

# On the Principal Eigencurve of the $p$ -Laplacian: Stability Phenomena

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*Abstract.* We show that each point of the principal eigencurve of the nonlinear problem

$$-\Delta_p u - \lambda m(x)|u|^{p-2}u = \mu|u|^{p-2}u \quad \text{in } \Omega,$$

is stable (continuous) with respect to the exponent  $p$  varying in  $(1, \infty)$ ; we also prove some convergence results of the principal eigenfunctions corresponding.

## 1 Introduction

In this paper we study the stability (continuity) of the two parameter nonlinear problem

$$(\mathcal{E}_p) \quad -\Delta_p u - \lambda m(x)|u|^{p-2}u = \mu|u|^{p-2}u \quad \text{in } \Omega,$$

with respect to the exponent  $p$ . The solution  $u$  is scalar valued and required to belong to the Sobolev space  $W_0^{1,p}(\Omega)$ , where  $\Omega$  is a bounded domain in the Euclidean space  $\mathbb{R}^N$  and the real parameters  $\lambda$  and  $\mu$  play the role of eigenvalues. The weight  $m \in L^\infty(\Omega) \setminus \{0\}$  can change its sign. Here  $1 < p < \infty$  is considered as a *variable exponent*.

The nonlinear degenerate operator  $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ , is the so-called  $p$ -Laplacian, which is well known to mathematicians for being a prototype for monotone operators, see [20, 24]. This operator has attracted growing interest, and occurs in a large variety of application in applied fields and many mathematical models of physical topics, according to values of the exponent  $p$ . For example, in the mathematical modelling of non-Newtonian fluids (dilatant fluids for  $p > 2$ , pseudo-plastics for  $1 < p < 2$ ) [4], reaction-diffusion problems, nonlinear elasticity and glaciology for  $p = \frac{4}{3}$  [22], in power-lower materials [3], in nonlinear diffusion and filtration problems [23], in flow porous media, in petroleum extraction and torsional creep problems [10] and the thermal radiation of a hydrogen bomb [6]. Note that the  $p$ -Laplacian counterpart of Smagornsky's model corresponds to  $p = 3$  [25]. For a discussion of some physical background, see [10]. For the linear case ( $p = 2$ ), there is an extensive literature.

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Note that when  $p = 2$ , we just get the usual Laplacian and we are back to a well-understood familiar linear equation

$$\Delta u + \lambda m(x)u + \mu u = 0,$$

appearing in connection with Schrödinger’s equation [9].

The above shows the importance of studying the dependence of the principal eigenvalue of  $(\mathcal{E}_p)$  with the exponent  $p$ . This is the main objective of the present paper. The difficulty is that the Sobolev spaces  $W_0^{1,p}(\Omega)$  change with the exponent  $p$ . We overcome this obstacle by assuming that the underlying domain  $\Omega$  has the so-called segment property, a sufficient regularity condition related to the geometry.

The problem  $(\mathcal{E}_p)$  was considered by Hess and Kato [17] (for the linear case  $p = 2$ ) and by Binding and Huang [7], in a smooth bounded domain. It was also considered by various authors for the particular cases  $\lambda = 0$  or  $\mu = 0$ . We cite the works of Anane [2], Azorero and Alonso [5], Otani and Theshima [21] and Lindqvist [18], just to mention a few authors. Note that the stability question is not considered in the above references.

In the particular case  $\lambda = 0$  in  $(\mathcal{E}_p)$ , the stability phenomena were studied essentially by Lindqvist in [19]. See also [14] for  $\mu = 0$  and where  $\lambda$  plays the role of eigenvalue.

The present paper is partially motivated by recent works [7, 13, 15]. The simplicity of each point of the first eigencurve of  $(\mathcal{E}_p)$  in a general bounded domain  $\Omega$  was shown in [15]. Here the first (or principal) eigencurve of  $(\mathcal{E}_p)$  means the graph of the numerical function  $\mu_1: \lambda \rightarrow \mu_1(\lambda)$  from  $\mathbb{R}$  to  $\mathbb{R}$ , where  $\mu_1(\lambda)$ , for fixed  $\lambda$ , denotes the principal eigenvalue of  $(\mathcal{E}_p)$ .

We now describe some of the results of this paper. To prove our main results, *i.e.*, the stability of the problem  $(\mathcal{E}_p)$  with respect to  $p$ , which varies continuously (Theorem 3.7 and Theorem 4.1), we were able to adapt some techniques of [19] to our problems with some modifications. However, our situation is bit more complicated due to the fact that the first eigenvalue of  $(\mathcal{E}_p)$ ,  $\mu_p(\lambda) = \mu_1(\lambda)$  (indicating the dependence on  $p$ ) can take negative values for some  $\lambda \in \mathbb{R}$ , because the weight  $m$  is indefinite and can change its sign on  $\Omega$ . Lindqvist [19] proved that the stability of  $(\mathcal{E}_p)$ , for the particular case  $\lambda = 0$ , may fail in an irregular domain, but it is true for a smooth domain. Our first result establishes that this result also holds true for a bounded domain such that  $W^{1,p}(\Omega) \cap W_0^{1,p-\varepsilon}(\Omega) = W_0^{1,p}(\Omega)$ , for some  $\varepsilon > 0$ . So we include this equality (Theorem 2.2) when  $\Omega$  is a bounded domain having the so-called segment property. This class of domains is fairly large. This property is needed here to guarantee the right boundary values of the limit function. This is all the more interesting when  $p < N$  (because for  $p \geq N$ , any function in  $W_0^{1,p}(\Omega)$  is continuous on  $\overline{\Omega}$ , by Sobolev embedding, and consequently the stability is required).

The other main objective of our paper is to study the convergence of the principal eigenfunctions in connection with the inequalities

$$\lim_{s \rightarrow p_-} \mu_s(\lambda) \leq \mu_p(\lambda) = \lim_{s \rightarrow p_+} \mu_s(\lambda),$$

proved in Corollary 3.5, by using a direct method based essentially on the variational

characterization of corresponding principal eigenvalues. For  $\lambda = 0$  the states' inequalities are due to Lindqvist [19].

The outline of this paper is as follows. In Section 2, we establish some definitions, notations and basic properties. In Section 3, we first give some general stability results with respect to  $p$  for each point of the first eigencurve of problem  $(\mathcal{E}_p)$  and then we restrict ourselves to a bounded domain having the segment property. In Section 4, we establish some convergence results of the principal eigenfunctions denoted  $\mu_p(\lambda)$ ,  $p \in (1, +\infty)$ , corresponding to  $\mu_p(\lambda)$ .

## 2 Preliminaries, Notations and Definitions

In defining the eigenpairs of  $(\mathcal{E}_p)$  (in a given bounded domain  $\Omega \subset \mathbb{R}^N$ ) we shall interpret the equation  $(\mathcal{E}_p)$  in the weak sense. We say that  $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}$  is an eigenpair of  $(\mathcal{E}_p)$ , if there exists a function  $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ , such that the pair  $((\lambda, \mu), u)$  satisfies  $(\mathcal{E}_p)$  in the weak sense, *i.e.*, such that

$$(2.1) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\Omega} (\lambda m(x) + \mu) |u|^{p-2} u \varphi \, dx,$$

whenever  $\varphi \in W_0^{1,p}(\Omega)$ . The function  $u$  is called an eigenfunction.

We define

$$(2.2) \quad U_p = \{ u \in W_0^{1,p}(\Omega), \|u\|_p = 1 \}.$$

For  $\lambda \in \mathbb{R}$ , let  $\mu_p(\lambda)$  denote the first eigenvalue of  $(\mathcal{E}_p)$ . We recall from [15] that  $\mu_p(\lambda)$  can be variationally characterized as

$$(2.3) \quad \mu_p(\lambda) = \inf \left\{ \frac{\|\nabla u\|_p^p - \lambda \int_{\Omega} m(x) |u|^p \, dx}{\|u\|_p^p}; u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\},$$

that is,

$$\mu_p(\lambda) = \min \left\{ \|\nabla u\|_p^p - \lambda \int_{\Omega} m(x) |u|^p \, dx; u \in W_0^{1,p}(\Omega), \|u\|_p = 1 \right\}.$$

The graph of the function  $\lambda \rightarrow \mu_p(\lambda)$  from  $\mathbb{R}$  into  $\mathbb{R}$  is called the first *eigencurve* of the  $p$ -Laplacian, which is also called the *principal eigencurve* of the  $p$ -Laplacian.

Throughout this paper, the first eigenfunctions are the eigenfunctions corresponding to eigenpair  $(\lambda, \mu_p(\lambda))$ . And the *principal eigenfunction*, denoted  $u_p(\lambda)$ , is the first eigenfunction required to be positive and belongs to  $U_p$ . Hence, for any  $\lambda \in \mathbb{R}$  we have

$$(2.4) \quad \mu_p(\lambda) = \|\nabla u_p(\lambda)\|_p^p - \lambda \int_{\Omega} m(x) \{u_p(\lambda)\}^p \, dx.$$

We end this paragraph by recalling the following properties that can be found in [15]:

- (P1)  $\mu_p(\lambda)$  can have negative values and  $\mu_p(0) = \lambda_1$  (i.e., the first eigenvalue of the  $p$ -Laplacian );
- (P2) for any  $\lambda \in \mathbb{R}$ , the first eigenfunctions are essentially unique for any bounded domain, i.e., they are merely constant multiples of each other;
- (P3) for any  $\lambda \in \mathbb{R}$ , the first eigenfunctions have no zeros in the domain and they are the only eigenfunctions not changing sign;
- (P4) for any  $\lambda \in \mathbb{R}$ ,  $\mu_p(\lambda)$  is the only eigenvalue of  $(\mathcal{E}_p)$  having an eigenfunction not changing sign;
- (P5) for any  $\lambda \in \mathbb{R}$ ,  $\mu_p(\lambda)$  is simple and isolated.

Now, we define a class of domains for which the boundary is sufficiently regular to guarantee that

$$W^{1,p}(\Omega) \bigcap_{q < p} W_0^{1,q}(\Omega) = W_0^{1,p}(\Omega).$$

**Definition 2.1** An open subset  $\Omega$  of  $\mathbb{R}^N$  is said to have the segment property if, given any  $x \in \partial\Omega$ , there exists an open set  $G_x$  in  $\mathbb{R}^N$  with  $x \in G_x$ , and a point  $y_x$  of  $\mathbb{R}^N \setminus \{0\}$  such that if  $z \in \overline{\Omega} \cap G_x$  and  $t \in (0, 1)$ , then  $z + ty_x \in \Omega$ .

This property allows us to push the support of a function  $u$  in  $\Omega$  via a translation. The following result is essential here.

**Theorem 2.2** ([13, 14]) *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  having the segment property. If  $u \in W^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$  for some  $q \in (1, p)$ , then  $u \in W_0^{1,p}(\Omega)$ .*

**Remark 2.3** (i) If  $\partial\Omega$  is of class  $C^1$ , then the proof is simple, since we can use the local carts, cf. [8]. (ii) The theorem holds if  $\Omega$  is not necessarily bounded. It suffices to approximate  $u$  arbitrarily closely in  $W^{1,p}(\Omega)$  by functions with bounded supports. (iii) If  $\Omega$  does not have the segment property, then the result of the theorem cannot hold. (iv) Note that a bounded domain  $\Omega \subset \mathbb{R}^N$  has the segment property if and only if it is in the class  $\mathcal{C}$ , cf. [12]. This means that locally the boundary has the continuous equation  $x_N = f(x_1, x_2, \dots, x_{N-1})$ , after a notation of the coordinate axis. (v) We also note that  $F(\Omega_1)$  has the segment property if  $\Omega_1$  has this property and  $F$  and  $F^{-1}$  are Lipschitz. This result is a consequence of (iv).

**Example 2.4** The set  $\{x \in \mathbb{R}^N; 0 < |x| < 1\}$  is a domain not having the segment property.

In what follows, we will need the next lemma which can be found in [15].

**Lemma 2.5** *For any  $\lambda \in \mathbb{R}$  and for any bounded domain  $\Omega$ ,*

$$(2.5) \quad \|u\|_\infty \leq 4^N (|\mu_p(\lambda)| + |\lambda| \|m\|_\infty)^{\frac{N}{p}} \|u\|_1$$

*for each  $p \in (1, \infty)$  and for each eigenfunction  $u$  associated to eigenpair  $(\lambda, \mu_p(\lambda))$ .*

**Remark 2.6**

(i) Remark that if we take  $u$  in  $U_p$ , then Hölder's inequality yields

$$(2.6) \quad \|u\|_\infty \leq 4^N (|\mu_p(\lambda)| + |\lambda| \|m\|_\infty)^{\frac{N}{p}} |\Omega|^{\frac{p-1}{p}},$$

where  $|\Omega|$  denotes the Lebesgue measure of  $\Omega$ .

(ii) From (2.6) it is clear that each eigenfunction of  $(\mathcal{E}_p)$  belongs to  $L^\infty(\Omega)$ . By an advanced result in regularity theory, see [11], it follows that each eigenfunction is of class  $C_{loc}^{1,\alpha}$ , for some  $\alpha \in (0, 1)$ . Note that  $\alpha$  depends on  $\lambda$  and on the exponent  $p$ .

**3 Stability Results**

First, from (2.4) we deduce

$$(3.1) \quad \|\nabla u_p(\lambda)\|_p^p = \mu_p(\lambda) + \lambda \int_\Omega m(x) \{u_p(\lambda)\}^p dx.$$

**Lemma 3.1** For any  $\lambda \in \mathbb{R}$  and for any bounded range  $I \subset (1, \infty)$ , the sets  $\{\mu_p(\lambda); p \in I\}$  and  $\{\|\nabla u_p(\lambda)\|_p^p; p \in I\}$  are uniformly bounded, that is,

$$\max_{p \in I} \{ \max(|\mu_p(\lambda)|, \|\nabla u_p(\lambda)\|_p^p) \} < \infty.$$

**Proof** Let  $u_p(0)$  be the principal eigenfunction associated to  $\mu_p(0) = \lambda_1(p)$  (the first eigenvalue of  $p$ -Laplacian). Thus  $u_p(0) \in U_p$ ,  $u(0) > 0$  and

$$\lambda_1(p) = k \|\nabla u_p(0)\|_p^p.$$

Hence  $u_p(0)$  is admissible in the definition of  $\mu_p(\lambda)$ . It follows that

$$(3.2) \quad \mu_p(\lambda) \leq \lambda_1(p) - \lambda \int_\Omega m(x) \{u_p(0)\}^p dx \leq \lambda_1(p) + |\lambda| \|m\|_\infty.$$

Remark that  $u_p(\lambda)$  is admissible in the definition of  $\lambda_1(p)$ . Therefore

$$\lambda_1(p) \leq \|\nabla u_p(\lambda)\|_p^p.$$

Hence

$$(3.3) \quad -\mu_p(\lambda) \leq \lambda_1(p) + |\lambda| \|m\|_\infty.$$

This and (3.2) yield

$$(3.4) \quad |\mu_p(\lambda)| \leq \lambda_1(p) + |\lambda| \|m\|_\infty.$$

On the other hand, from (2.4) we deduce that

$$(3.5) \quad \|\nabla u_p(\lambda)\|_p^p \leq \lambda_1(p) + |\lambda| \|m\|_\infty.$$

Using [19, (2.5)], we can show that  $\max_{p \in I} \lambda_1(p)$  is bounded by a finite constant. This fact, (3.4) and (3.5) achieve the proof. ■

**Remark 3.2** In a smooth domain one can use the continuity of the function  $p \rightarrow \lambda_1(p)$  to conclude that  $\max_{p \in I} \lambda_1(p)$  is finite, see [19]. Then from (3.4) and (3.5) we deduce the result.

**Proposition 3.3** For any  $\lambda \in \mathbb{R}$  and for any bounded domain  $\Omega$ , we have

$$(3.6) \quad \mu_p(\lambda) \leq \mu_s(\lambda) + \left[ \left( \frac{s}{p} \right)^p (\lambda_1(s))^{\frac{s}{p}-1} - 1 \right] \|\nabla u_s(\lambda)\|_s^s,$$

for any  $s, p \in (1, \infty)$  such that  $p < s$ .

**Proof** The principal eigenfunctions  $(u_s(\lambda))_{s>p}$  associated to  $(\mu_s(\lambda))_{s>p}$  verify

$$u_s(\lambda) \in U_s, \quad u_s(\lambda) > 0$$

and

$$(3.7) \quad \mu_s(\lambda) = \|\nabla u_s(\lambda)\|_s^s - \lambda \int_{\Omega} m(x) \{u_s(\lambda)\}^s dx.$$

If we set  $\varphi = \{u_s(\lambda)\}^{\frac{s}{p}}$ , then  $\varphi \in W_0^{1,p}(\Omega)$ , because  $p < s$ . Thus  $\varphi$  is admissible in the definition of  $\mu_p(\lambda)$  and  $\|\varphi\|_p = 1$ . Therefore

$$\begin{aligned} \mu_p(\lambda) &\leq \|\nabla \varphi\|_p^p - \lambda \int_{\Omega} m(x) \varphi^p dx \\ &= \left( \frac{s}{p} \right)^p \int_{\Omega} \{u_s(\lambda)\}^{s-p} |\nabla u_s(\lambda)|^p dx - \lambda \int_{\Omega} m(x) \{u_s(\lambda)\}^s dx, \end{aligned}$$

because  $\varphi^p = \{u_s(\lambda)\}^s$  and  $\nabla \varphi = \frac{s}{p} \{u_s(\lambda)\}^{\frac{s}{p}-1} \nabla u_s(\lambda)$ . Hölder's inequality yields

$$\mu_p(\lambda) \leq \left( \frac{s}{p} \right)^p \left( \int_{\Omega} \{u_s(\lambda)\}^s dx \right)^{1-\frac{p}{s}} \left( \int_{\Omega} |\nabla u_s(\lambda)|_s^s dx \right)^{\frac{p}{s}} - \lambda \int_{\Omega} m(x) \{u_s(\lambda)\}^s dx.$$

Hence

$$\mu_p(\lambda) \leq \left( \frac{s}{p} \right)^p \left( \int_{\Omega} |\nabla u_s(\lambda)|^s dx \right)^{\frac{p}{s}} - \lambda \int_{\Omega} m(x) \{u_s(\lambda)\}^s dx,$$

because  $u_s(\lambda) \in U_s$ . Since for  $\frac{p}{s} < 1$ ,

$$\left( \int_{\Omega} |\nabla u_s(\lambda)|^s dx \right)^{\frac{p}{s}} \leq \left( \int_{\Omega} |\nabla u_s(\lambda)|^s dx \right)^{\frac{p}{s}-1} \left( \int_{\Omega} |\nabla u_s(\lambda)|^s dx \right)$$

and

$$\left( \int_{\Omega} |\nabla u_s(\lambda)|^s dx \right)^{\frac{p}{s}-1} \leq (\lambda_1(s))^{\frac{p}{s}-1},$$

we obtain

$$(3.8) \quad \mu_p(\lambda) \leq \left( \frac{s}{p} \right)^p (\lambda_1(s))^{\frac{p}{s}-1} \left( \int_{\Omega} |\nabla u_s(\lambda)|^s dx \right) - \lambda \int_{\Omega} m(x) \{u_s(\lambda)\}^s dx.$$

The desired inequality follows from (3.8) and (3.7). ■

**Remark 3.4** If  $\lambda = 0$ , then  $\mu_p(0) = \lambda_1(p)$  and  $\mu_s(0) = \lambda_1(s) = \|\nabla u_s(0)\|_s^s$ . Thus, we find the same result stated by Lindqvist in [19]. Indeed, the inequality (3.6) becomes

$$p(\lambda_1(p))^{1/p} \leq s(\lambda_1(s))^{1/s}.$$

**Corollary 3.5** For any  $\lambda \in \mathbb{R}$ , we have

$$(3.9) \quad \limsup_{s \rightarrow p-} \mu_s(\lambda) \leq \mu_p(\lambda) \leq \liminf_{s \rightarrow p+} \mu_s(\lambda).$$

**Proof** Let us  $s > p$ , letting  $s \rightarrow p+$ , we find

$$\mu_p(\lambda) \leq \liminf_{s \rightarrow p+} \mu_s(\lambda),$$

because  $\lim_{s \rightarrow p+} \lambda_1(s) = \lambda_1(p) > 0$ , cf. [19].

For the case  $s \rightarrow p-$ , (3.6) can be written for  $s < p$  as

$$\mu_s(\lambda) \leq \mu_p(\lambda) + \left[ \left( \frac{p}{s} \right)^s (\lambda_1(p))^{\frac{p}{s}-1} - 1 \right] \|\nabla u_p(\lambda)\|_p^p.$$

This yields

$$\limsup_{s \rightarrow p-} \mu_s(\lambda) \leq \mu_p(\lambda).$$

Altogether, this achieves the proof. ■

**Remark 3.6** (i) Observe that if  $\lim_{s \rightarrow p} \mu_s(\lambda)$  exists, then this limit is necessarily equal to  $\mu_p(\lambda)$ . Thus in this case we say that we have the stability (continuity) of the point  $(\lambda, \mu_p(\lambda))$  of the principal eigencurve with respect to  $p$ . (ii) The first inequality in (3.9) can be strict in an irregular domain and  $1 < p \leq N$ .

**Theorem 3.7** For any  $\lambda \in \mathbb{R}$  and for any bounded domain having the segment property, the function  $p \rightarrow \mu_p(\lambda)$  is continuous from  $(1, \infty)$  into  $\mathbb{R}$ .

**Proof** Fix  $\lambda$  in  $\mathbb{R}$ . Let  $\{s_j\}_{j=1}^{j=\infty}$  be a sequence in  $(1, \infty)$  such that  $\lim_{j \rightarrow \infty} s_j = p$ . We claim that  $\lim_{j \rightarrow \infty} \mu_{s_j}(\lambda) = \mu_p(\lambda)$ . Indeed, let  $\varphi \in C_0^\infty(\Omega) \setminus \{0\}$ . Therefore

$$\mu_{s_j}(\lambda) \leq \frac{\|\nabla \varphi\|_{s_j}^{s_j} - \lambda \int_\Omega m(x) |\varphi|^{s_j} dx}{\|\varphi\|_{s_j}^{s_j}}.$$

Passing to the minimum over  $\varphi$ , we find

$$(3.10) \quad \limsup_{j \rightarrow \infty} \mu_{s_j}(\lambda) \leq \mu_p(\lambda).$$

To achieve the claim, it suffices to show that

$$\mu_p(\lambda) \leq \liminf_{j \rightarrow \infty} \mu_{s_j}(\lambda).$$

Indeed, let  $\{s_k\}_{k \geq 1}$  be a subsequence of  $\{s_j\}_{j \geq 1}$  such that

$$\lim_{k \rightarrow \infty} \mu_{s_k}(\lambda) = \liminf_{j \rightarrow \infty} \mu_{s_j}(\lambda).$$

From the minimizing property of  $\mu_{s_j}(\lambda)$ , there exists  $v_k \in W_0^{1,s_k}(\Omega)$  such that  $v_k \in U_{s_k}$ ,  $\int_{\Omega} |\nabla v_k|^{s_k} dx > 0$  and  $v_k$  is an eigenfunction associated to eigenpair  $(\lambda, \mu_{s_k}(\lambda))$ . Set

$$w_k = \frac{v_k}{\|\nabla v_k\|_{s_k}}.$$

Then  $w_k \in W_0^{1,s_k}(\Omega)$  and

$$(3.11) \quad \int_{\Omega} |\nabla w_k|^{s_k} dx = 1,$$

$$(3.12) \quad \int_{\Omega} |w_k|^{s_k} dx > 0,$$

$$(3.13) \quad \mu_{s_k}(\lambda) \int_{\Omega} |w_k|^{s_k} dx = 1 - \lambda \int_{\Omega} m(x) |w_k|^{s_k} dx.$$

On the other hand, we have  $s_k \rightarrow p$ , as  $k \rightarrow \infty$ . Thus for every  $\varepsilon > 0$  there exists  $k_{\varepsilon} \in \mathbb{N}^*$  such that

$$p - \varepsilon < s_k < p + \varepsilon, \quad \text{when } k \geq k_{\varepsilon}.$$

We have  $p > 1$ , let  $\varepsilon_1 > 0$  such that  $p - \varepsilon_1 > 1$ . Hence there exists  $k_0 \in \mathbb{N}^*$  such that

$$1 < p - \varepsilon_1 < s_k < p + \varepsilon_1, \quad \text{when } k \geq k_0.$$

Now let  $\varepsilon_2 > 0$  small enough so that  $0 < \varepsilon_2 < \sqrt{N^2 + p^2} - N$ . Fix  $\varepsilon_0 > 0$  such that  $0 < \varepsilon_0 \leq \min(\varepsilon_1, \varepsilon_2)$ . Then each  $\varepsilon \in (0, \varepsilon_0)$  satisfies

$$(3.14) \quad 1 < p - \varepsilon < s_k < p + \varepsilon < (p - \varepsilon)^*, \quad \text{when } k \geq k_0,$$

where for any  $q \in (1, \infty)$ ,  $q^* = \frac{Nq}{N-q}$  if  $q < N$  and  $q^* = \infty$  if  $q > N$ . Note that we are interested in  $k$  large enough since we have  $j \rightarrow +\infty$

From (3.11) and Hölder's inequality, we obtain

$$(3.15) \quad \|\nabla w_k\|_{p-\varepsilon} \leq |\Omega|^{\frac{s_k-p+\varepsilon}{s_k}}, \quad \text{when } 0 < \varepsilon < \varepsilon_0.$$

Hence  $\{w_k\}_{k \geq 1}$  is a bounded sequence in  $W_0^{1,p-\varepsilon}(\Omega)$ , whenever  $0 < \varepsilon < \varepsilon_0$ . Thus there is a subsequence noted also  $\{w_k\}_{k \geq 1}$  and a function  $u \in W_0^{1,p-\varepsilon}(\Omega)$  such that  $w_k \rightharpoonup u$  (weakly) in  $W_0^{1,p-\varepsilon}(\Omega)$ , as  $k \rightarrow \infty$ . By compactness, we have  $w_k \rightarrow u$  in  $L^{p+\varepsilon}(\Omega)$  in view of (3.14). Moreover, we can assume that  $w_k \rightarrow u$  a.e. on  $\Omega$  as  $k \rightarrow \infty$  (for a subsequence if necessary). In particular  $u \in L^p(\Omega)$  and is independent of  $\varepsilon$ . Moreover, by the Dominated Convergence Theorem and using Remark 2.6 and Lemma 2.5, we obtain

$$\int_{\Omega} |w_k|^{s_k} dx \rightarrow \int_{\Omega} |u|^p dx \quad \text{and} \quad \int_{\Omega} m(x) |w_k|^{s_k} dx \rightarrow \int_{\Omega} m(x) |u|^p dx$$

as  $k \rightarrow \infty$ . Letting  $k \rightarrow \infty$  in (3.13), we deduce that

$$(3.16) \quad \left( \lim_{k \rightarrow \infty} \mu_{s_k}(\lambda) \right) \int_{\Omega} |u|^p dx = 1 - \lambda \int_{\Omega} m(x)|u|^p dx.$$

On the other hand, the weak lower semicontinuity of the norm yields

$$(3.17) \quad \|\nabla u\|_{p-\varepsilon} \leq |\Omega|^{\frac{\varepsilon}{p}},$$

when  $0 < \varepsilon < \varepsilon_0$ . Fatou's lemma implies that

$$(3.18) \quad \int_{\Omega} \|\nabla u\|^p dx \leq 1.$$

It follows that  $u \in W^{1,p}(\Omega)$ . (3.16) yields that  $\int_{\Omega} |u|^p dx > 0$ . From this, (3.16) and (3.18), we find

$$(3.19) \quad \left( \lim_{k \rightarrow \infty} \mu_{s_k}(\lambda) \right) \int_{\Omega} |u|^p dx \geq \int_{\Omega} \|\nabla u\|^p dx - \lambda \int_{\Omega} m(x)|u|^p dx.$$

We have  $u \in W^{1,p}(\Omega) \cap W_0^{1,p-\varepsilon}(\Omega)$ , then from Theorem 2.2,  $u \in W_0^{1,p}(\Omega)$ . Consequently,  $u$  is an admissible function in the definition of  $\mu_p(\lambda)$ . Hence

$$(3.20) \quad \mu_p(\lambda) \int_{\Omega} |u|^p dx \leq \int_{\Omega} \|\nabla u\|^p dx - \lambda \int_{\Omega} m(x)|u|^p dx.$$

Finally, (3.19) and (3.20) imply that

$$\lim_{k \rightarrow \infty} \mu_{s_k}(\lambda) \geq \mu_p(\lambda).$$

That is,

$$(3.21) \quad \liminf_{j \rightarrow \infty} \mu_{s_j}(\lambda) \geq \mu_p(\lambda).$$

(3.10) and (3.21) show that

$$\lim_{j \rightarrow \infty} \mu_{s_j}(\lambda) = \mu_p(\lambda).$$

This concludes the proof of the claim. The theorem follows.  $\blacksquare$

**Remark 3.8** For the proof of the continuity on the right, the regularity of the domain is not necessary. It suffices to use (3.10) and Corollary 3.5. In the case of an irregular bounded domain and  $p < N$ , we can have  $\lim_{s \rightarrow p_-} \mu_s(\lambda) < \mu_p(\lambda)$ .

### 4 Convergence Results

Throughout this section, we assume that  $\Omega$  satisfies the segment property.

**Theorem 4.1** For any  $\lambda \in \mathbb{R}$ , we have the following:

$$(4.1) \quad \lim_{s \rightarrow p^+} \int_{\Omega} |\nabla u_s(\lambda) - \nabla u_p(\lambda)|^p dx = 0,$$

$$(4.2) \quad \lim_{s \rightarrow p^-} \int_{\Omega} |\nabla u_s(\lambda) - \nabla u_p(\lambda)|^s dx = 0,$$

where  $u_s(\lambda)$  denotes the principal eigenfunction associated to  $(\lambda, \mu_s(\lambda))$ .

Let us mention that for  $\lambda = 0$  this result is due to Lindqvist, see [19].

**Proof** For  $s \in (p, p + 1)$ , Hölder’s inequality yields

$$(4.3) \quad \int_{\Omega} |\nabla u_s(\lambda)|^p dx \leq |\Omega|^{1-\frac{p}{s}} \left( \int_{\Omega} |\nabla u_s(\lambda)|^s dx \right)^{p/s}.$$

From Lemma 3.1, we deduce that the set  $\{u_s; s \in (p, p + 1)\}$  is bounded in  $W_0^{1,p}(\Omega)$ . Hence there exist a sequence  $\{s_j\}_j$  converging to  $p_+$  and a function  $u \in W_0^{1,p}(\Omega)$  such that  $u_{s_j}(\lambda) \rightharpoonup u$  (weakly) in  $W_0^{1,p}(\Omega)$  as  $j \rightarrow \infty$ . Using the Rellich–Kondrachov Compactness Theorem, cf. [1, 16], we obtain that  $u_{s_j}(\lambda) \rightarrow u$  in  $L^p(\Omega)$ , as  $j \rightarrow \infty$ . Passing to a subsequence if necessary, we can assume that  $u_{s_j}(\lambda) \rightarrow u$  a.e. on  $\Omega$ . Thus  $u \geq 0$  a.e. on  $\Omega$ . We will prove that  $u \equiv u_p(\lambda)$ . From (4.3) we obtain that

$$(4.4) \quad \left( \int_{\Omega} |\nabla u_s(\lambda)|^p dx \right)^{\frac{s}{p}} \leq |\Omega|^{\frac{s}{p}-1} (\mu_s(\lambda) + \lambda \int_{\Omega} m(x) \{u_s(\lambda)\}^s dx).$$

The weak lower semicontinuity of the norm, the Dominated Convergence Theorem and the fact that  $\lim_{s \rightarrow p^+} \mu_s(\lambda) = \mu_p(\lambda)$  imply that

$$(4.5) \quad \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} m(x) u^p dx \leq \mu_p(\lambda).$$

On the other hand,  $u \in U_p$ . Therefore,  $u$  is admissible in the definition of  $\mu_p(\lambda)$ . Thus we conclude from (4.5) and the variational characterization of  $\mu_p(\lambda)$  that

$$\mu_p(\lambda) = \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} m(x) u^p dx.$$

Finally, by the uniqueness of the principal eigenfunction associated to the eigenpair  $(\lambda, \mu_p(\lambda))$ , we assert that  $u \equiv u_p(\lambda)$ . Thus the limit function  $u$  does not depend on the particular (sub)sequence  $s_1, s_2, \dots$ . Therefore  $u_s(\lambda) \rightarrow u_p(\lambda)$  at least in  $L^p(\Omega)$ , as  $s \rightarrow p_+$ . The strong convergence (4.1) can be obtained from Clarkson’s inequalities in  $W_0^{1,p}(\Omega)$ , cf. [1].

For the convergence (4.2), we proceed as follows. Let  $\varepsilon_0 > 0$  be small enough so that for any  $\varepsilon \in (0, \varepsilon_0)$  and each  $s$  close enough to  $p$ , we have

$$(4.6) \quad 1 < p - \varepsilon < s < p + \varepsilon < (p - \varepsilon)^*.$$

Such an  $\varepsilon_0$  exists because we suppose that as  $s \rightarrow p_-$  Hölder’s inequality yields

$$(4.7) \quad \int_{\Omega} |\nabla u_s(\lambda)|^{p-\varepsilon} dx \leq |\Omega|^{\frac{1-p+\varepsilon}{s}} \left( \int_{\Omega} |\nabla u_s(\lambda)|^s dx \right)^{\frac{p-\varepsilon}{s}},$$

whenever  $0 < \varepsilon < \varepsilon_0$ . We use the same argument in the proof of the stability to the left ( $s \rightarrow p_-$ ). It is not difficult to have  $u_s(\lambda)$  converge to  $u(\lambda)$  weakly in  $W_0^{1,p-\varepsilon}(\Omega)$ , strongly in  $L^p(\Omega)$  and almost everywhere in  $\Omega$ , as  $s \rightarrow p_-$ , when  $\varepsilon > 0$  satisfying (4.6), for some  $u(\lambda) \in W^{1,p}(\Omega) \cap W_0^{1,p-\varepsilon}(\Omega)$ . By Theorem 2.2,  $u(\lambda) \in W_0^{1,p}(\Omega)$  and by the uniqueness of the principal eigenfunction associated to the eigenpair  $(\lambda, \mu_p(\lambda))$ , we deduce that  $u \equiv u_p(\lambda)$ .

To achieve the proof, we use Clarkson’s inequalities related to uniform convexity of  $W_0^{1,s}(\Omega)$ . Consider the case  $p > 2$  first. Since  $s \rightarrow p_-$ , we take  $s$  near to  $p$  so that  $2 \leq s < p$  and we use Clarkson’s inequality associated to  $W_0^{1,s}(\Omega)$ . The case  $1 < p \leq 2$  is similar, the only change being to choose the other Clarkson inequality associated to  $W_0^{1,s}(\Omega)$ , since in this case  $1 < s \leq 2$ . The theorem follows. ■

**Remark 4.2** (i) The convergence (4.1) is also true for any bounded domain. (ii) In any arbitrary bounded domain, we can show that the convergence (4.2) implies *stability to the left*, i.e., for any  $\lambda \in \mathbb{R}$   $\lim_{s \rightarrow p_-} \mu_s(\lambda) = \mu_p(\lambda)$ . The converse is an open question when  $p < N$ .

Using the regularity  $C_{loc}^{1+\alpha}(\Omega)$  of the principal eigenfunctions  $u_s(\lambda)$  and the  $L^\infty$ -estimation established in Remark 2.6, we can state the following important result generalizing [19].

**Theorem 4.3** For any  $\lambda \in \mathbb{R}$  and  $p \in (1, \infty)$  and for any bounded domain  $\Omega$ , each sequence converging to  $p$ , contains a subsequence  $s_1, s_2, \dots, s_j$  such that  $u_{s_j}(\lambda) \rightarrow u(\lambda)$  and  $\nabla u_{s_j}(\lambda) \rightarrow \nabla u(\lambda)$ , locally uniformly in  $\Omega$ , where  $u(\lambda)$  is some function in  $C^1(\Omega)$ . Moreover,  $u(\lambda)$  is a weak solution to the equation

$$(4.8) \quad \Delta_p u + \lambda m(x)|u|^{p-2}u + \mu(\lambda)|u|^{p-2}u = 0,$$

where  $\mu(\lambda) = \lim_{j \rightarrow +\infty} \mu_{s_j}(\lambda)$ .

**Remarks**

- (1) (i) The limit  $\lim_{s \rightarrow p_-} \mu_s(\lambda)$  can fail to exist, because the limit function  $u(\lambda)$  can depend on the particular choice of  $s_1, s_2, \dots, s_j$ .
- (ii) The limit function  $u(\lambda)$  is positive. Indeed, since each  $u_{s_j}(\lambda) > 0$ , we have  $u(\lambda) \geq 0$ . Moreover  $u(\lambda) \in W^{1,p}(\Omega) \cap W_0^{1,p-\varepsilon}(\Omega)$ , for  $\varepsilon > 0$  small enough. Applying the Maximum Principle to (4.8), (cf. [26]), we conclude that  $u(\lambda) > 0$  in  $\Omega$ .

- (iii) We know that only the first eigenfunctions are not changing signs. Thus from (ii), if  $\mu(\lambda)$  is an eigenvalue associated to  $\lambda$  then  $\mu(\lambda) = \mu_p(\lambda)$ , and by normalization, we have  $u(\lambda) \equiv u_p(\lambda)$ .
- (2) An important point: even though for any  $\varepsilon > 0$  chosen sufficiently small the limit function  $u(\lambda)$  is in  $W^{1,p}(\Omega) \cap W_0^{1,p-\varepsilon}(\Omega)$ , it is not always the right eigenfunction  $u_p(\lambda)$ , i.e.,  $u(\lambda)$  is not necessarily in  $W_0^{1,p}(\Omega)$ . Thus  $u(\lambda)$  is not admissible in the definition of  $\mu_p(\lambda)$ .
- (3) If  $\Omega$  has the segment property as regularity, then  $u(\lambda) \equiv u_p(\lambda)$ ,  $\mu(\lambda) = \mu_p(\lambda)$  and  $\lim_{s \rightarrow p} \mu_s(\lambda) = \mu_p(\lambda)$ .

**Remark 4.4** (Generalization) Many results are readily extended to equations of the more general form

$$(4.9) \quad -\Delta_p u + h(x)|u|^{p-2}u - \lambda m(x)|u|^{p-2}u = \mu g(x)|u|^{p-2}u$$

$u \in W_0^{1,p}(\Omega)$ , where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$ ,  $N \geq 1$ ,  $h, m \in L^\infty(\Omega)$ , with  $m \not\equiv 0$  and  $g \in L^\infty(\Omega)$  with  $g \geq \varepsilon_0 > 0$  a.e. on  $\Omega$ , for some  $\varepsilon_0$ .

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