SOME NOTES ON THE THEORY OF HOLOMORPHIC CURVES

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§ 1. Introduction

In this paper, we shall give some notes on the order functions of holomorphic curves and, applying these facts, we may formulate the theory of holomorphic curves more precisely than those in Ahlfors [2], Cowen and Griffiths [4], Weyl [7] or Wu [8] in several cases.

Let \( x : |z| < R \to P^n(C) \) \((n \geq 1, 0 < R \leq \infty)\) be a non-degenerate holomorphic curve and \( x^p \) be the associated curve of rank \( p \) of \( x \) \((p = 1, \cdots, n; x^1 = x)\).

Let

\[
X = (x_1(z), \cdots, x_{n+1}(z))
\]

be a reduced representation of \( x \) where \( x_1, \cdots, x_{n+1} \) are holomorphic functions in \(|z| < R\) without common zero for all and

\[
X^p = [X, dX/dz, \cdots, d^{p-1}X/dz^{p-1}] \quad (p = 1, \cdots, n)
\]

the osculating \( p \)-element of \( X (X^1 = X) \) (see Weyl [7]), which is a representation of \( x^p \). Let \( T_p(r) \) be the order function of \( x^p \) \((p = 1, \cdots, n)\). Then it is known that

\[
V_p(r) + \{T_{p+1}(r) - 2T_p(r) + T_{p-1}(r)\} = \Omega_p(r) - \Omega_p(r_0) \quad (r_0 \leq r < R)
\]

where \( r_0 \) is a fixed positive number, \( V_p(r) \) is the valence in \(|z| < r\) of the stationary points of rank \( p \) of \( x \) and

\[
\Omega_p(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|X^{p+1}||X^{p-1}||X^p\| \, d\theta \quad (z = re^{i\theta})
\]

(Weyl [7], p. 123).

In the theory of holomorphic curves, it is essential to evaluate \( \Omega_p \) in terms of \( T_p \). For example,

"When \( R = \infty \), for any number \( \alpha > 1 \), the inequality

\[
\sum_{p=1}^{n} \Omega_p(r) \leq \frac{1}{2\pi} \int_0^{2\pi} \log \left( \sum_{p=1}^{n} \|X^{p+1}||X^{p-1}||X^p\| \right) \, d\theta
\]

is true."

Received June 15, 1979.
holds except for \( r \) of an open set \( E \subset [r_0, \infty) \) such that
\[
\int_E r^{-1}dr < \infty.
\]
(See Weyl [7], Chapter III.) There is a similar result for \( R < \infty \) (Weyl [7], Chapter IV).

We note that, in this paper, we always use \( r = |z| \) as the independent variable instead of \( \log r \). (cf. Weyl [7], Wu [8].)

In the proof of the defect relations for holomorphic curves, this estimate of \( \Omega_p \) plays a fundamental role and necessarily it comes out exceptional sets in many inequalities of which we are in need to prove the defect relations. (See Ahlfors [2], Weyl [7] etc.)

On the other hand, in the Nevanlinna theory of meromorphic functions in \( |z| < R \) (Hayman [5], Nevanlinna [6]), the second fundamental theorem tells us that, for a non-constant meromorphic function \( f(z) \) in \( |z| < R \), the exceptional set does not come out if the order of \( f \) is finite and the defect relation holds either
(I) for any non-constant meromorphic function when \( R = \infty \), or
(II) for any meromorphic function \( f \) such that
\[
\limsup_{r \to R} T(r, f)/\log (R - r)^{-1} = \infty
\]
when \( R < \infty \).

For the systems of holomorphic functions in \( |z| < R \), similar results are known (Cartan [3]).

In this paper, after the model of the case of meromorphic functions stated above, we shall remove the exceptional sets in the case of holomorphic curves applying the method used in Ahlfors [1] when \( T \) is of finite order and weaken "the hypothesis \( H' \)" (Weyl [7], p. 201) in the case of \( R < \infty \).

We will use the notation used in Weyl [7] in the main.

§ 2. Lemmas

We prepare some lemmas for later use.

**Lemma 1.** Let \( f(r) \) be a function defined on \( [r_0, R) \) with continuous non-negative derivative and \( f(r_0) \geq 1 \). Then for any numbers \( \alpha > 1 \) and \( \mu \geq 0 \),

\[
2\Omega_p(r) \leq \alpha \log T_p(r) - 2 \log r + O(1)
\]
(I) when $R = \infty$, the inequality
\[ f'(r) \leq \{f(r)\}^s r^{s-1} \]
holds except in an open set $E \subset [r_o, \infty)$ such that
\[ \int_E r^{s-1} dr \leq (\alpha - 1)^{-1}; \]
(II) when $R < \infty$, the inequality
\[ f'(r) \leq \{f(r)\}^s (R - r)^{-s-1} \]
holds except in an open set $E \subset [r_o, R)$ such that
\[ \int_E (R - r)^{-s-1} dr \leq (\alpha - 1)^{-1}. \]
(See Weyl [7] p. 155 and p. 197.)

**Lemma 2.** Let two functions $f(r)$ and $F(r)$ be given on $[r_o, R)$ ($r_o > 0$) such that $f(r)$ is continuous and
\[ 1 + \log (r/r_o) + \int_{r_o}^r \{(\log r - \log t) \exp (f(t))\}/tdt \leq F(r) \quad (r_o \leq r < R). \]
Then, for any numbers $\alpha > 1$ and $\mu \geq 0$,
(I) when $R = \infty$,
\[ f(r) \leq \alpha^2 \log F(r) + \mu(\alpha + 1) \log r \]
for all $r \geq r_o$ except in an open set $E \subset [r_o, \infty)$ such that
\[ \int_E r^{s-1} dr \leq 2(\alpha - 1)^{-1}; \]
(II) when $R < \infty$,
\[ f(r) \leq \alpha^2 \log F(r) + (\mu + 1)(\alpha + 1) \log (R - r)^{-1} + (\alpha + 1) \log r \]
for all $r$ in $[r_o, R)$ except in an open set $E \subset [r_o, R)$ such that
\[ \int_E (R - r)^{-s-1} dr \leq 2(\alpha - 1)^{-1}. \]

We may prove this lemma as in Weyl [7], p. 156 or p. 197 using Lemma 1.

**Definition.** We say that a holomorphic curve $x$ in $|z| < R$ is admissible if either $R = \infty$ and $x$ is non-degenerate, or if $R < \infty$, $x$ is non-degenerate and the following condition holds:
\[ \lim_{r \to R} T_p(r) = \infty \quad (p = 1, \ldots, n). \]

Note that, if \( R = \infty \) and \( x \) is non-degenerate, the condition (2) holds. (See Weyl [7].)

**Lemma 3.** For any admissible curve \( x \) in \(|z| < R\), the following inequality holds for all \( r \) in \([R_o, R) \) \( (r_e \leq R)\):

\[
1 + \log \left(\frac{r}{r_0}\right) + \int_{r_0}^r \left\{ \left(\log \frac{r}{t} - \log t \right) \exp \left(2\tilde{\Omega}_p(t)\right) \right\} dt \leq 2T_p(r)
\]

where \( \tilde{\Omega}_p(t) = \Omega_p(t) + \log \left(\frac{t}{r_0}\right) \) and \( R_o \) depends on the curve. (See Weyl [7], p. 154 and p. 196.)

From now on in § 2 and § 3, we assume that all curves in our mind are admissible.

Applying Lemmas 1 and 2 to our curves, we obtain the following by Lemma 3.

**Proposition 1.** For any numbers \( \alpha > 1 \) and \( \mu \geq 0 \),

(I) when \( R = \infty \), the inequality

\[
2\tilde{\Omega}_p(r) \leq \alpha^2 \log T_p(r) + \mu(\alpha + 1) \log r + O(1)
\]

holds for all \( r \geq R_o \) except in an open set \( E \subset [R_o, \infty) \) such that

\[
\int_E r^{s-1}dr < \infty ;
\]

(II) when \( R < \infty \), the inequality

\[
2\tilde{\Omega}_p(r) \leq \alpha^2 \log T_p(r) + (\mu + 1)(\alpha + 1) \log (R - r)^{-1} + O(1)
\]

holds for all \( r \) in \([R_o, R)\) except in an open set \( E \subset [R_o, R) \) such that

\[
\int_E (R - r)^{-s-1}dr < \infty .
\]

(See Weyl [7], p. 198.)

Using this proposition, we obtain the following as in Weyl [7].

**Proposition 2.** For any numbers \( \epsilon > 0 \) and \( \mu \geq 0 \), there exists an \( r_1 \) such that

(I) when \( R = \infty \),

\[ T_{p+1}(r) < (1 + \frac{1}{p} + \epsilon)T_p(r) + O(\log r), \]


\( T_p^{-}(r) < (1 + 1/(n + 1 - p) + \varepsilon)T_p(r) + O(\log r) \)

for all \( r \geq r_i \), except in an open set \( E \subset [r_i, \infty) \) such that

\[
\int_E r^{\rho_p}dr < \infty ;
\]

(II) when \( R < \infty \),

\( T_p^{+}(r) < (1 + 1/p + \varepsilon)T_p(r) + O(\log (R - r)^{-1}) \),

\( T_p^{-}(r) < (1 + 1/(n + 1 - p) + \varepsilon)T_p(r) + O(\log (R - r)^{-1}) \)

for all \( r \in [r_i, R) \) except in an open set \( E \subset [r_i, R) \) such that

\[
\int_E (R - r)^{-\rho_p}dr < \infty .
\]

§ 3. Theorems and applications

In this section, we are going to investigate the orders of \( T_p(r) \) and improve Propositions 1 and 2.

The order \( \rho_p \) of \( T_p(r) \) is defined by

\[
\limsup_{r \to \infty} \frac{\log T_p(r)}{\log r} = \rho_p \quad (R = \infty)
\]

or

\[
\limsup_{r \to R} \frac{\log T_p(r)}{\log (R - r)^{-1}} = \rho_p \quad (R < \infty)
\]

and the lower order \( \lambda_p \) of \( T_p(r) \) by

\[
\liminf_{r \to \infty} \frac{\log T_p(r)}{\log r} = \lambda_p \quad (R = \infty)
\]

or

\[
\liminf_{r \to R} \frac{\log T_p(r)}{\log (R - r)^{-1}} = \lambda_p \quad (R < \infty).
\]

Theorem 1. All \( T_p(r) \) are of the same order.

Proof. When \( R = \infty \), this was proved by Ahlfors [2]. We prove this theorem when \( R < \infty \) applying the method used in Ahlfors [1].

Now, let \( \rho_p \) be finite, then for any \( \rho > \rho_p \), there is an \( r_i(\geq r_i) \) such that, for all \( r \in [r_i, R) \),

\[
T_p(r) \leq O((R - r)^{-\rho}) .
\]

Here, we apply Proposition 2, (II), (5).
(i) When \( r \in E \) and \( r_2 \leq r < R \), we have
\[
T_{p+1}(r) \leq O((R - r)^{-\rho}).
\]

(ii) When \( r \in E \) and \( r_2 \leq r < R \), let \( r' \) be the right hand end point of the maximal interval included in \( E \) and containing \( r \). Then, putting \( \mu = \rho \), we have
\[
(R - r')^{-\rho} - (R - r)^{-\rho} \leq \rho \int_E (R - r)^{-\rho - 1} dr = O(1)
\]
so that
\[
(R - r')^{-\rho} \leq (R - r)^{-\rho} + O(1)
\]
and
\[
\log (R - r')^{-1} = \log (R - r)^{-1} + O(1).
\]

By (i) and (ii), for all \( r \) sufficiently near \( R \)
\[
T_{p+1}(r) < O((R - r)^{-\rho}).
\]
This means \( \rho_{p+1} \leq \rho \). As \( \rho \) is arbitrarily greater than \( \rho_p \), we obtain \( \rho_{p+1} \leq \rho_p \). Similarly, we obtain \( \rho_{p-1} \leq \rho_p \) applying (6). It follows that, if one of \( \rho_p \) is finite, then all \( \rho_p \) are finite and same. This means also that if one of \( \rho_p \) is infinite, then all \( \rho_p \) are infinite. That is, all \( T_p(r) \) are of the same order.

**Theorem 2.** All \( T_p(r) \) are of the same lower order.

**Proof.** (I) \( R = \infty \). We have only to prove the case when one of \( T_p(r) \) is of positive or infinite lower order. Otherwise, all \( T_p(r) \) are of lower order zero.

Now, let \( \lambda_p \) be positive or infinite. Then, for any \( 0 < \lambda < \lambda_p \), there is an \( r_3(\geq r_1) \) such that
\[
T_p(r) \geq r^i \quad (r \geq r_3).
\]

We apply Proposition 2, (i), (3).

(i) For \( r \in E \) and \( r \geq r_3 \),


\[ r^i \leq (1 + 1/(p - 1) + \varepsilon) T_{p-1}(r) + O(\log r) \, . \]

(ii) For \( r \in E \) and \( r \geq r_\alpha \), let \( \bar{r} \) be the left hand end point of the maximal interval included in \( E \) and containing \( r \). Then, putting \( \mu = \lambda \), we have

\[ r^i - \bar{r}^i = \lambda \int_{\bar{r}}^r t^{i-1}dt \leq \lambda \int_{E} t^{i-1}dt = O(1) \, , \]

so that

\[ r^i \leq \bar{r}^i + O(1) \, . \]

As \( \bar{r} \in E \) for sufficiently large \( r \) and \( T_{p-1}(r) \) is increasing,

\[ \bar{r}^i \leq (1 + 1/(p - 1) + \varepsilon) T_{p-1}(\bar{r}) + O(\log \bar{r}) \]
\[ \leq (1 + 1/(p - 1) + \varepsilon) T_{p-1}(r) + O(\log r) \]

and

\[ r^i \leq (1 + 1/(p - 1) + \varepsilon) T_{p-1}(r) + O(\log r) + O(1) \, . \]

By (i) and (ii), for all sufficiently large \( r \),

\[ r^i \leq (1 + 1/(p - 1) + \varepsilon) T_{p-1}(r) + O(\log r) + O(1) \, . \]

This means \( \lambda \leq \lambda_{p-1} \). As \( \lambda < \lambda_p \) and \( \lambda \) is arbitrary, we have \( \lambda_p \leq \lambda_{p-1} \).

Similarly, using (4) instead of (3), we have \( \lambda_p \leq \lambda_{p+1} \). It follows that, if one of \( \lambda_p \) is not zero, then all \( \lambda_p \) are not zero and same.

(II) \( R < \infty \). We can prove this theorem by using Proposition 2, (II) as in the case \( R = \infty \).

THEOREM 3. If \( T_p(r) \) is of finite order, then, for any number \( \alpha > 1 \),

(I) when \( R = \infty \), the inequality

\[ 2\tilde{\omega}_p(r) \leq \alpha^2 \log T_p(r) + O(\log r) + O(1) \]

holds for all sufficiently large values \( r \);

(II) when \( R < \infty \), the inequality

\[ 2\tilde{\omega}_p(r) \leq \alpha^2 \log T_p(r) + O(\log (R - r)^{-1}) + O(1) \]

holds for all \( r \) sufficiently near \( R \).

Proof. (I) \( R = \infty \). Let \( T_p(r) \) be of finite order \( \rho \). Then, by Theorem 1, all \( T_p(r) \) are of order \( \rho \). Thus, for any \( \rho > \rho \), there is an \( r_\alpha(\geq R_\alpha) \) such that
and for any \( p \)-ad \( A^p \) (Weyl [7]), as
\[
N_p(r, A^p) \leq T_p(r) + C_p,
\]
\( C_p \) being independent of \( A^p \) (Ahlfors [2], p. 7 or Wu [8], p. 105),

\[
(7) \quad n_p(r, A^p) \log 2 \leq \int_r^r n_p(t, A^p)/tdt \leq N_p(2r, A^p) = O(r^s)
\]
for \( r \geq r_0 \). Now, putting \( \mu = \rho \) in Proposition 1,

\[
(8) \quad 2\mathcal{Q}_p(r) \leq \alpha^2 \log T_p(r) + \rho(\alpha + 1) \log r + O(1)
\]
for all \( r \geq R_0 \) except in an open set \( E \subset [R_0, \infty) \) such that
\[
\int_E r^{-1}dr < \infty.
\]

For \( r \in E \) and \( r \geq r_0 \), let \( \bar{r} \) be the right hand end point of the maximal interval included in \( E \) and containing \( r \). Then,

\[
(9) \quad \bar{r} - r = \rho \int_r^r t^{-1}dt \leq \rho \int_E t^{-1}dt = O(1)
\]
and

\[
(10) \quad \log \bar{r} = \log r + O(1).
\]

Further, by (7) and (9),

\[
N_p(\bar{r}, A^p) - N_p(r, A^p) = \int_r^r n_p(t, A^p)/tdt \leq O\left(\int_r^r t^{-1}dt\right) = O(1),
\]
that is,

\[
N_p(\bar{r}, A^p) \leq N_p(r, A^p) + O(1).
\]

As
\[
\max_{A^p} N_p(r, A^p) = T_p(r)
\]
(Ahlfors [2], p. 8 or Wu [8], p. 107) and \( O(1) \) is independent of \( A^p \),

\[
(11) \quad T_p(\bar{r}) \leq T_p(r) + O(1).
\]

On the other hand, by (1), we have the equality:
\[
V_p(r) + T_p-1(r) + T_p,1(r) + \log r/r_o = \mathcal{Q}_p(r) - \mathcal{Q}_p(r_o) + 2T_p(r)
\]
and this shows that
\[ 2T_p(r) + \tilde{O}_p(r) \]
is increasing. Thus, we have
\[ 2T_p(r) + \tilde{O}_p(r) \leq 2T_p(\tilde{r}) + \tilde{O}_p(\tilde{r}) , \]
so that by (11),
\[ \tilde{O}_p(r) \leq \tilde{O}_p(\tilde{r}) + O(1) . \]
As \( \tilde{r} \in E \), by (8), (10) and (11), we have
\[ 2\tilde{O}_p(r) \leq \alpha^2 \log T_p(r) + \rho(\alpha + 1) \log r + O(1) . \]
Thus, for any \( r \geq r_* \), the inequality
\[ 2\tilde{O}_p(r) \leq \alpha^2 \log T_p(r) + O(\log r) + O(1) \]
holds.

(II) \( R < \infty \). We can carry out the proof parallel to the case \( R = \infty \).
Corresponding to Proposition 2, we have

**Corollary 1.** If \( T_1(r) \) is of finite order, for any \( \varepsilon > 0 \), there is an \( r_* \)
such that

(I) when \( R = \infty \),
\[
T_{p+1}(r) < (1 + 1/p + \varepsilon)T_p(r) + O(\log r) ,
\]
\[
T_{p-1}(r) < (1 + 1/(n + 1 - p) + \varepsilon)T_p(r) + O(\log r) ,
\]
for all \( r \geq r_* \);

(II) when \( R < \infty \),
\[
T_{p+1}(r) < (1 + 1/p + \varepsilon)T_p(r) + O(\log (R - r)^{-1}) ,
\]
\[
T_{p-1}(r) < (1 + 1/(n + 1 - p) + \varepsilon)T_p(r) + O(\log (R - r)^{-1})
\]
for all \( r \in [r_*, R) . \)

When \( R = \infty \), it is well-known (see Wu [8]) that if the original curve \( x \) is transcendental:
\[ \lim_{r \to \infty} T_1(r)/\log r = \infty , \]
then all the associated curves \( x^p \) are also transcendental.

Similarly to this fact, we give the following for \( R < \infty \).
THEOREM 4. When $R$ is finite, if

$$\limsup_{r \to R} T(r)/\log (R - r)^{-1} = \infty,$$

then there exists a sequence $\{s_n\}$ outside an exceptional set $E$ such that

$$\lim_{n \to \infty} s_n = R$$

and

$$\lim_{n \to \infty} T_p(s_n)/\log (R - s_n)^{-1} = \infty \quad (p = 1, 2, \ldots, n).$$

Proof. When the order of $T(r) = \rho_1$ is finite, we have the result easily by Corollary 1. When the order of $T(r)$ is infinite, then $T_1(r), \ldots, T_n(r)$ are also of order infinite by Theorem 1. In this case, we apply Proposition 2, (II), (6) for $\mu = 0$. First of all, we note that

$$\rho'_1 = \limsup_{r \in E} \log T(r)/\log (R - r)^{-1} = \infty.$$ 

In fact, suppose that $\rho'_1$ is finite. For any $\tilde{r} \in E$ sufficiently near $R$, let $t_1$ be the left hand end point of the maximal interval $I$ included in $E$ and containing $\tilde{r}$ and $t_2$ the right hand end point of $I$. Then, $t_1, t_2 \in E$ and

$$\log (R - t_1)^{-1} - \log (R - t_2)^{-1} = \int_{t_1}^{t_2} (R - r)^{-1} dr \leq \int_0^{\tilde{r}} (R - r)^{-1} dr = O(1),$$

$$\log (R - t_1)^{-1} < \log (R - \tilde{r})^{-1} < \log (R - t_2)^{-1}.$$ 

From this and $T(r)$ being increasing, we have the following:

$$\log T(t_1)/\log (R - t_1)^{-1} \leq \log T(\tilde{r})/\log (R - \tilde{r})^{-1} \leq \log T(t_2)/\log (R - t_2)^{-1}$$

and

$$\lim_{r \to R} \log (R - t_1)^{-1}/\log (R - t_2)^{-1} = 1$$

so that

$$\limsup_{r \in E} \log T(\tilde{r})/\log (R - \tilde{r})^{-1} \leq \limsup_{r \in E} \log T(r)/\log (R - r)^{-1} = \rho'_1.$$ 

This means that the order of $T(r)$ is finite. This is a contradiction. $\rho'_1$ must be $\infty$. "$\rho'_1 = \infty"$ means that there is a sequence $\{s_n\} \subset [r, R) - E$ such that $s_n \to R$ ($n \to \infty$) and
Applying this fact to Proposition 2, (II), (6) for \( p = 2, \ldots, n \), we have the desired result.

**COROLLARY 2.** "The hypothesis H" in Weyl [7], p. 201 may be changed by the following:

"When \( R \) is finite,

\[
x \text{ is admissible and } \limsup_{r \to R} T_i(r)/\log(R - r)^{-1} = \infty.
\]

**REFERENCES**


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