$$\sum_1^{\infty}|\operatorname{Log}(1+u_j(z))-v_j(z)|$$

converges uniformly on $C(0,R)$. The continuity of u_j and (ii) suffice to insure that $\sum_1^n \operatorname{Log}(1+u_j(z))$ is bounded uniformly in n and z on $C(0,R)$, so the result follows from (2.4).

LEMMA 2.5. *Suppose there is a positive integer k such that*
(i) $\sum_{n=1}^{N} d_n^j$ *is bounded in N for each j* ($j=1,\ldots,k$), *and*
(ii) $\sum_{n=1}^{\infty}|d_n|^{k+1}<\infty$.

Then $(1+d_N/z)!$ is bounded and bounded away from 0 uniformly in N and z on any $C(0,R)$ to which the d_n's are interior.

Proof. Apply Lemma 2.3 with $u_n(z)=d_n/z$. We have

$$\sum_{n=1}^{\infty}|d_n/z|^{k+1}=(1/R)^{k+1}\sum_1^{\infty}|d_n|^{k+1}<\infty,$$

and

$$\begin{aligned}\left|\sum_{n=1}^{N} v_n(z)\right| &= \left|\sum_{n=1}^{N}\sum_{\nu=1}^{k}\frac{(-1)^{\nu-1}}{\nu}(d_n/z)^{\nu}\right| \\ &= \left|\sum_{\nu=1}^{k}\frac{(-1)^{\nu-1}}{\nu z^{\nu}}\sum_{n=1}^{N} d_n^{\nu}\right|\leq kM,\end{aligned}$$

where

$$M=\max_{\nu,N}(1/R)^{\nu}\left|\sum_{n=1}^{N} d_n^{\nu}\right|,$$

so the hypotheses of the lemma are met.

THEOREM 2.6. *If $|d_n|<1$ for every n, and there exists a positive integer k such that $\sum_{n=1}^{\infty}|d_n|^{k+1}$ converges, and $\sum_{n=1}^{N} d_n^j$ is bounded in N for each j, $j=1,\ldots,k$, then the (F, d_n) method is regular and Mercerian.*

Proof. Choose R so that $|d_n|<R<1$ for each n. By Lemma 2.5 there is an M such that for every n we have $|1+d_n/z|!\le M$ on $C(0, R)$, i.e., $|z+d_n|!\le M|z|^n$. It is also true that there is an $\varepsilon>0$ such that $|1+d_n|!\ge\varepsilon$ for every n. Then, from (1.3) and Cauchy's estimate, follows

$$|a_{n\nu}|\le MR^{n-\nu}/\varepsilon, \tag{2.7}$$

whence

$$\|A\|=\sup_n\sum_{\nu}|a_{n\nu}|\le\sup_n(M/\varepsilon)\sum_{\nu=0}^{n}R^{n-\nu}\le\frac{M/\varepsilon}{1-R},$$

as well as $\lim_n|a_{n\nu}|=0$. It follows that the (F, d_n) method is regular. The above lemma implies that $|1-d_n/z|!\ge\delta>0$ on $C(0, R)$ for each n, so from (1.4) follows

$$|b_{n\nu}|\le|1+d_{\nu}|!\,R^{n-\nu}/\delta,$$

so

$$\|A^{-1}\|=\sup_n\sum_{\nu}|b_{n\nu}|=0(1)/\delta(1-R).$$

Thus (F, d_n) is Mercerian.

This now leads to our main result,

COROLLARY 2.8. *If each d_n is real, then necessary and sufficient conditions for the (F, d_n) method to be regular and Mercerian are that $|d_n|<1$ for every n, $\sum_1^n d_\nu$ be bounded, and $\sum_1^\infty d_\nu^2<\infty$.*

Proof. The necessity follows immediately from Theorem 2.1. The sufficiency follows from Theorem 2.6 with $k=1$.

This corollary generalizes Theorem 3.18 in [3]. However, this is not the strongest form we can obtain.

COROLLARY 2.9. *If d_n is eventually in the closed upper half-plane or any rotation of it, then (F, d_n) is regular and Mercerian if and only if $|d_n|<1$ for each n, $\sum_1^n d_\nu$ is bounded, and $\sum_1^n d_\nu^2$ is bounded.*

Proof. The necessity follows from Theorem 2.1. To see that the conditions are also sufficient, suppose at first that d_n is eventually in the closed upper half-plane (UHP), so that if $d_n = x_n + iy_n$, then $y_n \geq 0$ for all large n. The boundedness of $\sum_1^n d_\nu^2$ implies that of its real part $\sum_1^n (x_\nu^2 - y_\nu^2)$. Similarly, the boundedness of $\sum_1^n d_\nu$ implies the convergence of $\sum_1^\infty y_\nu$, and hence that of $\sum_1^\infty y_\nu^2$. But then $\sum_1^n x_\nu^2$ is bounded and $\sum_1^\infty x_\nu^2$ converges. It follows that $\sum_1^\infty |d_n|^2 < \infty$. The result now follows from Theorem 2.6 with $k = 1$. Now suppose d_n is eventually in a rotation of UHP through the angle α. Let $d_n' = e^{-i\alpha} d_n$; then d_n' is eventually in UHP. Clearly, the boundedness of $\sum_1^n d_\nu'$ and $\sum_1^n (d_\nu')^2$ follow from that of $\sum_1^n d_\nu$ and $\sum_1^n d_\nu^2$, so $\sum_1^\infty |d_\nu'|^2 = \sum_1^\infty |d_\nu|^2 < \infty$. Theorem 2.6 gives the result.

A result of a similar nature which substitutes sectors of a circle for a half-plane is

COROLLARY 2.10. *Let $|d_n| < 1$ for each n and suppose that there is a positive integer k such that $\sum_{\nu=1}^n d_\nu^j$ is bounded in n for each j, $j = 1, \ldots, k$. Let $0 < \varepsilon < 1$ and suppose that if* $\arg d_n = \theta_n$, *then θ_n eventually lies in the union of the k sectors defined by*

$$\frac{(4j-1)\pi}{2k} + \frac{\varepsilon}{k} \leq \operatorname{Arg} z \leq \frac{(4j+1)\pi}{2k} - \frac{\varepsilon}{k}, \qquad j = 0, 1, \ldots, k-1. \tag{2.11}$$

Then (F, d_n) is regular and Mercerian. The result also holds for any rotation of these k sectors.

Proof. If $j = k$ and $d_n = r_n \exp(i\theta_n)$, then $|\sum_{\nu=1}^n d_\nu^k| = |\sum_{\nu=1}^n r_\nu^k \exp(ik\theta_\nu)| \leq M$ for each n, so $|\sum_{\nu=1}^n r_\nu^k \cos k\theta_\nu| \leq M$ for each n. But, because $k\theta_\nu$ is eventually in $[-(\pi/2)+\varepsilon, (\pi/2)-\varepsilon]$ (mod 2π), $\cos(k\theta_\nu) \geq \delta > 0$ for all large ν. It follows that $\sum_1^\infty r_\nu^k = \sum_1^\infty |d_\nu|^k < \infty$, and Theorem 2.6 gives the result. If the k sectors are rotated through the angle α, then an argument similar to that used in the proof of the above corollary shows that the result still holds.

Theorem 2.6 and its corollaries require that $\sum_n |d_n|^k$ converges for some positive integer k. It turns out, however, that this is not a necessary condition for a generalized Lototsky method to be regular and Mercerian. In order to present the example which demonstrates this, we need

THEOREM 2.12. *If $d_n \to 0$, then (F, d_n) is regular.*

Proof. If $0 < R < 1$, (1.3) implies that

$$|a_{nk}| = \frac{1}{2\pi} \left| \int_{C(0,R)} \frac{(z+d_n)!}{(1+d_n)!} \cdot \frac{1}{z^{k+1}} \, dz \right|$$

Because $1+x\le e^x$ for all real x, we may write

$$\left|\frac{z+d_\nu}{1+d_\nu}\right|\le\exp\left(-1+\left|\frac{z+d_\nu}{1+d_\nu}\right|\right)=\exp\left(\frac{-|1+d_\nu|+|z+d_\nu|}{|1+d_\nu|}\right)$$

$$\le\exp\left(\frac{|z|+|d_\nu|-|1-|d_\nu||}{|1+d_\nu|}\right)$$

$$=\begin{cases}\exp\left(\dfrac{R+1}{|1+d_\nu|}\right), & |d_\nu|\ge 1\\[2mm] \exp\left(\dfrac{R-1+2\,|d_\nu|}{|1+d_\nu|}\right), & |d_\nu|<1.\end{cases}$$

There exist N and $\varepsilon>0$ such that $R-1+2\,|d_\nu|\le-\varepsilon$ and $|d_\nu|<1$ when $\nu>N$. Moreover, $R-1+2\,|d_\nu|<R+1$ when $1\le\nu\le N$ and $|d_\nu|<1$. Thus,

$$\text{(2.13)}\qquad \begin{aligned}|a_{nk}|&\le\frac{1}{R^k}\exp\left(\sum_{\nu=1}^{N}\frac{R+1}{|1+d_\nu|}+\sum_{N+1}^{n}\frac{-\varepsilon}{|1+d_\nu|}\right)\\ &=0(1)\frac{1}{R^k}\exp\left(-\varepsilon\sum_{N+1}^{n}\frac{1}{|1+d_\nu|}\right)\to 0\quad\text{as}\quad n\to\infty,\end{aligned}$$

so the (F, d_n) matrix A has null columns. From (2.13) follows

$$\text{(2.14)}\qquad \begin{aligned}\sum_k|a_{nk}|&=0(1)\exp\left(-\varepsilon\sum_{N+1}^{n}\frac{1}{|1+d_\nu|}\right)\cdot\sum_{0}^{n}\frac{1}{R^k}\\ &=0(R^{-n})\exp\left(-\varepsilon\sum_{N+1}^{n}\frac{1}{|1+d_\nu|}\right).\end{aligned}$$

Because $|1+d_n|^{-1}\to 1$, so does its Cesàro transform:

$$\lim\frac{1}{n}\sum_{\nu=1}^{n}\frac{1}{|1+d_\nu|}=1;$$

it follows that there is an $\alpha>0$ such that the transform exceeds α for all large n, whence

$$\text{(2.15)}\qquad \varepsilon\sum_{1}^{n}|1+d_\nu|^{-1}\ge n\varepsilon\alpha>0\quad\text{for all large } n.$$

We may assume that R has been chosen in $(0, 1)$ so that $\log(1/R)=\varepsilon\alpha$. Then from (2.15) follows

$$\text{(2.16)}\qquad \varepsilon\sum_{N+1}^{n}|1+d_\nu|^{-1}\ge-\varepsilon\sum_{1}^{N}|1+d_\nu|^{-1}+n\cdot\log(1/R)$$

for large n. Choose δ in $(0, 1)$ so that $-\varepsilon\sum_1^N|1+d_\nu|^{-1}=\log(\delta)$. Then (2.16) implies

$$\exp\left(\varepsilon\sum_{N+1}^{n}|1+d_\nu|^{-1}\right)\ge\delta/R^n$$

for large n, whence (2.14) gives $\sum_k |a_{nk}| = 0(1)/\delta$, so $\|A\| < \infty$. It follows that A is regular.

By choosing $d_n = e^{in\pi\sqrt{2}}/\log(n+2)$ $(n = 1, 2, \ldots)$, it is possible to show (making use of Theorem 2.12) that (F, d_n) is regular and Mercerian, even though in this case we have $\sum_n |d_n|^k = +\infty$ for every positive integer k. This example shows that the hypotheses of Theorem 2.6, while sufficient, are not all necessary.

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The Pennsylvania State University
Berks Campus,
Reading, PA

Bell Laboratories
Holmdel, NJ

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