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UNIMODAL EXPANDING MAPS OF THE INTERVAL

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Let $I = [0,1]$ and let f be an unimodal expanding map in $C^0(I,I)$. If f has an expanding constant $\lambda \geqq (\lambda_n)^{1/2^m}$ for some integers $m \geqq 0$ and $n \geqq 1$, where λ_n is the unique positive zero of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$, then we show that f has a periodic point of period $2^m(2n+1)$. The converse of the above result is trivially false. The condition $\lambda \geqq (\lambda_n)^{1/2^m}$ in the above result is the best possible in the sense that we cannot have the same conclusion if the number λ_n is replaced by any smaller positive number and the generalisation of the above result to arbitrary piecewise monotonic expanding maps in $C^0(I,I)$ is not possible.

1. INTRODUCTION

Let I denote the unit interval $[0,1]$ of the real line and let $f \in C^0(I,I)$. For any positive integer n, let f^n denote the nth iterate of f. A point $x_0 \in I$ is called a periodic point of f if $f^m(x_0) = x_0$ for some positive integer m and the smallest such positive integer m is called the period of x_0 (under f).

The continuous map f is said to be piecewise monotonic if I can be divided into finite number of non-degenerate subintervals $I_1, I_2, \ldots, I_k$ on each of which f is either strictly increasing or strictly decreasing. If f is piecewise monotonic and there is a constant $\lambda > 1$ such that $|f(x) - f(y)| \geqq \lambda\,|x - y|$ whenever both x and y belong to some interval on which f is monotonic, then we call f an expanding map and, in this case, call λ an expanding constant for f.

The main result of this note is the following

THEOREM. *Assume that $f \in C^0(I,I)$ is an unimodal expanding map. If f has an expanding constant $\lambda \geqq (\lambda_n)^{1/2^m}$ for some integers $m \geqq 0$ and $n \geqq 1$, where λ_n is the unique positive zero of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$, then f has a periodic point of period $2^m(2n+1)$.*

The above result improves the main result in [4] and answers the question posed in [6, p. 437].

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Remark 1. It is easy to see that the converse of the above result is false. In the following, we present one such example. For $1/2 < \alpha \leqq 1$, let g_α be the map in $C^0(I,I)$ defined by letting $g_\alpha(x) = 2x$ for $0 \leqq x \leqq 1/2$ and $g_\alpha(x) = -2\alpha x + \alpha + 1$ for $1/2 \leqq x \leqq 1$. Then it is clear that g_α is an unimodal expanding map with 2α as an expanding constant which is close to 1 when α is close to $1/2$. However, it is shown in [7] that, for $1/2 < \alpha \leqq 1$, g_α has a periodic point of period 6.

Remark 2. The above result is the best possible in the sense that, if $m \geqq 0$ and $n \geqq 1$ are integers and λ is a real number with $1 < \lambda < (\lambda_n)^{1/2^m}$, where λ_n is the unique positive zero of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$, then there is an unimodal expanding map in $C^0(I,I)$ with λ as an expanding constant which has no periodic point of period $2^m(2n+1)$. To give one such example, let, for $1 < \lambda \leqq 2$, f_λ be the unimodal expanding map in $C^0(I,I)$ defined by letting $f_\lambda(x) = \lambda x + 2 - \lambda$ for $0 \leqq x \leqq 1 - 1/\lambda$ and $f_\lambda(x) = -\lambda x + \lambda$ for $1 - 1/\lambda \leqq x \leqq 1$. Then it is shown in [**10**, p. 227] that f_λ has a periodic point of period $2^m(2n+1)$ where $m \geqq 0$ and $n \geqq 1$ are integers if and only if $\lambda \geqq (\lambda_n)^{1/2^m}$

Remark 3. The above result does not hold for arbitrary piecewise monotonic expanding maps in $C^0(I,I)$. To be more precise, there exist, for integers $m \geqq 1$ and $n \geqq 2$, piecewise monotonic expanding maps $f_{m,n}$ in $C^0(I,I)$ such that (i) the integer n is an expanding constant for $f_{m,n}$; (ii) the topological entropy (see [**1**] for definition) of $f_{m,n}$ is greater than or equal to $\log n$; and (iii) $f_{m,n}$ has periodic points of period $2^m \cdot 3$, but no periodic points of period $2^{m-1}(2k+1)$ for any positive integer k. See [**8**] for some examples.

2. Preliminary results

We now introduce some preliminary results which will be used in the proof of our main result. The following result is a well-known result of Sharkovskii ([**14**], see also [**2, 3, 5, 9, 13, 15, 16**]).

THEOREM 1. *Rearrange the set of all positive integers according as the following new ordering (called Sharkovskii's ordering):* $3 \to 5 \to 7 \to \ldots. \to 2\cdot 3 \to 2 \cdot 5 \to 2 \cdot 7 \to \ldots. \to 2^k \cdot 3 \to 2^k \cdot 5 \to 2^k \cdot 7 \to \ldots. \to 2^3 \to 2^2 \to 2 \to 1$. *Assume that* $f \in C^0(I,I)$ *has a periodic point of period* m. *Then* f *also has a periodic point of period* n *precisely when* $m \to n$

The following result ([**11, 12**] of Li *et al* is useful in showing the existence of periodic points of certain odd periods > 1.

LEMMA 2. *Let* $f \in C^0(I,I)$ *and let* $n \geqq 3$ *be an odd integer. If there is a point* x_0 *such that* $f^n(x_0) \leqq x_0 < f(x_0)$ *or* $f^n(x_0) \geqq x_0 > f(x_0)$, *then* f *has a periodic point of period* n.

A proof of the following result can be found in [**17**].

LEMMA 3. *Assume that $f \in C^0(I,I)$ is a piecewise monotonic expanding map with λ as an expanding constant. Then, for every positive integer m, f^m is a piecewise monotonic expanding map with λ^m as an expanding constant.*

The following result improves [**4**, Lemma 3].

LEMMA 4. *Let $h \in C^0(I,I)$ be a piecewise monotonic expanding map with λ as an expanding constant. Assume that h admits a point y with $h(y) < y < h^2(y)$ such that h is decreasing on $[h(y), y]$ and increasing on $[y, h^2(y)]$. If h has no periodic point of period 3 and $g = h^2$, then there is a point z with $h(y) = g(z) < z < g^2(z) < y$ such that g is decreasing on $[g(z), z]$ and increasing on $[z, g^2(z)]$.*

PROOF: Since h is strictly decreasing on $[h(y), y]$ and $y \in [h(y), h^2(y)] = h([h(y), y])$, there is a unique point $z \in (h(y), y)$ such that $h(z) = y$. So, $g(z) = h(y) < z$. If $g^2(z) \geqq y$, then $h(y) < z < y \leqq g^2(z) = h^3(y)$. By Lemma 2, h has a periodic point of period 3 which is a contradiction. So, $h^3(y) = g^2(z) < y$. Since $g^2(z) - g(z) = h^4(z) - h^2(z) \geqq \lambda\big(h^3(z) - h(z)\big) \geqq \lambda^2\big(z - h^2(z)\big) > z - h^2(z) = z - g(z)$, we obtain $g^2(z) > z$. Consequently, we have shown that $h(y) = g(z) < z < g^2(z) < y$.

On the other hand, since $g_{|[g(z),z]}$ is the composition of $h_{|[g(z),z]}$ which is decreasing and $h_{|h([g(z),z])}$ which is increasing, it is decreasing. Similarly, since $g_{|[z,g^2(z)]}$ is the composition of $h_{|[z,g^2(z)]}$ which is decreasing and $h_{|h\left([z,g^2(z)]\right)}$ which is also decreasing, it is increasing. This completes the proof. ∎

The following lemma is crucial in proving our main result.

LEMMA 5. *Assume that $f \in C^0(I,I)$ admits a point y with $f(y) < y < f^2(y)$ such that f is decreasing on $[f(y), y]$ and increasing on $[y, f^2(y)]$. For every positive integer k, let λ_k denote the unique positive zero of the polynomial $x^{2k+1} - 2x^{2k-1} - 1$. If, for some positive integer n, $|f(u) - f(v)| \geqq \lambda_n |u - v|$ whenever both u and v lie in $[f(y), y]$ or in $[y, f^2(y)]$, then f has a periodic point of period $2n+1$.*

PROOF: If $f^3(y) \geqq f^2(y)$, then it is clear that f has a periodic point of period 3 and hence, by Theorem 1, f has a periodic point of period $2n+1$. So, we assume that $f^3(y) < f^2(y)$. Note that, since f is increasing on $[y, f^2(y)]$, $f(y) \leqq f^3(y)$. For simplicity, we let $\lambda = \lambda_n$ in the sequel.

Assume that $n = 1$. Then we have $f^3(y) - f(y) \geqq \lambda[f^2(y) - y] = \lambda\{[f^2(y) - f(y)] - [y - f(y)]\} \geqq \lambda\{\lambda[y - f(y)] - [y - f(y)]\} = \lambda(\lambda - 1)[y - f(y)] = \{[(\lambda^3 - 2\lambda - 1)/(\lambda + 1)] + 1\}[y - f(y)] = y - f(y)$. So, $f^3(y) \geqq y$. By Lemma 2, f has a periodic point of period 3.

Assume that $n \geqq 2$. If $f^3(y) \geqq y$, then by Lemma 2, f has a periodic point of period 3, and hence, by Theorem 1, f has a periodic point of period $2n+1$. So, we assume that $f^3(y) < y$. Consequently, f maps $[f(y), f^2(y)]$ onto itself. On the other hand, $y - f^4(y) = [f^2(y) - f^4(y)] - [f^2(y) - y] \geqq (\lambda^2 - 1)[f^2(y) - y] > 0$. Now, we have two cases to consider:

Case 1. $n = 2$.

In this case, we have $f^5(y) - f(y) \geqq \lambda[y - f^4(y)] \geqq \lambda(\lambda^2 - 1)[f^2(y) - y] = \lambda(\lambda^2 - 1)\{[f^2(y) - f(y)] - [y - f(y)]\} \geqq \lambda(\lambda^2 - 1)(\lambda - 1)[y - f(y)] = \{[(\lambda^5 - 2\lambda^3 - 1)/(\lambda + 1)] + 1\}[y - f(y)] = y - f(y)$. So, $f^5(y) \geqq y$. By Lemma 2, f has a periodic point of period 5.

Case 2. $n > 2$.

In this case, if $f^{2k+1}(y) \geqq y$ for some $1 \leqq k \leqq n - 1$, then by Lemma 2, f has a periodic point of period $2k + 1$, and hence, by Theorem 1, f has a periodic point of period $2n + 1$. So we assume that $f^{2k+1}(y) < y$ for all $1 \leqq k \leqq n - 1$. Also, for all $2 \leqq k \leqq n$, we define $A_k(\lambda)$ recursively, by putting $A_2(\lambda) = \lambda^2 - 1$ and $A_{j+1}(\lambda) = \lambda^2 A_j(\lambda) - 1$ for $2 \leqq j \leqq n - 1$. Then, since $\lambda = \lambda_n \geqq \sqrt{2}$, we have $A_k(\lambda) \geqq 1$ for all $2 \leqq k \leqq n$.

Assume that $y - f^{2k}(y) \geqq A_k(\lambda)[f^2(y) - y]$ for some $2 \leqq k \leqq n - 1$. Then

$$\begin{aligned} y - f^{2k+2}(y) &= [f^2(y) - f^{2k+2}(y)] - [f^2(y) - y] \\ &\geqq \lambda^2[y - f^{2k}(y)] - [f^2(y) - y] \geqq \lambda^2 A_k(\lambda)[f^2(y) - y] - [f^2(y) - y] \\ &= A_{k+1}(\lambda)[f^2(y) - y] > 0. \end{aligned}$$

Since we have already shown that $y - f^4(y) \geqq A_2(\lambda)[f^2(y) - y]$, the above implies, by induction on k, that $y > f^{2n}(y)$. Consequently

$$\begin{aligned} f^{2n+1}(y) - f(y) &\geqq \lambda[y - f^{2n}(y)] \geqq \lambda A_n(\lambda)[f^2(y) - y] \\ &\geqq \lambda A_n(\lambda)\{[f^2(y) - f(y)] - [y - f(y)]\} \\ &\geqq \lambda A_n(\lambda)\{\lambda[y - f(y)] - [y - f(y)]\} \\ &= \lambda(\lambda - 1)A_n(\lambda)[y - f(y)] \\ &= \{[(\lambda^{2n+1} - 2\lambda^{2n-1} - 1)/(\lambda + 1)] + 1\}[y - f(y)] \\ &= y - f(y) \end{aligned}$$

Thus, $f^{2n+1}(y) \geqq y$. By Lemma 2, f has a periodic point of period $2n + 1$.

This completes the proof. ∎

3. Proof of the theorem

Without loss of generality, we may assume that there is a point c with $0 < c < 1$ such that f is increasing on $[0, c]$ and decreasing on $[c, 1]$.

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If $m = 0$, the desired result follows from Lemma 5 above. So, from now on, we assume that $m \geqq 1$. Since $f(c) - f^2(c) \geqq \lambda[f(c) - c] > f(c) - c$, we have $f^2(c) < c$. If $f^3(c) \leqq c$, then since $f^3(c) \leqq c < f(c)$, f has, by Lemma 2, a periodic point of period 3. By Theorem 1, f has a periodic point of period $2^m(2n+1)$. So, we assume that $f^3(c) > c$. Since $f^4(c) - f^2(c) \geqq \lambda^2(c - f^2(c)) > c - f^2(c)$, if follows that $f^2(c) < c < f^4(c)$.

If $f^5(c) \leqq c$, then since $f^5(c) \leqq c < f(c)$, f has, by Lemma 2, a periodic point of period 5. By Theorem 1, f has a periodic point of period $2^m(2n+1)$. So, we assume that $f^5(c) > c$. Since $f^2 |_{[f^2(c),c]}$ is the composition of $f |_{[f^2(c),c]}$ which is increasing and $f |_{f([f^2(c),c])}$ which is decreasing, it is decreasing. Similarly, since $f^2 |_{[c,f^4(c)]}$ is the composition of $f |_{[c,f^4(c)]}$ which is decreasing and $f |_{f([c,f^4(c)])}$ which is also decreasing, it is increasing. If f^2 has a periodic point of period 3, then, by Theorem 1, f has a periodic point of period $2^m(2n+1)$. Otherwise, we can apply Lemma 4 to the map $h = f^2$ with $y = c$. So, without loss of generality, we may assume that f has no periodic point of period $2^k \cdot 3$ for any integer k with $0 < k < m$ and let $g = f^{2^m}$, By Lemma 4, there is a point z with $f^2(c) = g(z) < z < g^2(z) < c$ such that g is decreasing on $[g(z), z]$ and increasing on $[z, g^2(z)]$. By Lemma 3, g is a piecewise monotonic expanding map in $C^0(I,I)$ with λ^{2^m} as an expanding constant. Since $\lambda^{2^m} \geqq \lambda_n$, it follows from Lemma 5 that g has a periodic point of period $2n+1$. Therefore, by Theorem 1, f has a periodic point of period $2^m(2n+1)$. This completes the proof. ∎

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