MULTIPLICITIES IN SYLOW SEQUENCES AND THE SOLVABLE RADICAL

GIL KAPLAN and DAN LEVY

(Received 3 March 2008)

Abstract

A complete Sylow sequence, \( P = P_1, \ldots, P_m \), of a finite group \( G \) is a sequence of \( m \) Sylow \( p_i \)-subgroups of \( G \), one for each \( p_i \), where \( p_1, \ldots, p_m \) are all of the distinct prime divisors of \( |G| \). A product of the form \( P_1 \cdots P_m \) is called a complete Sylow product of \( G \). We prove that the solvable radical of \( G \) equals the intersection of all complete Sylow products of \( G \) if, for every composition factor \( S \) of \( G \), and for every ordering of the prime divisors of \( |S| \), there exist a complete Sylow sequence \( P \) of \( S \), and \( g \in S \) such that \( g \) is uniquely factorizable in \( P \). This generalizes our results in Kaplan and Levy [‘The solvable radical of Sylow factorizable groups’, Arch. Math. 85(6) (2005), 490–496].


Keywords and phrases: Sylow products, Sylow sequences, solvable radical, Sylow multiplicity, unique.

1. Introduction

Let \( G \) be a finite group\(^1\) and let \( \pi(G) = \{p_1, \ldots, p_m\} \) denote the set of all distinct prime divisors of \( |G| \). If \( A \) is a subgroup or a quotient group of \( G \), and \( \tau \) is a permutation of \( 1, \ldots, m \), then a complete Sylow sequence of \( A \) of type \( \tau \) is a sequence of the form \( P = P_{\tau(1)} \cdots P_{\tau(m)} \), where each \( P_j \) is a Sylow \( p_j \)-subgroup of \( A \) and trivial Sylow subgroups of \( A \) (those for which \( p_j \) does not divide \( |A| \)) are ignored. Note that the length of such a sequence is \( |\pi(A)| \). The corresponding product \( \Pi(P) = P_{\tau(1)} \cdots P_{\tau(m)} \) (which is a subset of \( A \)) is called a complete Sylow product of \( A \) of type \( \tau \). Whenever the precise ordering is immaterial, we just use the term complete Sylow sequence (product), and \( P = P_1 \cdots P_m \) will refer to a fixed but otherwise arbitrary ordering of the primes in \( \pi(G) \). We denote by \( CSS(A) \) (\( CSS_\tau(A) \)) the set of all complete Sylow sequences of \( A \) (of type \( \tau \)).

The earliest reference known to us that considers complete Sylow products is a paper by Miller [10]. Miller proves that if \( G \) is solvable, then \( \Pi(P) = G \) for every complete Sylow sequence \( P \) of \( G \), and raises the question whether the reverse

\(^1\) All groups considered in this paper are assumed to be finite.

© 2009 Australian Mathematical Society 0004-9727/09 $A2.00 + 0.00

477
implication is also true. The same claim and question are also discussed, much later and independently, by Hall in [5]. Thanks to the work of Thompson on the classification of \(N\)-groups [11, Corollary 3] the following solvability criterion is established: A group \(G\) is solvable if and only if \(\Pi(\mathcal{P}) = G\) for every complete Sylow sequence \(\mathcal{P}\) of \(G\) [1, Theorem 1], [8, Theorem A]. A closely related result already appears in [3] (see also [7, 17.14]).

In [8, 9] we found a connection between complete Sylow products of an arbitrary group and its solvable radical. For a given group \(G\), we denote by \(H_\tau(G)\) the intersection of all complete Sylow products of \(G\) of type \(\tau\), and by \(H(G)\), the intersection of all complete Sylow products of \(G\) (obviously \(H(G) \subseteq H_\tau(G)\)). Let \(R(G)\) denote the solvable radical of \(G\). We proved that \(H_\tau(G)\) and \(H(G)\) are characteristic subgroups of \(G\), and that \(R(G) \leq H(G)\). We have also given certain sufficient conditions for equality. It is still an open question whether \(H(G) = R(G)\) for every group \(G\), or equivalently, whether \(H(G)\) is always solvable.

In the present paper we consider these issues using the concept of multiplicity of an element in a Sylow sequence. This concept also appears in [10] under a different name.

**Definition 1.1.** Let \(G\) be a group and let \(\mathcal{P} = P_1,\ldots, P_m\) be a complete Sylow sequence of \(G\). A factorization of \(g \in G\) in \(\mathcal{P}\) is a sequence \(g_1,\ldots, g_m\) where \(g_i \in P_i\) such that \(g = g_1 \cdots g_m\). When convenient, we shall refer to an expression of the form \(g = g_1 \cdots g_m\) as a factorization of \(g \in G\). The multiplicity of \(g\) in \(\mathcal{P}\) is the number of distinct factorizations of \(g\) in \(\mathcal{P}\). This nonnegative integer will be denoted \(m_\mathcal{P}(g)\).

Observe that, for any complete Sylow sequence \(\mathcal{P}\) of \(G\), we have \(\sum_{g \in G} m_\mathcal{P}(g) = |G|\). Hence, \(\Pi(\mathcal{P}) = G\) if and only if \(m_\mathcal{P}(g) = 1\) for all \(g \in G\). If such a \(\mathcal{P}\) exists we say that \(G\) is Sylow factorizable. In [9] we have proved that if \(G\) is Sylow factorizable then \(R(G) = H(G)\). However, it is known (see [6] and Proof of Proposition 1.5 here) that not all groups are Sylow factorizable. This motivates the examination of weaker conditions on multiplicities in Sylow sequences.

**Definition 1.2.** Let \(G\) be a group and let \(\tau\) be an arbitrary ordering of \(\pi(G)\). Then \(G\) is \(\tau\)-unique if there exists a complete Sylow sequence \(\mathcal{P}\) of type \(\tau\) such that \(m_\mathcal{P}(g) = 1\) for some \(g \in G\). We shall say that \(G\) is unique if \(^1\) it is \(\tau\)-unique for all \(\tau\).

Our first result shows that the \(R(G) = H(G)\) question is related to the question of which simple groups are unique.

**Theorem 1.3.** Let \(G\) be a group such that all composition factors of \(G\) are unique. Then \(R(G) = H(G)\).

The question of whether a group \(G\) is unique is itself related to the question of whether its composition factors are unique.

**Theorem 1.4.** The group property unique is closed under extensions.

\(^1\) The trivial group is defined to be unique.
Determining which simple nonabelian groups are unique (simple abelian groups are trivially unique) does not seem to be a particularly easy problem, and we shall not attempt to address it here in its general form. Nevertheless, the following very limited result may serve as an appetizer.

**Proposition 1.5.** All simple nonabelian groups $G$ such that $|G|$ is divisible by exactly three primes, are unique.

Now we turn our attention to transformations which preserve the multiplicity of an element in a Sylow sequence.

**Definition 1.6.** Let $G$ be a group and let $\tau$ be some ordering of $\pi(G)$. Then

$$M_\tau(G) \overset{\text{def}}{=} \{ g \in G \mid m_\mathcal{P}(gx) = m_\mathcal{P}(x), \text{ for all } x \in G, \text{ for all } \mathcal{P} \in \text{CSS}_\tau(G) \}$$

$$M(G) \overset{\text{def}}{=} \{ g \in G \mid m_\mathcal{P}(gx) = m_\mathcal{P}(x), \text{ for all } x \in G, \text{ for all } \mathcal{P} \in \text{CSS}(G) \}.$$  

**Remark 1.7.** It is easy to verify that replacing $gx$ by $xg$ in Definition 1.6 yields the same two sets.

As a first application we rephrase the solvability criterion mentioned above.

**Theorem 1.8.** A group $G$ is solvable if and only if $G = M_\tau(G)$ for some ordering $\tau$ of $\pi(G)$.

**Proof.** If $G$ is solvable then $m_\mathcal{P}(x) = 1$ for all $x \in G$ and all $\mathcal{P} \in \text{CSS}_\tau(G)$, and therefore $G = M_\tau(G)$. In the other direction, let $\mathcal{P}$ be an arbitrary Sylow sequence of $G$ of type $\tau$. Then $\Pi(\mathcal{P})$ is not an empty set and hence $m_\mathcal{P}(x) > 0$ for some $x \in G$. Now $G = M_\tau(G)$ implies $m_\mathcal{P}(x) > 0$ for all $x \in G$ and hence $\Pi(\mathcal{P}) = G$. By [8, Theorem A], $G$ is solvable.

**Theorem 1.9.** Let $G$ be a group and let $\tau$ be some ordering of $\pi(G)$. Then $M_\tau(G)$ and $M(G)$ are characteristic subgroups of $G$. Moreover, $R(G) \leq M_\tau(G) \leq H_\tau(G)$ and $R(G) \leq M(G) \leq H(G)$.

We call a subgroup of $M(G)$ a Sylow multiplicity preserving subgroup of $G$. Thus, Theorem 1.9 allows us to view the solvable radical of $G$ as a Sylow multiplicity preserving subgroup of $G$. Moreover, it is immediate from Theorem 1.9 that if $G$ is such that $R(G) = H(G)$ then $R(G)$ is the unique maximal Sylow multiplicity preserving subgroup of $G$. Our last theorem shows that this already follows from assuming that $G$ is $\tau$-unique.

**Theorem 1.10.** Let $G$ be a group and let $\tau$ be some ordering of $\pi(G)$. If $G$ is $\tau$-unique then $M_\tau(G) = M(G) = R(G)$.

**Remark 1.11.** Note that the conclusion of Theorem 1.10 follows from the assumption of Theorem 1.3 (use Theorems 1.3 and 1.9). However, the assumption of Theorem 1.3 implies the assumption of Theorem 1.10, while we do not know whether the reverse implication holds.
2. Proofs

2.1. Basic properties  We summarize some useful concepts and results concerning multiplicities of elements in Sylow sequences, which will be used in the proofs of the main theorems.

**Definition 2.1.** Let $G$ be a group and let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of $G$. Let $b = b_1 \cdots b_m$ be a factorization of $b$ in $\mathcal{P}$. We define the complete Sylow sequence $\mathcal{P} b_{m}^{-1} \cdots b_{1}^{-1}$ by

\[ \mathcal{P} b_{m}^{-1} \cdots b_{1}^{-1} \text{ def } P_1, P_2^{b_{1}}, \ldots, P_i^{b_{i-1} \cdots b_{1}}, \ldots, P_m^{b_{m-1} \cdots b_{1}}. \]

**Remark 2.2.** Since each $b_i$ above takes any value in the group $P_i$, $\mathcal{P} b_{m}^{-1} \cdots b_{1}^{-1}$ is also well defined. One can verify the equality $\Pi(\mathcal{P} b_{m}^{-1} \cdots b_{1}^{-1}) = (\Pi \mathcal{P}) b^{-1}$. We stress that the sequence $\mathcal{P} b_{m}^{-1} \cdots b_{1}^{-1}$ depends on the particular factorization chosen for $b$ although its product depends only on $b$. Also note that a complete Sylow sequence $b_{m}^{-1} \cdots b_{1}^{-1} \mathcal{P}$ can be defined similarly, and that $\mathcal{P} b_{m}^{-1} \cdots b_{1}^{-1}$ is of the same type as $\mathcal{P}$.

**Lemma 2.3.** Let $G$ be a group and let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of $G$. Let $a, b \in \Pi(\mathcal{P})$. Let $a = a_1 \cdots a_m$ and $b = b_1 \cdots b_m$ be factorizations of $a$ and $b$ in $\mathcal{P}$ respectively. Then

\[ (\mathcal{P} b_{m}^{-1} \cdots b_{1}^{-1})(a_{m}^{-1} b_{m-1}^{-1} \cdots b_{1}^{-1}) \cdots (a_{2}^{-1} b_{1}^{-1}) a_{1}^{-1} = \mathcal{P} b_{m}^{-1} a_{m}^{-1} \cdots b_{1}^{-1} a_{1}^{-1}. \]

**Proof.** Routine computation. \qed

**Lemma 2.4.** Let $G$ be a group and let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of $G$. For any $x \in \Pi(\mathcal{P})$ and $y \in G$, and for any factorization $x = x_1 \cdots x_m$ of $x$ in $\mathcal{P}$,

\[ m_{\mathcal{P} x_{m}^{-1} \cdots x_{1}^{-1}}(yx^{-1}) = m_{\mathcal{P}}(y). \]

**Proof.** Let $y = y_{1}^{(i)} \cdots y_{m}^{(i)}$, $i = 1, 2$, be two distinct factorizations of $y$ in $\mathcal{P}$. Then

\[ yx^{-1} = (y_{1}^{(i)} x_{1}^{-1})(y_{2}^{(i)} x_{2}^{-1}) x_{i}^{-1} \cdots (y_{m}^{(i)} x_{m}^{-1}) x_{m-1}^{-1} \cdots x_{1}^{-1}, \quad i = 1, 2, \]

are two distinct factorizations of $yx^{-1}$ in $\mathcal{P} x_{m}^{-1} \cdots x_{1}^{-1}$. Hence $m_{\mathcal{P} x_{m}^{-1} \cdots x_{1}^{-1}}(yx^{-1}) \geq m_{\mathcal{P}}(y)$ (including the cases $m_{\mathcal{P}}(y) = 0, 1$). However, since $1_G \in \Pi(\mathcal{P})$, we have $x^{-1} \in \Pi(\mathcal{P}) x_{1}^{-1} = \Pi(\mathcal{P} x_{m}^{-1} \cdots x_{1}^{-1})$. In fact,

\[ x^{-1} = x_{1}^{-1} (x_{2}^{-1}) x_{3}^{-1} \cdots (x_{m}^{-1}) x_{m-1}^{-1} \cdots x_{1}^{-1} \]

is a factorization of $x^{-1}$ in $\mathcal{P} x_{m}^{-1} \cdots x_{1}^{-1}$. Hence, we can repeat the argument above with $\mathcal{P}$ replaced by $\mathcal{P} x_{m}^{-1} \cdots x_{1}^{-1}$, $y$ replaced by $yx^{-1}$ and $x$ replaced by $x^{-1}$. We get

\[ m_{\mathcal{P} x_{m}^{-1} \cdots x_{1}^{-1}}(yx^{-1}) x^{-1} \geq m_{\mathcal{P} x_{m}^{-1} \cdots x_{1}^{-1}}(yx^{-1}). \]

By Lemma 2.3, the left-hand side is $m_{\mathcal{P}}(y)$ and this concludes the proof. \qed
COROLLARY 2.5. Let $G$ be a group and let $\mathcal{P}$ be a complete Sylow sequence of type $\tau$ of $G$. Let $g \in G$. If $m_\mathcal{P}(g) = 1$ then there exists a complete Sylow sequence $Q$ of $G$ of the same type $\tau$ such that $m_Q(1_G) = 1$.

PROOF. Since $m_\mathcal{P}(g) = 1$, $g \in \Pi(\mathcal{P})$ and hence there exists a factorization $g = g_1 \cdots g_m$ of $g$ in $\mathcal{P}$. Now, by taking in Lemma 2.4 $y = x = g$, we get $Q = \mathcal{P}g_{m}^{-1} \cdots g_1^{-1}$. □

LEMMA 2.6. Let $G$ be a group and let $\mathcal{P}$ be a complete Sylow sequence of $G$. Let $g \in G$ and suppose that $g = g_1^{(1)} \cdots g_m^{(i)}$, $i = 1, 2$, are two distinct factorizations of $g$ in $\mathcal{P}$. Then there are at least three distinct values of $1 \leq k \leq m$ such that $g_k^{(1)} \neq g_k^{(2)}$.

PROOF. First observe that if $1_G = x_1 \cdots x_m$ is a nontrivial Sylow factorization of $1_G$, then there are at least three distinct values of $1 \leq k \leq m$ such that $x_k \neq 1_G$. The claim of the lemma now follows from the fact that

$$1_G = (g_1^{(1)}(g_2^{(2)})^{-1})(g_2^{(1)}(g_2^{(2)})^{-1})(g_3^{(1)})^{-1} \cdots (g_m^{(1)}(g_m^{(2)})^{-1})(g_m^{(2)})^{-1}$$

is a nontrivial Sylow factorization of $1_G$. □

2.2. Proof of Theorem 1.3. The key result which relates the $R(G) = H(G)$ question to the unique property is formulated in the following lemma.

LEMMA 2.7. Let $G$ be a group and let $\tau$ be an arbitrary ordering of $\pi(G)$. Let $N \trianglelefteq G$ be such that $G = G/N$ is $\tau$-unique. Then $H_\tau(G) \cap N = H_\tau(N)$.

PROOF. Observe that $H_\tau(N) \leq H_\tau(G) \cap N$ since every complete Sylow product of type $\tau$ of $G$ contains a complete Sylow product of type $\tau$ of $N$. It therefore remains to prove $H_\tau(G) \cap N \leq H_\tau(N)$. By assumption there exist a complete Sylow sequence $\mathcal{P}$ of type $\tau$ of $G$ and $\overline{g} \in \overline{G}$ such that $m_\overline{\mathcal{P}}(\overline{g}) = 1$. By Corollary 2.5 we can assume that $\overline{g} = 1_{\overline{G}}$. We have $\overline{\mathcal{P}} = P_1N/N, \ldots, P_mN/N$ where the $P_i$ are Sylow $p_i$-subgroups of $G$. Denote $\mathcal{P} = P_1, \ldots, P_m$. Let $n_1, \ldots, n_m$ be $m$ arbitrary elements of $N$ (not necessarily $p_i$-elements). We claim that

$$(P_1^{n_1} \cdots P_m^{n_m}) \cap N = (P_1 \cap N)^{n_1} \cdots (P_m \cap N)^{n_m}.$$

The reason for this is as follows. The subset on the r.h.s. is easily seen to be contained in the subset on the left-hand side. (Here the assumption that $\overline{G}$ is $\tau$-unique is not required.) For the reverse inclusion we make use of the assumption $m_\overline{\mathcal{P}}(1_{\overline{G}}) = 1$. Let $g \in (P_1^{n_1} \cdots P_m^{n_m}) \cap N$. Then $g = g_1 \cdots g_m$, with $g_i \in P_i^{n_i}$. Since also $g \in N$, its image under the canonical homomorphism $G \to \overline{G}$ is $1_{\overline{G}}$. Hence $\overline{g}_1 \cdots \overline{g}_m = 1_{\overline{G}}$ where $\overline{g}_i \in P_i^{n_i}N/N = P_iN/N$ is the image of $g_i$. The assumption $m_\overline{\mathcal{P}}(1_{\overline{G}}) = 1$ now implies $\overline{g}_i = 1_{\overline{G}}$ for all $1 \leq i \leq m$. Hence $g_i \in P_i^{n_i} \cap N = (P_i \cap N)^{n_i}$ for all $1 \leq i \leq m$, concluding the proof of the reverse inclusion.

Note that the intersection of $(P_1 \cap N)^{n_1} \cdots (P_m \cap N)^{n_m}$ over all choices of $n_1, \ldots, n_m \in N$ is $H_\tau(N)$. On the other hand, the intersection of $P_1^{n_1} \cdots P_m^{n_m} \cap N$
over all choices of \( n_1, \ldots, n_m \in N \) contains \( H_\tau(G) \cap N \). We obtain \( H_\tau(G) \cap N \leq H_\tau(N) \). 

**Proof of Theorem 1.3.** Let \( \tau \) be an arbitrary ordering of \( \pi(G) \). If \( G \) is simple then \( H_\tau(G) = H(G) = R(G) \) (see [8, remarks after Theorem B]). Henceforth, we assume that \( G \) is not simple and we prove the claim by induction on \(|G|\). Since \( R(G) \leq H(G) \leq H_\tau(G) \), it is sufficient to prove that \( H_\tau(G) \) is solvable. Let \( N \) be a maximal normal subgroup of \( G \). Then \( G/N \) is simple. Hence, either \( H_\tau(G/N) = 1 \) (if \( G/N \) is simple nonabelian) or \( H_\tau(G/N) = G/N \) (if \( G/N \) is cyclic of prime order) and in both cases \( H_\tau(G/N) \) is solvable. One can prove (see [8, Lemma 11]) that 

\[
H_\tau(G)/(H_\tau(G) \cap N) \cong H_\tau(G)N/N \leq H_\tau(G/N).
\]

Hence \( H_\tau(G)/(H_\tau(G) \cap N) \) is solvable. By assumption, the composition factor \( G/N \) of \( G \) is \( \tau \)-unique1. Thus, by Lemma 2.7, \( H_\tau(G) \cap N = H_\tau(N) \). However, \( H_\tau(N) \) is solvable by induction. Since \( H_\tau(G)/H_\tau(N) \) is also solvable, we get that \( H_\tau(G) \) is solvable. 

2.3. Proofs of Theorem 1.4 and Proposition 1.5

**Proof of Theorem 1.4.** Let \( G \) be an arbitrary group and let \( \tau \) be an arbitrary ordering of \( \pi(G) \). Let \( N \) be any nontrivial proper normal subgroup of \( G \) such that \( N \) and \( G/N \) are \( \tau \)-unique1 for the same \( \tau \). We shall prove that \( G \) is \( \tau \)-unique1. Suppose to the contrary that \( G \) is not \( \tau \)-unique1. Because \( G \) is \( \tau \)-unique1 we have a complete Sylow sequence \( \overline{\mathcal{P}} = \overline{P}_1, \ldots, \overline{P}_m \) of \( G \) of type \( \tau \) such that \( m_{\overline{\mathcal{P}}}(1_G) = 1 \). Let \( \mathcal{P} = P_1, \ldots, P_m \) be a complete Sylow sequence of \( G \) of type \( \tau \) such that \( \overline{P}_i = P_iN/N \). Let \( \mathcal{Q} = Q_1, \ldots, Q_m \) be any complete Sylow sequence of \( G \) such that \( Q_1 = P_1^{n_1}, \ldots, Q_m = P_m^{n_m} \) where \( n_1, \ldots, n_m \in N \) are arbitrary. Note that any such sequence maps, under the canonical homomorphism \( G \to \overline{G} \), to \( \overline{\mathcal{P}} = \overline{P}_1, \ldots, \overline{P}_m \) of \( \overline{G} \). Since \( G \) is not \( \tau \)-unique1, there exists a nontrivial factorization \( 1_G = g_1 \cdots g_m \) in \( \mathcal{Q} \) for any \( \mathcal{Q} \) as above. In \( \overline{G} = G/N \) we get \( \overline{g}_1 \cdots \overline{g}_m = 1_{\overline{G}} \) where \( \overline{g}_i \in \overline{P}_i \) is the image of \( g_i \). Now \( m_{\overline{\mathcal{P}}}(1_{\overline{G}}) = 1 \) implies \( \overline{g}_i = 1_{\overline{G}} \) for all \( 1 \leq i \leq m \). Hence \( g_i \in N \) and \( g_i \in P_i^{n_i} \cap N = (P_i \cap N)^{n_i} \) for all \( 1 \leq i \leq m \). Thus \( 1_G \) is not uniquely factorizable in any complete Sylow sequence \( ((P_i \cap N)^{n_i})_{1 \leq i \leq m} \) of \( N \). However, these are all of the complete Sylow sequences of \( N \) of type \( \tau \), contradicting the assumption that \( N \) is \( \tau \)-unique1.

Another property of unique1 which is worth observing is given in the following proposition.

**Proposition 2.8.** The property unique1 is inherited by normal subgroups.

**Proof.** Let \( G \) be a group with the property unique1 and let \( N \trianglelefteq G \). Let \( \tau \) be an arbitrary ordering of \( \pi(G) \). Then, by Corollary 2.5, there exists a complete Sylow sequence \( \mathcal{P} = P_1, \ldots, P_m \) of \( G \) of type \( \tau \) such that \( m_{\mathcal{P}}(1_G) = 1 \). Hence \( \mathcal{Q} = P_1 \cap N, \ldots, P_m \cap N \) satisfies \( m_{\mathcal{Q}}(1_G) = 1 \). 

\[ \square \]
REMARK 2.9. Every group property $\alpha$ which is inherited by normal subgroups and extensions has a residual, namely, for every group $G$, the set $\{N \trianglelefteq G \mid G/N \text{ is } \alpha\}$ has a unique minimal element. Thus, the property unique1 has a residual.

**Lemma 2.10.** Let $G$ be a group such that $|G|$ is divisible by exactly three primes. Let $\tau$ be any ordering of $\pi(G)$. If $G$ is $\tau$-unique1 then $G$ is unique1.

**Proof.** Let $\mathcal{P} = P_1, P_2, P_3$ be a complete Sylow sequence of $G$ of type $\tau$. Suppose that $1_G = abc$ with $a \in P_1$, $b \in P_2$ and $c \in P_3$ all nontrivial. Then $1_G = abc = cab = bca$ and $1_G = c^{-1}b^{-1}a^{-1} = a^{-1}c^{-1}b^{-1} = b^{-1}a^{-1}c^{-1}$ are all nontrivial Sylow factorizations of $1_G$, corresponding to all possible orderings of the $P_i$. Thus $m_\mathcal{P}(1_G) = 1$ if and only if $m_\mathcal{Q}(1_G) = 1$ where $\mathcal{Q} = P_\sigma(1), P_\sigma(2), P_\sigma(3)$ and $\sigma$ is any ordering of $\pi(G)$. Now the claim follows from Corollary 2.5.

**Proof of Proposition 1.5.** There are exactly eight simple nonabelian groups $G$ such that $|G|$ is divisible by exactly three primes [4]: $PSL(2,5) \cong A_5$, $PSL(2,7)$, $PSL(2,8)$, $PSL(2,17)$, $PSL(3,5)$, $A_6$, $SU(4,2) \cong O(5,3)$ and $SU(3,3)$. The first seven groups are Sylow factorizable. The Sylow factorizability of the first six follows from the results of [6] and from the fact that Sylow factorizability is inherited by normal subgroups and by quotient groups. The Sylow factorizability of $SU(4,2)$ can be proved as follows. First note that $|SU(4,2)| = 2^6 \cdot 3^4 \cdot 5$. $SU(4,2)$ has a maximal subgroup $M = V \rtimes A_5$ (see [2, p. 26]), where $V \cong \mathbb{Z}_2^4$. Since $A_5$ is equal to a complete Sylow product where the primes are ordered $(3,2,5)$, and $V \leq M$, we get that $M$ is equal to a complete Sylow product of the same type. Now, since $|G:M| = 3^3$, $SU(4,2) = P M$, where $P$ is a Sylow $3$-subgroup of $SU(4,2)$ containing a Sylow $3$-subgroup of $M$. It follows that $SU(4,2)$ is Sylow factorizable. Finally, $SU(3,3)$ is not Sylow factorizable but it has a complete Sylow sequence $\mathcal{P}$ such that $|\Pi(\mathcal{P})| > \frac{1}{2}|G|$ [6]. Clearly, in such $\mathcal{P}$ we have at least one $g \in G$ for which $m_\mathcal{P}(g) = 1$. Thus, all eight groups are $\tau$-unique1 for some $\tau$. We now use Lemma 2.10 in order to deduce that each of these groups is unique1.

**2.4. Proofs of Theorems 1.9 and 1.10** Lemmas 2.11 and 2.12 below are part of the content of Theorem 1.9.

**Lemma 2.11.** Let $G$ be a group and let $\tau$ be some ordering of $\pi(G)$. Then $M_\tau(G)$ and $M(G)$ are characteristic subgroups of $G$.

**Proof.** Clearly $1 \in M_\tau(G)$ so $M_\tau(G) \neq \phi$. Let $g_1, g_2 \in M_\tau(G)$. Let $\mathcal{P}$ be any complete Sylow sequence of $G$ of type $\tau$, and let $x \in G$ be arbitrary. Then $m_\mathcal{P}(g_1g_2) = m_\mathcal{P}(g_1g_2x) = m_\mathcal{P}(g_2x) = m_\mathcal{P}(x)$ and thus $g_1g_2 \in M_\tau(G)$. This proves that $M_\tau(G)$ is a subgroup of $G$. It is characteristic in $G$ since any automorphism of $G$ defines a bijection $CSS_\tau(G) \rightarrow CSS_\tau(G)$. The claims about $M(G)$ now follow from

$$M(G) = \bigcap_{\tau \in \text{Sym}(\{1,...,m\})} M_\tau(G).$$

\[\square\]
Lemma 2.12. Let \( G \) be a group and let \( \tau \) be some ordering of \( \pi(G) \). Then \( M_{\tau}(G) \leq H_{\tau}(G) \) and \( M(G) \leq H(G) \).

Proof. It is sufficient to prove that if \( g \in M_{\tau}(G) \) and if \( \mathcal{P} \) is any complete Sylow sequence of \( G \) of type \( \tau \), then \( g \in \Pi(\mathcal{P}) \). We have \( 1_G \in \Pi(\mathcal{P}) \). Hence \( m_{\mathcal{P}}(1_G) > 0 \). Now \( m_{\mathcal{P}}(g) = m_{\mathcal{P}}(g1_G) = m_{\mathcal{P}}(1_G) > 0 \). Therefore \( g \in \Pi(\mathcal{P}) \).

Lemma 2.13. Let \( G \) be a group, \( \pi(G) = \{p_1, \ldots, p_m\} \) and let \( \mathcal{P} = P_1, \ldots, P_m \) be an arbitrary complete Sylow sequence of \( G \) of type \( \tau \). Fix \( 1 \leq j \leq m \), and \( 1 \neq n \in O_{P_j}(G) \). Then, for all \( x \in G \), one can pair the factorizations of \( x \) and \( xn \) in \( \mathcal{P} \) so that the pair members are identical in all but the \( j \)th factors, the \( j \)th factors differ by an element of \( O_{P_j}(G) \), and each factorization of \( x \) and \( xn \) belongs to exactly one pair. It follows that \( O_{P_j}(G) \leq M_{\tau}(G) \).

Proof. Let \( x = x_1 \cdots x_j \cdots x_m \) be a factorization of \( x \) in \( \mathcal{P} \). Then

\[
\begin{align*}
xn &= x_1 \cdots x_{j-1}(x_j n^{x_{m-1}x_{m-2} \cdots x_{j+1}}) x_{j+1} \cdots x_m,
\end{align*}
\]

is a factorization of \( xn \) in \( \mathcal{P} \). This factorization differs from the given factorization of \( x \) in \( \mathcal{P} \) only in the \( j \)th factor and the difference between the factors is

\[
\begin{align*}
x_{j-1}^{-1} x_j n^{x_{m-1}x_{m-2} \cdots x_{j+1}} &= n^{x_{m-1}x_{m-2} \cdots x_{j+1}} \in O_{P_j}(G)
\end{align*}
\]

(note that \( 1 \neq n \) implies \( 1 \neq n^{x_{m-1}x_{m-2} \cdots x_{j+1}} \)). We define a pairing of factorizations of \( x \) and \( xn \) in \( \mathcal{P} \) by associating to the factorization of \( x = x_1 \cdots x_j \cdots x_m \) of \( x \) in \( \mathcal{P} \) the factorization

\[
\begin{align*}
xn &= x_1 \cdots x_{j-1}(x_j n^{x_{m-1}x_{m-2} \cdots x_{j+1}}) x_{j+1} \cdots x_m
\end{align*}
\]

of \( xn \) in \( \mathcal{P} \). Since two distinct factorizations of the same element in \( \mathcal{P} \) must differ by at least three factors (Lemma 2.6), it is clear that two distinct factorizations of \( x \) are paired with two distinct factorizations of \( xn \). Thus, this pairing defines an injective function from the set of all factorizations of \( x \) in \( \mathcal{P} \) to the set of all factorizations of \( xn \) in \( \mathcal{P} \). Since \( x = (xn)n^{-1} \), we have a bijection. Hence \( m_{\mathcal{P}}(x) = m_{\mathcal{P}}(xn) \), implying (see Remark 1.7) that \( n \in M_{\tau}(G) \).

Definition 2.14. Let \( G \) be a group, \( \pi(G) = \{p_1, \ldots, p_m\} \), \( N \trianglelefteq G \), and let \( \mathcal{P} = P_1, \ldots, P_m \) be a complete Sylow sequence of \( G \). Set \( \overline{G} = G/N \). For any \( x \in G \) and any \( A \trianglelefteq G \) denote by \( \overline{x} \) and \( \overline{A} \), respectively, their images under the canonical homomorphism from \( G \) onto \( \overline{G} \). Let \( \overline{\mathcal{P}} = \overline{P_1}, \ldots, \overline{P_m} \). Let \( x = x_1 \cdots x_m \) be a factorization of \( x \) in \( \mathcal{P} \). The image of the factorization \( x = x_1 \cdots x_m \) in \( \overline{G} \) is the factorization \( \overline{x} = \overline{x_1} \cdots \overline{x_m} \) of \( \overline{x} \) in \( \overline{\mathcal{P}} \). Conversely, a preimage of a factorization \( \overline{x} = \overline{x_1} \cdots \overline{x_m} \) of \( \overline{x} \) in \( \overline{\mathcal{P}} \) is any factorization, \( xn = y_1 \cdots y_m \) in \( \mathcal{P} \), where \( n \in N \), such that \( \overline{y_i} = \overline{x_i} \) for all \( 1 \leq i \leq m \).
Lemma 2.15. Let $G$ be a group, $\pi(G) = \{p_1, \ldots, p_m\}$ and $\mathcal{P} = P_1, \ldots, P_m$ a complete Sylow sequence of $G$. Let $1 \leq j \leq m$ and let $N$ be a normal $p_j$-subgroup of $G$. Using the notation of Definition 2.14 we have, for any $x \in G$,

$$m_{\mathcal{P}}(x) = m_{\overline{\mathcal{P}}}(\overline{x}).$$

Proof. If $m_{\mathcal{P}}(x) = 0$ then $x \notin \Pi(\mathcal{P})$. Since $\Pi(\mathcal{P})N = \Pi(\mathcal{P})$ (see [8, proof of Theorem B parts (c) and (d)]) we get $\overline{x} \notin \Pi(\overline{\mathcal{P}})$ and therefore $m_{\overline{\mathcal{P}}}(\overline{x}) = 0$. Now suppose that $m_{\mathcal{P}}(x) > 0$. The preimage of $\overline{x}$ is the set $xN$. Thus, any factorization of $\overline{x}$ in $\overline{\mathcal{P}}$ is the image of a factorization in $\mathcal{P}$ of an element of $xN$ (the same factorization of $\overline{x}$ in $\overline{\mathcal{P}}$ can be the image of more than one factorization of more than one element of $xN$). Let $x = x_1^{(s)} \cdots x_m^{(s)}$, $1 \leq s \leq m_{\mathcal{P}}(x)$ be all of the $m_{\mathcal{P}}(x)$ distinct factorizations of $x$ in $\mathcal{P}$. By Lemma 2.13, every factorization of $xn$ in $\mathcal{P}$, where $n \in N$, can be paired with one of the $m_{\mathcal{P}}(x)$ factorizations of $x$ in $\mathcal{P}$ so that the pair members are identical in all but the $j$th factors, and the $j$th factors differ by an element of $N$. Thus, when we vary $n$ over $N$, the image in $\overline{\mathcal{P}}$ of all factorizations of $xn$ which pair with the same $s$th factorization $x = x_1^{(s)} \cdots x_m^{(s)}$ of $x$ in $\mathcal{P}$ is the same. On the other hand, two distinct factorizations of $x$ in $\mathcal{P}$ (two distinct $s$ values) must differ by at least three factors (Lemma 2.6) and hence their images in $\overline{\mathcal{P}}$ are distinct. The claim follows. \qed

Proof of Theorem 1.9. It remains to prove (see Lemmas 2.11 and 2.12) that $R(G) \leq M_{\tau}(G)$ for an arbitrary ordering $\tau$ of $\pi(G) = \{p_1, \ldots, p_m\}$. We do this by induction on $|G|$. We can assume that $R(G) > 1$. Hence there exists $1 \leq j \leq m$ such that $O_{p_j}(G) > 1$. Let $\overline{G} = G/O_{p_j}(G)$ and, for all $x \in G$, let $\overline{x}$ denote its image under the canonical homomorphism $G \to \overline{G}$. For any $A \leq G$ let $\overline{A} = AO_{p_j}(G)/O_{p_j}(G)$. Let $x \in G$ and let $y \in R(G)$. Let $\mathcal{P} = P_1, \ldots, P_m$ be a complete Sylow sequence of $G$ of type $\tau$. We have to prove that $m_{\mathcal{P}}(x \cdot y^{-1}) = m_{\mathcal{P}}(x)$. By Lemma 2.15, $m_{\mathcal{P}}(x) = m_{\overline{\mathcal{P}}}(\overline{x})$ and $m_{\mathcal{P}}(x \cdot y^{-1}) = m_{\overline{\mathcal{P}}}(x \cdot y^{-1}) = m_{\overline{\mathcal{P}}}(\overline{x} \cdot \overline{y}^{-1})$. Since $R(G) = R(\overline{G})$, we have $\overline{y} \in R(\overline{G})$ and by induction assumption $m_{\overline{\mathcal{P}}}(\overline{x} \cdot \overline{y}^{-1}) = m_{\overline{\mathcal{P}}}(\overline{x})$. The claim follows. \qed

Proof of Theorem 1.10. Set $N = M_{\tau}(G)$. By Theorem 1.9 it is sufficient to prove that $N$ is solvable. By assumption, and by Corollary 2.5, there exists a complete Sylow sequence $\mathcal{P} = P_1, \ldots, P_m$ of type $\tau$ such that $m_{\mathcal{P}}(1_G) = 1$. It follows that $m_{\mathcal{P}}(h) = 1$ for all $h \in N$. Now set $\overline{G} = G/N$ and $\overline{\mathcal{P}} = (P_1N)/N, \ldots, (P_mN)/N$. We claim $m_{\overline{\mathcal{P}}}(1_{\overline{G}}) = 1$. Suppose to the contrary that $\overline{g_1} \cdots \overline{g_m} = 1_{\overline{G}}$ where $\overline{g_i} \in (P_iN)/N$ and at least one of the $\overline{g_i}$ is nontrivial. Let $g_i$ be preimages of the $\overline{g_i}$ in $P_i$. Then $g_1 \cdots g_m = h$ for some $h \in N$. Since the $\overline{g_i}$ are not all trivial, there exists some $g_i$ which is not in $N$. Since $m_{\mathcal{P}}(h) = 1$, the factorization $g_1 \cdots g_m = h$ is the only factorization of $h$ in $\mathcal{P}$ and hence $h$ is not factorizable in the Sylow sequence $N \cap P_1, \ldots, N \cap P_m$ of $N$. It follows that there exists $\tilde{h} \in N$ which has at least two distinct factorizations in $N \cap P_1, \ldots, N \cap P_m$ in contradiction to $m_{\mathcal{P}}(h) = 1$. Thus, $m_{\overline{\mathcal{P}}}(1_{\overline{G}}) = 1$ and we can apply Lemma 2.7 and get $H_{\tau}(G) \cap N = H_{\tau}(N)$. By Theorem 1.9, we have $N \leq H_{\tau}(G)$ and thus $N = H_{\tau}(N)$ implying [8, Theorem A] that $N$ is solvable. \qed
References


GIL KAPLAN, The School of Computer Sciences, The Academic College of Tel-Aviv-Yaffo, 2 Rabenu Yeruham St., Tel-Aviv 61083, Israel e-mail: gilk68@gmail.com

DAN LEVY, The School of Computer Sciences, The Academic College of Tel-Aviv-Yaffo, 2 Rabenu Yeruham St., Tel-Aviv 61083, Israel e-mail: danlevy@trendline.co.il