CHARACTERIZATION OF CERTAIN DIFFERENTIAL OPERATORS IN THE SOLUTION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS†

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1. Introduction. In this paper we consider differential equations of the form

$$w_{z_1 z_2} + A_1(z_1, z_2)w_{z_1} + A_2(z_1, z_2)w_{z_2} + A_3(z_1, z_2)w = 0, \tag{1}$$

where the coefficients A_i are holomorphic functions in a domain $G_1 \times G_2 \subset \mathbf{C} \times \mathbf{C}$. We restrict our attention to those equations for which it is possible to represent the solutions in the form

$$w(z_1, z_2) = \sum_{k=0}^{n_1} a_{1,k}(z_1, z_2)g_1^{(k)}(z_1) + \sum_{k=0}^{n_2} a_{2,k}(z_1, z_2)g_2^{(k)}(z_2), \tag{2}$$

where $g_1(z_1)$ and $g_2(z_2)$ are arbitrary holomorphic functions in G_1 and G_2 respectively. The coefficients $a_{1,k}$ and $a_{2,k}$ depend on the given differential equation. Within the last ten years a number of publications have been devoted to this kind of representation of solutions.

When a representation (2) has the property,

$$a_{1,n_1} \neq 0 \quad \text{and} \quad a_{2,n_2} \neq 0 \quad \text{in} \quad G_1 \times G_2, \tag{3}$$

we give relations between the associated functions g_1 and g_2 in (2) and the associated functions of Vekua's integral operators (Lemmas 1 and 2). We should note here that the functions g_1 and g_2 are called "erzeugende Funktionen" in German papers; this notation has been introduced by K. W. Bauer. In order to avoid confusion with the "generating function" in S. Bergman's sense, we use here the term "*associated function*".

From Lemmas 1 and 2 we deduce that (3) is a sufficient condition on (2) for all solutions of the considered equation to be representable in this form. (It should be mentioned that the set of equations (1), for which representations (2) with the property (3) exist, is not empty and indeed contains important cases.) Finally, as an example, we consider a special differential equation, for which a representation (2) is given by K. W. Bauer and H. Florian in [**1**].

2. Vekua's representation theorem. We now consider the general equation (1), where the coefficients A_i, $i = 1, 2, 3$, are holomorphic functions in the bicylinder $G_1 \times G_2$. By G_1 and G_2 we denote simply connected domains of the complex plane. We define integral operators I_1 and I_2 by,

$$(I_1 f)(z_1, z_2) = \int_{z_1^0}^{z_1} R(t_1, z_2^0, z_1, z_2)f(t_1)\, dt_1,$$

$$(I_2 f)(z_1, z_2) = \int_{z_2^0}^{z_2} R(z_1^0, t_2, z_1, z_2)f(t_2)\, dt_2,$$

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where $R(z_1, z_2, t_1, t_2)$ is the Riemann function of (1) as defined on p. 16 of [**3**]; here and in the following $z_1^0 \in G_1$ and $z_2^0 \in G_2$ are arbitrary, but fixed points and the paths of integration are piecewise smooth curves in G_1 and G_2 respectively. The following theorem is well known, see [**3**, p. 23].

THEOREM 1 (Vekua).

(a) *For every solution w of equation* (1), *holomorphic in* $G_1 \times G_2$, *there exist two associated functions* $f_1(z_1)$ *and* $f_2(z_2)$, *holomorphic in* G_1 *and* G_2 *respectively, and an associated constant* $c \in \mathbf{C}$, *such that*

$$w(z_1, z_2) = cR(z_1^0, z_2^0, z_1, z_2) + I_1 f_1 + I_2 f_2. \tag{4}$$

(b) *Conversely, for arbitrary functions* $f_1(z_1)$ *and* $f_2(z_2)$, *holomorphic in* G_1 *and* G_2 *respectively, and arbitrary* $c \in \mathbf{C}$, (4) *represents solutions w of* (1), *holomorphic in* $G_1 \times G_2$.

(c) *For a given solution w of* (1), $f_1(z_1)$, $f_2(z_2)$ *and c are uniquely determined by*

$$\left.\begin{aligned} f_1(z_1) &= w_{z_1}(z_1, z_2^0) + A_2(z_1, z_2^0) w(z_1, z_2^0), \\ f_2(z_2) &= w_{z_2}(z_1^0, z_2) + A_1(z_1^0, z_2) w(z_1^0, z_2), \\ c &= w(z_1^0, z_2^0). \end{aligned}\right\} \tag{5}$$

3. Relations between Vekua and Bauer operators. In order to formulate Lemma 1, we give the following definitions, in which $\mathbf{N}_0$ denotes $\mathbf{N} \cup \{0\}$.

DEFINITION 1. H_i denotes the set of all functions $f(z_i)$, holomorphic in the domain G_i, and $H_i(s, z_i^0)$ is the set of all functions $f \in H_i$, such that $f^{(k)}(z_i^0) = 0$ for all $k = 0, 1, 2, \ldots, s-1$, where $s \in \mathbf{N}$ and $i = 1, 2$.

DEFINITION 2. Let T_i, $i = 1$ or $i = 2$, be a differential operator of the form

$$T_i = \sum_{k=0}^{r_i} a_{i,k}(z_1, z_2) \frac{\partial^k}{\partial z_i^k},$$

where $r_i \in \mathbf{N}_0$, $a_{i,k}$ is holomorphic in $G_1 \times G_2$ for all $k = 0, 1, 2, \ldots, r_i$ and $a_{i,r_i} \not\equiv 0$. If $T_i g_i$ is a solution of equation (1) for all $g_i \in H_i$, then T_i will be termed a *Bauer operator* of equation (1) and r_i the *order* of this operator.

Now we can establish the lemma.

LEMMA 1. *Let*

$$T_i = \sum_{k=0}^{n_i} a_{i,k} \frac{\partial^k}{\partial z_i^k}, \; i = 1 \text{ or } i = 2,$$

be a given Bauer operator of order n_i *of equation* (1), *such that* $a_{i,n_i} \neq 0$ *in* $G_1 \times G_2$. *Then the following statements hold.*

(a) *For every given function* $g_i \in H_i(n_i + 1, z_i^0)$ *there exists a uniquely determined function*

$f_i \in H_i$, such that $T_i g_i = I_i f_i$. This function f_i is given by

$$f_i(z_i) = \sum_{k=0}^{n_i+1} g_i^{(k)}(z_i)\left[a_{i,k-1}(z_1,z_2)+\frac{\partial}{\partial z_i}a_{i,k}(z_1,z_2)+A_{3-i}(z_1,z_2)a_{i,k}(z_1,z_2)\right]\Bigg|_{z_{3-i}=z_{3-i}^0}, \quad (6)$$

where $a_{i,-1} \equiv 0$ and $a_{i,n_i+1} \equiv 0$.

(b) *Conversely, for every given function $f_i \in H_i$ there exists a uniquely determined function $g_i \in H_i(n_i+1, z_i^0)$, such that $I_i f_i = T_i g_i$. This function g_i is the uniquely determined solution of the ordinary differential equation* (6) *with the initial conditions $g_i^{(k)}(z_i^0) = 0$ for all $k = 0, 1, 2, \ldots, n_i$.*

Proof. We give the proof for $i = 1$. For $i = 2$ the proof is completely analogous.

(a) Every solution w of the form $w = T_1 g_1$, $g_1 \in H_1(n_1+1, z_1^0)$, of equation (1) can be represented (see Theorem 1) by $w = cR(z_1^0, z_2^0, z_1, z_2)+I_1 f_1+I_2 f_2$, where c, f_1 and f_2 are given by (5). Since z_1^0 is a zero of order (at least) n_1+1 of g_1, it follows that $c = 0, f_2 \equiv 0$ and that

$$f_1(z_1) = \sum_{k=0}^{n_1+1} g_1^{(k)}(z_1)\left[a_{1,k-1}(z_1,z_2^0)+\frac{\partial}{\partial z_1}a_{1,k}(z_1,z_2^0)+A_2(z_1,z_2^0)a_{1,k}(z_1,z_2^0)\right], \quad (7)$$

where the coefficients $a_{1,-1}$ and a_{1,n_1+1} are defined by $a_{1,-1} \equiv 0$ and $a_{1,n_1+1} \equiv 0$. All other coefficients $a_{1,k}$, $k = 0, 1, 2, \ldots, n_1$, are determined by the form of the given Bauer operator

$$T_1 = \sum_{k=0}^{n_1} a_{1,k}\frac{\partial^k}{\partial z_1^k}.$$

A_2 is given by the differential equation (1). Since $g_1 \in H_1(n_1+1, z_1^0) \subset H_1$, $A_2(z_1, z_2^0) \in H_1$ and $a_{1,k}(z_1, z_2^0) \in H_1$, $k = 0, 1, 2, \ldots, n_1$, it is obvious that $f_1(z_1) \in H_1$.

(b) It only remains to show that there exists one and only one solution $g_1 \in H_1(n_1+1, z_1^0)$ of (7) for every $f_1 \in H_1$. This actually follows from the theory of linear ordinary differential equations with complex variables. Let us divide equation (7) by $a_{1,n_1}(z_1, z_2^0)$. Then we find an equation of the form

$$g_1^{(n_1+1)}+\alpha_{n_1}(z_1)g_1^{(n_1)}+\ldots+\alpha_0(z_1)g_1(z_1) = f_1(z_1)/a_{1,n_1}(z_1,z_2^0). \quad (8)$$

Let us assume that $U_\varepsilon(z_1^0) \subset G_1$ is an ε-neighbourhood of z_1^0, such that $a_{1,n_1}(z_1, z_2^0) \neq 0$ in $U_\varepsilon(z_1^0)$. Then all coefficients in (8) are holomorphic in $U_\varepsilon(z_1^0)$. So the initial value problem (8) with $g_1^{(k)}(z_1^0) = 0$ for $k = 0, 1, 2, \ldots, n_1$ certainly has a uniquely determined solution g_1, holomorphic in $U_\varepsilon(z_1^0)$; see [2, p. 20]. If $a_{1,n_1} \neq 0$ in $G_1 \times G_2$, this solution can be extended by unrestricted analytic continuation to the whole domain G_1. As G_1 is simply connected, it follows from the Monodromy Theorem that, for every $f_1 \in H_1$, there exists one and only one solution $g_1 \in H_1(n_1+1, z_1^0)$ of equation (7).

LEMMA 2. *Let*

$$T_1 = \sum_{k=0}^{n_1} a_{1,k}\frac{\partial^k}{\partial z_1^k}, \quad T_2 = \sum_{k=0}^{n_2} a_{2,k}\frac{\partial^k}{\partial z_2^k}$$

be given Bauer operators of order n_1 and n_2 respectively of equation (1), *such that $a_{1,n_1} \neq 0$ and*

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$a_{2,n_2} \neq 0$ in $G_1 \times G_2$. Then, for every constant $c \in \mathbf{C}$ there exist two functions $\tilde{g}_1 \in H_1(n_1, z_1^0)$ and $\tilde{g}_2 \in H_2(n_2, z_2^0)$, such that $cR(z_1^0, z_2^0, z_1, z_2) = T_1\tilde{g}_1 + T_2\tilde{g}_2$.

Proof. Every solution w of equation (1) of the form $w = cR(z_1^0, z_2^0, z_1, z_2)$, $c \in \mathbf{C}$, can be represented by $w = T_1\tilde{g}_1 + T_2\tilde{g}_2$, where $\tilde{g}_1(z_1)$ and $\tilde{g}_2(z_2)$ have the form:

$$\tilde{g}_1(z_1) = k_1(z_1 - z_1^0)^{n_1} - g_1(z_1), \tag{9}$$

$$\tilde{g}_2(z_2) = k_2(z_2 - z_2^0)^{n_2} - g_2(z_2). \tag{10}$$

Here k_1, $k_2 \in \mathbf{C}$ are numbers which satisfy the equation

$$c = k_1 a_{1,n_1}(z_1^0, z_2^0)n_1! + k_2 a_{2,n_2}(z_1^0, z_2^0)n_2! \tag{11}$$

and g_i, $i = 1, 2$, are the uniquely determined solutions of the initial value problems (6) with $g_i^{(k)}(z_i^0) = 0$ for $k = 0, 1, 2, \ldots, n_i$ and $f_i(z_i)$ given by

$$f_i(z_i) = \frac{\partial}{\partial z_i}[T_1 k_1(z_1 - z_1^0)^{n_1} + T_2 k_2(z_2 - z_2^0)^{n_2}]\big|_{z_{3-i} = z_{3-i}^0} +$$
$$+ [A_{3-i}(z_1, z_2)\{T_1 k_1(z_1 - z_1^0)^{n_1} + T_2 k_2(z_2 - z_2^0)^{n_2}\}]\big|_{z_{3-i} = z_{3-i}^0}.$$

All these results can be derived immediately from Theorem 1(c) (if we first only consider a solution w of the form $w = T_1 k_1(z_1 - z_1^0)^{n_1} + T_2 k_2(z_2 - z_2^0)^{n_2}$) and from Lemma 1. As $\tilde{g}_1 \in H_1(n_1, z_1^0)$ and $\tilde{g}_2 \in H_2(n_2, z_2^0)$, Lemma 2 follows.

REMARK 1. Let w be a solution of equation (1) of the form $w = cR(z_1^0, z_2^0, z_1, z_2)$, $c \in \mathbf{C}$. Then $k_1 \in \mathbf{C}$ and $k_2 \in \mathbf{C}$ in $cR(z_1^0, z_2^0, z_1, z_2) = T_1\tilde{g}_1 + T_2\tilde{g}_2$, where $\tilde{g}_1 \in H_1(n_1, z_1^0)$ and $\tilde{g}_2 \in H_2(n_2, z_2^0)$ are determined by (9) and (10) respectively, are not uniquely determined, but related by formula (11). Contrarily, $g_1 \in H_1(n_1 + 1, z_1^0)$ and $g_2 \in H_2(n_2 + 1, z_2^0)$ in (9) and (10) respectively are uniquely determined by a pair (k_1, k_2).

From these results, we can easily deduce the following theorem. In relation to fundamental systems in the theory of partial differential equations, see [**4, 5**], an equivalent result has been given by W. Watzlawek in [**5**, p. 203].

THEOREM 2. *Let*

$$T_1 = \sum_{k=0}^{n_1} a_{1,k}\frac{\partial^k}{\partial z_1^k}, \quad T_2 = \sum_{k=0}^{n_2} a_{2,k}\frac{\partial^k}{\partial z_2^k}$$

be given Bauer operators of order n_1 and n_2 respectively of equation (1), *such that $a_{1,n_1} \neq 0$ and $a_{2,n_2} \neq 0$ in $G_1 \times G_2$. Then for every solution w of equation* (1), *holomorphic in $G_1 \times G_2$, there exist two associated functions $g_1 \in H_1(n_1, z_1^0)$ and $g_2 \in H_2(n_2, z_2^0)$, such that $w(z_1, z_2) = T_1 g_1 + T_2 g_2$.*

Proof. This result follows immediately from Theorem 1(a), Lemma 1(b) and Lemma 2.

EXAMPLE. K. W. Bauer and H. Florian gave a representation of solutions of a certain differential equation in [**1**]. Theorem 2 enables us to state that all solutions of this equation can be represented in the given form. In [1] the following equation is considered:

$$\omega^2 S_1 S_2 w + (n-n^*)\omega S_2 w - n(n^*+1)w = 0 \quad (n, n^* \in \mathbf{N}_0), \tag{12}$$

where

$$\left.\begin{aligned} \omega(z_1, z_2) = \phi(z_1)+\psi(z_2)\,(\phi\in H_1, \psi \in H_2) \text{ such that } \omega\phi'\psi' \neq 0 \text{ in } G_1 \times G_2, \\ S_1 = \frac{1}{\phi'(z_1)}\frac{\partial}{\partial z_1}, \\ S_2 = \frac{1}{\psi'(z_2)}\frac{\partial}{\partial z_2}. \end{aligned}\right\} \tag{13}$$

It is shown that for all $g_1 \in H_1$ and $g_2 \in H_2$,

$$w = \sum_{k=0}^{n} \frac{(-1)^k (n+n^*-k)!}{k!(n-k)!\omega^{n-k}} S_1^k g_1 + \sum_{k=0}^{n^*} \frac{(-1)^k (n+n^*-k)!}{k!(n^*-k)!\omega^{n-k}} S_2^k g_2 \tag{14}$$

represents solutions of (12), holomorphic in $G_1 \times G_2$. If we arrange the right hand side of (14) as the sum of two series

$$\sum_{k=0}^{n} a_{1,k} g_1^{(k)} \quad \text{and} \quad \sum_{k=0}^{n^*} a_{2,k} g_2^{(k)}$$

we see that (14) is of the form $T_1 g_1 + T_2 g_2$, where T_1 and T_2 are Bauer operators of order n and n^* respectively. The coefficients $a_{1,n}$ and a_{2,n^*} of T_1 and T_2 respectively are of the form

$$a_{1,n} = \frac{(-1)^n n^*!}{n![\phi'(z_1)]^n} \neq 0 \quad \text{in} \quad G_1 \times G_2,$$

$$a_{2,n^*} = \frac{(-1)^{n^*} n!}{n^*!\omega^{n-n^*}[\psi'(z_2)]^{n^*}} \neq 0 \quad \text{in} \quad G_1 \times G_2.$$

From Theorem 2 it follows that (14) represents all solutions of (12), holomorphic in $G_1 \times G_2$, if $g_1 \in H_1(n, z_1^0)$ and $g_2 \in H_2(n^*, z_2^0)$.

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REFERENCES

1. K. W. Bauer and H. Florian, Bergman-Operatoren mit Polynomerzeugenden, *Applicable Anal.*; to appear.

2. L. Bieberbach, *Theorie der gewöhnlichen Differentialgleichungen. Auf funktionentheoretischer Grundlage dargestellt* (Springer, 1953).

3. I. N. Vekua, *New methods for solving elliptic equations* (North Holland Publishing Company, 1968).

4. W. Watzlawek, Über lineare partielle Differentialgleichungen zweiter Ordnung mit Fundamentalsystemen, *J. Reine Angew. Math.* **247** (1971), 69–74.

5. W. Watzlawek, Über Zusammenhänge zwischen Fundamentalsystemen, Riemannfunktion und Bergman-Operatoren, *J. Reine Angew. Math.* **251** (1971), 200–211.

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