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Hulls of Ring Extensions

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Abstract. We investigate the behavior of the quasi-Baer and the right FI-extending right ring hulls under various ring extensions including group ring extensions, full and triangular matrix ring extensions, and infinite matrix ring extensions. As a consequence, we show that for semiprime rings R and S, if R and S are Morita equivalent, then so are the quasi-Baer right ring hulls $\widehat{Q}_{\mathfrak{qB}}(R)$ and $\widehat{Q}_{\mathfrak{qB}}(S)$ of R and S, respectively. As an application, we prove that if unital C^*-algebras A and B are Morita equivalent as rings, then the bounded central closure of A and that of B are strongly Morita equivalent as C^*-algebras. Our results show that the quasi-Baer property is always preserved by infinite matrix rings, unlike the Baer property. Moreover, we give an affirmative answer to an open question of Goel and Jain for the commutative group ring $A[G]$ of a torsion-free Abelian group G over a commutative semiprime quasi-continuous ring A. Examples that illustrate and delimit the results of this paper are provided.

In [16] and [18], the ring hull concept with respect to a class of rings was introduced and developed. Let $H_{\mathfrak{R}}(R)$ denote a ring hull of R with respect to a class $\mathfrak{R}$ of rings and $\mathbf{X}(R)$ denote a ring extension of R. Then it is natural to ask: How does $H_{\mathfrak{R}}(\mathbf{X}(R))$ compare with $\mathbf{X}(H_{\mathfrak{R}}(R))$? In this paper, we consider this question for ring hulls mainly with respect to the class of quasi-Baer rings, the class of right FI-extending rings, and various ring extensions including monoid rings, full and triangular matrix rings, and infinite matrix rings.

Throughout this paper, all rings are associative with identity unless indicated otherwise, and R denotes such a ring. Subrings and overrings preserve the identity of the base ring. All modules are unitary and we use M_R to denote a right R-module. If N_R is a submodule of M_R, then N_R is *essential* (resp., *dense* also called *rational*) in M_R if for any $0 \neq x \in M$, there exists $r \in R$ such that $0 \neq xr \in N$ (resp., for any $x, y \in M$ with $x \neq 0$, there exists $r \in R$ such that $yr \in N$ and $xr \neq 0$). For a module M_R, we let $N_R \leq^{\text{ess}} M_R$ denote that N_R is an essential submodule of M_R. As in [16], we say that an overring T of R is a *right ring of quotients* (resp., *right essential overring*) of R if R_R is dense in T_R (resp., R_R is essential in T_R). The maximal right ring of quotients of R is denoted by $Q(R)$, and the injective hull of R_R is denoted by $E(R_R)$.

Recall the definitions of some classes that generalize the class of right self-injective rings or (von Neumann) regular right self-injective rings. A ring R is called right (*FI-*)*extending* if every (ideal) right ideal of R is essential in a right ideal generated by an idempotent (see [12, 14, 15, 21, 23]); it is right (*quasi-*)*continuous* if R is right extending and (if A_R and B_R are direct summands of R_R with $A \cap B = 0$, then $A_R \oplus B_R$ is a direct summand of R_R) if $X_R \cong Y_R$ and X_R is a direct summand of R_R, then Y_R is a direct summand of R_R (see [25, 30]); and it is (*quasi-*)*Baer* if the right annihilator of

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every (ideal) nonempty subset of R is generated by an idempotent as a right ideal (see [7–10,20,22,28,32]). From [12, Theorem 4.7], the right FI-extending and quasi-Baer conditions are equivalent for a semiprime ring. These classes have their roots in the study of right self-injective rings and in Functional Analysis, especially in the study of von Neumann algebras.

We use $\mathfrak{B}$, $\mathfrak{qB}$, $\mathfrak{FI}$, $\mathfrak{E}$, and $\mathfrak{qCon}$ to denote the class of Baer rings, the class of quasi-Baer rings, the class of right FI-extending rings, the class of right extending rings, and the class of right quasi-continuous rings, respectively.

For a ring R, we let $\mathbf{I}(R)$, $\mathbf{B}(R)$, $\mathrm{Mat}_n(R)$, $T_n(R)$, and $Q^m(R)$ denote the set of all idempotents of R, the set of all central idempotents of R, the n-by-n matrix ring over R, the n-by-n upper triangular matrix ring over R, and the Martindale right ring of quotients of R, respectively. For a nonempty subset X of a ring R, $r_R(X)$, $\ell_R(X)$, and $\langle X\rangle_R$ denote the right annihilator of X in R, the left annihilator of X in R, and the subring of R generated by X, respectively. Ideals without the adjective "right" or "left" are two-sided ideals. For a ring R, $I \trianglelefteq R$ denotes that I is an ideal of R. We let $R\mathbf{B}(Q(R))$ denote the subring of $Q(R)$ generated by R and $\mathbf{B}(Q(R))$, which is called the *idempotent closure* of R [5].

Henceforth we assume that all right essential overrings of a ring R are contained as right R-modules in a fixed injective hull $E(R_R)$ of R_R, and all right rings of quotients of R are subrings of a fixed maximal right ring of quotients $Q(R)$ of R.

In our next definition, we exploit the notion of a right essential overring that is minimal with respect to belonging to a class $\mathfrak{K}$ of rings.

Definition 1 ([16, Definition 2.1]) Let $\mathfrak{K}$ denote a class of rings. For a ring R, let S be a right essential overring of R and T an overring of R. Consider the following conditions.

(i) $S \in \mathfrak{K}$.
(ii) If $T \in \mathfrak{K}$ and T is a subring of S, then $T = S$.
(iii) If S and T are subrings of a ring V and $T \in \mathfrak{K}$, then S is a subring of T.
(iv) If $T \in \mathfrak{K}$ and T is a right essential overring of R, then S is a subring of T.

If S satisfies (i) and (ii), then we say that S is a $\mathfrak{K}$ *right ring hull* of R, denoted by $\widetilde{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iii), then we say that S is the $\mathfrak{K}$ *absolute to V right ring hull* of R, denoted by $Q^V_{\mathfrak{K}}(R)$; for the $\mathfrak{K}$ absolute to $Q(R)$ right ring hull, we use the notation $\widehat{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iv), then we say that S is the $\mathfrak{K}$ *absolute right ring hull* of R, denoted by $Q_{\mathfrak{K}}(R)$. Observe that if $Q(R) = E(R_R)$, then $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$. The concept of a $\mathfrak{K}$ absolute right ring hull was already implicit in [30] from their definition of a type III continuous (module) hull.

Theorem 2 ([18, Theorems 3.3 and 3.15]) *Let R be a semiprime ring. Then we have the following.*

(i) $\widehat{Q}_{\mathfrak{qB}}(R) = \widehat{Q}_{\mathfrak{FI}}(R) = R\mathbf{B}(Q(R))$.
(ii) *If T is a right essential overring of R such that $R\mathbf{B}(Q(R))$ is a subring of T, then T is right FI-extending and quasi-Baer.*

The following example shows that the semiprimeness of R in Theorem 2 is not superfluous.

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Example 3 ([29, p. 372, Example 13.26(4)]) There exists a quasi-Baer right and left nonsingular Artinian ring R such that $\mathbf{B}(Q(R)) \not\subseteq R = Q_{\mathfrak{q}\mathfrak{B}}(R)$. Let A be a semisimple Artinian ring, and let

$$R = \begin{pmatrix} A & A & A \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}.$$

Then $R \in \mathfrak{q}\mathfrak{B}$ by [15, Theorem 3.2]. Now $|\mathbf{B}(R)| = |\mathbf{B}(A)|$, where $|\cdot|$ is the cardinality of a set. But, $Q(R) \cong \mathrm{Mat}_2(A) \oplus \mathrm{Mat}_2(A)$. Hence, $|\mathbf{B}(Q(R))| = |\mathbf{B}(A)|^2 < \infty$. Thus, $\mathbf{B}(Q(R)) \not\subseteq R$.

A monoid G is called a *u.p.-monoid* (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq G$ there exists an element $x \in G$ that is uniquely presented in the form ab, where $a \in A$ and $b \in B$. This class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid is cancellative, and every u.p.-group is torsion-free.

Theorem 4 *Let $R[G]$ be a semiprime monoid ring of a monoid G over a ring R. Then we have the following.*

(i) $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)[G] \subseteq \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R[G])$.

(ii) *If G is a u.p.-monoid, then* $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R[G]) = \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)[G]$.

Proof (i) Note that R is semiprime. If $I \trianglelefteq R$ with $\ell_R(I) = 0$, then

$$I[G]_{R[G]} \trianglelefteq R[G]_{R[G]} \quad \text{and} \quad \ell_{R[G]}(I[G]) = 0.$$

Thus $Q^m(R) \subseteq Q^m(R[G])$, so $Q^m(R)[G] \subseteq Q^m(R[G])$. Let $c \in \mathbf{B}(Q^m(R))$. Then $c \in Q^m(R)[G] \subseteq Q^m(R[G])$ and $cr = rc$ for any $r \in R$, hence $cb = bc$ for any $b \in R[G]$. So $c \in \mathbf{B}(Q^m(R[G]))$. Thus,

$$\mathbf{B}(Q(R)) = \mathbf{B}(Q^m(R)) \subseteq \mathbf{B}(Q^m(R[G])) = \mathbf{B}(Q(R[G])).$$

By Theorem 2, $R\mathbf{B}(Q(R)) = \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$. Hence, $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)[G] \subseteq \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R[G])$.

The proof of (ii) is a consequence of part (i) and [13, Theorem 1.2]. ■

In [25] Goel and Jain posed the open question: *If G is an infinite cyclic group and A is a prime right quasi-continuous ring, is it true that $A[G] \in \mathfrak{q}\mathfrak{Con}$?* Since a semiprime right quasi-continuous ring is quasi-Baer (see [16, Proposition 1.3]) and $A[G]$ is semiprime, Theorem 4 and [16, Proposition 1.3] show that $A[G] \in \mathfrak{F}\mathfrak{I}$. Thus, from Theorem 4, when A is a commutative semiprime quasi-continuous ring and G is torsion-free Abelian, then $A[G] \in \mathfrak{E}$, hence $A[G] \in \mathfrak{q}\mathfrak{Con}$. This provides an affirmative answer to the question when A is a commutative semiprime quasi-continuous ring.

Corollary 5 *Let R be a semiprime ring. Then we have the following:*

(i) $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R[x, x^{-1}]) = \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)[x, x^{-1}]$.

(ii) $\widehat{Q}_{\mathfrak{qB}}(R[X]) = \widehat{Q}_{\mathfrak{qB}}(R)[X]$ *and* $\widehat{Q}_{\mathfrak{qB}}(R[[X]]) = \widehat{Q}_{\mathfrak{qB}}(R)[[X]]$ *for a nonempty set* X *of not necessarily commuting indeterminates.*

Proof (i) Note that $R[x, x^{-1}] \cong R[C_\infty]$, which is semiprime, where C_∞ is the infinite cyclic group. Therefore $\widehat{Q}_{\mathfrak{qB}}(R[x, x^{-1}]) = \widehat{Q}_{\mathfrak{qB}}(R)[x, x^{-1}]$ by Theorem 4.

(ii) Since R is semiprime, so is $R[X]$. Thus $\widehat{Q}_{\mathfrak{qB}}(R[X]) = \widehat{Q}_{\mathfrak{qB}}(R)[X]$ follows from Theorem 4. By using the fact that $R \in \mathfrak{qB}$ if and only if $R[[X]] \in \mathfrak{qB}$ in [11, Theorem 1.8] and by modifying the proof of Theorem 4, we have $\widehat{Q}_{\mathfrak{qB}}(R[[X]]) = \widehat{Q}_{\mathfrak{qB}}(R)[[X]]$. ■

Example 6 (i) ([18, Example 3.7]) Let $\mathbb{Z}[G]$ be the group ring of the group $G = \{1, g\}$ over the ring $\mathbb{Z}$ of integers. Then $\mathbb{Z}[G]$ is semiprime and

$$\widehat{Q}_{\mathfrak{qB}}(\mathbb{Z})[G] = \mathbb{Z}[G] \subsetneq \widehat{Q}_{\mathfrak{qB}}(\mathbb{Z}[G]) = \mathbb{Z}[G]\mathbf{B}(\mathbb{Q}[G]),$$

where $\mathbb{Q}$ is the field of rational numbers. Thus the "u.p.-monoid" condition is not superfluous in Theorem 4(ii).

(ii) Let F be a field. Then $F[x]$ is a semiprime u.p.-monoid ring and

$$F[x] = Q(F)[x] \neq Q(F[x]) = F(x),$$

where $F(x)$ is the field of fractions of $F[x]$. Thus "Q" cannot replace "$\widehat{Q}_{\mathfrak{qB}}$" in Theorem 4(ii).

Theorem 7 *Let $\mathfrak{K}$ be a class of rings such that $\Lambda \in \mathfrak{K}$ if and only if $\mathrm{Mat}_n(\Lambda) \in \mathfrak{K}$ for any positive integer n, and let $H_{\mathfrak{K}}(-)$ denote any of the right ring hulls indicated in Definition 1 for the class $\mathfrak{K}$. Then for a ring R, $H_{\mathfrak{K}}(R)$ exists if and only if $H_{\mathfrak{K}}(\mathrm{Mat}_n(R))$ exists for any n. In this case, $H_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(H_{\mathfrak{K}}(R))$.*

Proof We prove the case when $H_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$. The other cases are proved in a similar manner. Assume that $Q_{\mathfrak{K}}(R)$ exists. By hypothesis, $\mathrm{Mat}_n(Q_{\mathfrak{K}}(R)) \in \mathfrak{K}$. Let T be a right essential overring of $\mathrm{Mat}_n(R)$. Then T has a set of n-by-n matrix units. So $T = \mathrm{Mat}_n(A)$ for some ring A. A routine argument shows that A is a right essential overring of R. Thus if $T \in \mathfrak{K}$, then $A \in \mathfrak{K}$. Hence $Q_{\mathfrak{K}}(R)$ is a subring of A, so $\mathrm{Mat}_n(Q_{\mathfrak{K}}(R))$ is a subring of T. Therefore, $\mathrm{Mat}_n(Q_{\mathfrak{K}}(R)) = Q_{\mathfrak{K}}(\mathrm{Mat}_n(R))$. Thus if $Q_{\mathfrak{K}}(R)$ exists, then $Q_{\mathfrak{K}}(\mathrm{Mat}_n(R))$ exists.

Next assume that $Q_{\mathfrak{K}}(\mathrm{Mat}_n(R))$ exists. Then by the same argument as above, there is a right essential overring S of R with $Q_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(S)$. By hypothesis, $S \in \mathfrak{K}$. If W is a right essential overring of R and $W \in \mathfrak{K}$, then $Q_{\mathfrak{K}}(\mathrm{Mat}_n(R))$ is a subring of $\mathrm{Mat}_n(W)$. Hence S is a subring of W. Therefore, $S = Q_{\mathfrak{K}}(R)$. ■

Lemma 8 *Let $\delta \subseteq \mathbf{B}(Q(R))$ and $\Delta = \{1_n c \mid c \in \delta\}$, where 1_n is the identity matrix of $\mathrm{Mat}_n(R)$. Then we have the following:*

(i) $\mathrm{Mat}_n(\langle R \cup \delta \rangle_{Q(R)}) = \langle \mathrm{Mat}_n(R) \cup \Delta \rangle_{Q(\mathrm{Mat}_n(R))}$.
(ii) $Q(T_n(R)) = Q(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(Q(R))$.
(iii) $T_n(\langle R \cup \delta \rangle_{Q(R)}) = \langle T_n(R) \cup \Delta \rangle_{Q(\mathrm{Mat}_n(R))}$.

Proof (i) This part follows from straightforward calculations.

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(ii) Let $T = T_n(R)$. Then T_T is dense in $\mathrm{Mat}_n(R)_T$. So $Q(T_n(R)) = Q(\mathrm{Mat}_n(R))$. From [33, 2.3], $Q(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(Q(R))$. Thus,

$$Q(T_n(R)) = Q(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(Q(R)).$$

(iii) The proof follows from part (ii) and a routine calculation. ∎

Recall from [17, Definition 2.1] that a class of rings is said to be the *idempotent closure class*, denoted $\mathfrak{IC}$, if $\mathfrak{IC}$ is the $\mathfrak{D}$-$\mathfrak{E}$ class with

$$\mathfrak{D}_{\mathfrak{IC}}(R) = \{I \trianglelefteq R \mid I \cap \ell_R(I) = 0 \quad \text{and} \quad \ell_R(I) \cap \ell_R(\ell_R(I)) = 0\}$$

(see [16] for more details on $\mathfrak{D}$-$\mathfrak{E}$ classes). In other words, $R \in \mathfrak{IC}$ if and only if for each $I \in \mathfrak{D}_{\mathfrak{IC}}(R)$ there exists $e \in \mathbf{I}(R)$ such that $I_R \leq^{\mathrm{ess}} eR_R$. It was shown in [17, Remark 2.2(i)] that R is semiprime if and only if $\mathfrak{D}_{\mathfrak{IC}}(R)$ is the set of all ideals of R. The set $\mathfrak{D}_{\mathfrak{IC}}(R)$ was studied by Johnson and denoted by $\mathfrak{F}'(R)$ in [27]. In [17, Theorem 2.7] it was proved that:

(i) $R \in \mathfrak{IC}$ if and only if $R = R\mathbf{B}(Q(R))$;
(ii) $\widehat{Q}_{\mathfrak{IC}}(R) = R\mathbf{B}(Q(R))$; and
(iii) if $R \in \mathfrak{IC}$ and S is a right ring of quotients of R, then $S \in \mathfrak{IC}$.

Corollary 9 *Let R be a ring and n a positive integer. Then we have the following:*

(i) $\widehat{Q}_{\mathfrak{IC}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(\widehat{Q}_{\mathfrak{IC}}(R)) = \mathrm{Mat}_n(R\mathbf{B}(Q(R))$.
(ii) $\widehat{Q}_{\mathfrak{IC}}(T_n(R)) = T_n(\widehat{Q}_{\mathfrak{IC}}(R)) = T_n(R\mathbf{B}(Q(R)))$.
(iii) *If R is semiprime, then* $\widehat{Q}_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(\widehat{Q}_{\mathfrak{K}}(R))$, *where* $\mathfrak{K} = \mathfrak{qB}$ *or* $\mathfrak{FI}$.

Proof Theorem 2, Theorem 7, Lemma 8, [17, Theorem 2.7], [18, Theorems 3.3 and 3.15], and [32, Proposition 2] yield the result. ∎

The following example shows that Corollary 9(iii) does not hold when $\mathfrak{K} = \mathfrak{qCon}$.

Example 10 Let K be a field and n a positive integer such that $n > 1$. In [18, Example 3.12], $\mathrm{Mat}_n(K[x]) = \mathrm{Mat}_n(K)[x]$ is not right quasi-continuous, so

$$Q_{\mathfrak{qCon}}(\mathrm{Mat}_n(K)[x]) \neq \mathrm{Mat}_n(K)[x] = Q_{\mathfrak{qCon}}(\mathrm{Mat}_n(K))[x].$$

Hence $Q_{\mathfrak{qCon}}(\mathrm{Mat}_n(K)[x]) \neq Q_{\mathfrak{qCon}}(\mathrm{Mat}_n(K))[x]$. Also

$$Q_{\mathfrak{qCon}}(\mathrm{Mat}_n(K[x])) \neq \mathrm{Mat}_n(K[x]) = \mathrm{Mat}_n(Q_{\mathfrak{qCon}}(K[x])).$$

So $Q_{\mathfrak{qCon}}(\mathrm{Mat}_n(K[x])) \neq \mathrm{Mat}_n(Q_{\mathfrak{qCon}}(K[x]))$.

Theorem 11 *Let R be a semiprime ring. If R and a ring S are Morita equivalent, then* $\widehat{Q}_{\mathfrak{qB}}(R)$ *and* $\widehat{Q}_{\mathfrak{qB}}(S)$ *are Morita equivalent.*

Proof First we show the following.

Claim $\widehat{Q}_{\mathfrak{qB}}(eAe) = e\widehat{Q}_{\mathfrak{qB}}(A)e$ for any semiprime ring A and any $0 \neq e \in \mathbf{I}(A)$.

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Proof of Claim. Let $f \in \mathbf{B}(Q^m(A))$. Then there is $I \trianglelefteq A$ with $I_A \leq^{\text{ess}} A_A$ and $fI \subseteq A$. Let $0 \neq ete \in eAe$. Then, $0 \neq eteI$ because $\ell_A(I) = 0$. Since A is semiprime, $(eteI)^2 \neq 0$. Hence, $0 \neq eteIe \subseteq eIe$, so $eIe_{eAe} \leq^{\text{ess}} eAe_{eAe}$. Since

$$fI \subseteq A, \quad efIe = efeIe = efe \cdot eIe \subseteq eAe$$

with $eIe \trianglelefteq eAe$ and $eIe_{eAe} \leq^{\text{ess}} eAe_{eAe}$. Hence, $efe \in Q^m(eAe)$ and $(efe)^2 = efe$. Also for $eae \in eAe$, note that $efe \cdot eae = eae \cdot efe$, thus $efe \in \mathbf{B}(Q^m(eAe))$, so $e\mathbf{B}(Q^m(A))e \subseteq \mathbf{B}(Q^m(eAe))$. By Theorem 2,

$$e\widehat{Q}_{\mathfrak{qB}}(A)e = e(A\mathbf{B}(Q^m(A)))e = eAe \cdot e\mathbf{B}(Q^m(A))e \subseteq eAe \cdot \mathbf{B}(Q^m(eAe)) = \widehat{Q}_{\mathfrak{qB}}(eAe)$$

because $\mathbf{B}(Q(A)) = \mathbf{B}(Q^m(A))$ and $e\mathbf{B}(Q^m(A))e \subseteq \mathbf{B}(Q^m(eAe))$. Since $\widehat{Q}_{\mathfrak{qB}}(A) \in \mathfrak{qB}$, $e\widehat{Q}_{\mathfrak{qB}}(A)e \in \mathfrak{qB}$ by [9, Proposition 16]. Also, since

$$eAe \subseteq e\widehat{Q}_{\mathfrak{qB}}(A)e \subseteq \widehat{Q}_{\mathfrak{qB}}(eAe) \subseteq Q(eAe), \quad \widehat{Q}_{\mathfrak{qB}}(eAe) \subseteq e\widehat{Q}_{\mathfrak{qB}}(A)e$$

from Theorem 2. Thus $\widehat{Q}_{\mathfrak{qB}}(eAe) = e\widehat{Q}_{\mathfrak{qB}}(A)e$, proving the claim.

Now, since R and S are Morita equivalent, S is semiprime. Also there exists a positive integer n and $e^2 = e \in \operatorname{Mat}_n(R)$ such that $S = e\operatorname{Mat}_n(R)e$ and $\operatorname{Mat}_n(R)e\operatorname{Mat}_n(R) = \operatorname{Mat}_n(R)$ (see [29, p. 491]). Thus by the claim and Corollary 9(iii), we have that

$$\widehat{Q}_{\mathfrak{qB}}(S) = \widehat{Q}_{\mathfrak{qB}}(e\operatorname{Mat}_n(R)e) = e\widehat{Q}_{\mathfrak{qB}}(\operatorname{Mat}_n(R))e = e\operatorname{Mat}_n(\widehat{Q}_{\mathfrak{qB}}(R))e.$$

Also

$$\begin{aligned}
\operatorname{Mat}_n(\widehat{Q}_{\mathfrak{qB}}(R))e\operatorname{Mat}_n(\widehat{Q}_{\mathfrak{qB}}(R)) &= \operatorname{Mat}_n(R\mathbf{B}(Q(R)))e\operatorname{Mat}_n(R\mathbf{B}(Q(R))) \\
&= \operatorname{Mat}_n(R)e\operatorname{Mat}_n(R)\mathbf{B}(Q(R)) \\
&= \operatorname{Mat}_n(R)\mathbf{B}(Q(R)) = \operatorname{Mat}_n(R\mathbf{B}(Q(R))) \\
&= \operatorname{Mat}_n(\widehat{Q}_{\mathfrak{qB}}(R)).
\end{aligned}$$

Moreover, $e \in \operatorname{Mat}_n(\widehat{Q}_{\mathfrak{qB}}(R))$. Thus $\widehat{Q}_{\mathfrak{qB}}(R)$ is Morita equivalent to $\widehat{Q}_{\mathfrak{qB}}(S)$ by [29, p. 491]. ■

Theorem 11 does not hold for the case of Baer absolute to $Q(R)$ right ring hulls. Let $R = F[x, y]$, where F is a field, and let $S = \operatorname{Mat}_n(R)$ with $n > 1$. Then $\widehat{Q}_{\mathfrak{B}}(R) = R$, however, $\widehat{Q}_{\mathfrak{B}}(S)$ does not even exist (see [18, Example 3.10]).

A C^*-algebra is called *unital* if it has an identity. Let A be a (not necessarily unital) C^*-algebra. Then the set $\mathfrak{I}_{ce}$ of all norm closed essential ideals of A forms a filter directed downwards by inclusion. The ring $Q_b(A)$ denotes the algebraic direct limit of $\{M(I)\}_{I \in \mathfrak{I}_{ce}}$, where $M(I)$ denotes the C^*-algebra multipliers of I; and $Q_b(A)$ is called the *bounded symmetric algebra of quotients* of A [3, p. 57, Definition 2.23]. The norm closure $M_{\text{loc}}(A)$ of $Q_b(A)$ (*i.e.*, the C^*-algebra direct limit $M_{\text{loc}}(A)$ of

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$\{M(I)\}_{I\in\mathcal{I}_{ce}}$) is called the *local multiplier algebra* of A [3, p. 65, Definition 2.3.1]. The local multiplier algebra $M_{\rm loc}(A)$ was first used in [24, 31] to show the innerness of certain $*$-automorphisms and derivations. For more details on $M_{\rm loc}(A)$ and $Q_b(A)$, see [3, 24, 31].

Following [3, p. 73, Definition 3.2.1], for a C^*-algebra A, the C^*-subalgebra $\overline{A\,{\rm Cen}(Q_b(A))}$ (*i.e.*, the norm closure of $A\,{\rm Cen}(Q_b(A))$ in $M_{\rm loc}(A)$) of $M_{\rm loc}(A)$ is called the *bounded central closure* of A, where ${\rm Cen}(-)$ is the center of a ring. If $A = \overline{A\,{\rm Cen}(Q_b(A))}$, then A is said to be *boundedly centrally closed*. Boundedly centrally closed algebras have been used to obtain a complete description of all centralizing additive mappings on C^*-algebras [1] and for investigating the central Haagerup tensor product of multiplier algebras [2].

Let A be a unital C^*-algebra. Then it is shown in [18, Lemma 4.9(i)] that $Q_{\mathfrak{qB}}(A)$ is a $*$-subalgebra of $Q_b(A)$, so $Q_{\mathfrak{qB}}(A)$ is a $*$-subalgebra of $M_{\rm loc}(A)$. By [18, Lemma 4.12(i)], A is boundedly centrally closed if and only if $A \in \mathfrak{qB}$. Let $\overline{Q_{\mathfrak{qB}}(A)}$ be the norm closure of $Q_{\mathfrak{qB}}(A)$ in $M_{\rm loc}(A)$. From [18, Lemma 4.12(ii) and (iii)], $\overline{Q_{\mathfrak{qB}}(A)} = \overline{A\,{\rm Cen}(Q_b(A))}$, and it is the smallest boundedly centrally closed intermediate C^*-algebra between A and $M_{\rm loc}(A)$.

Corollary 12 *Let A and B be unital C^*-algebras. Assume that A and B are Morita equivalent. Then we have the following:*

(i) *$Q_{\mathfrak{qB}}(A)$ and $Q_{\mathfrak{qB}}(B)$ are Morita equivalent.*
(ii) *The bounded central closure of A is Morita equivalent to the bounded central closure of B as rings.*
(iii) *The bounded central closure of A is strongly Morita equivalent to the bounded central closure of B as C^*-algebras.*

Proof (i) This follows from Theorem 11, since C^*-algebras are semiprime and right nonsingular.

(ii) Assume that A and B are Morita equivalent. Then, as in the proof of Theorem 11, there exist a positive integer n and $e = e^2 \in {\rm Mat}(A)$ such that

$$Q_{\mathfrak{qB}}(B) = e\,{\rm Mat}_n(Q_{\mathfrak{qB}}(A))e \text{ and } {\rm Mat}_n(Q_{\mathfrak{qB}}(A))e\,{\rm Mat}_n(Q_{\mathfrak{qB}}(A)) = {\rm Mat}_n(Q_{\mathfrak{qB}}(A)).$$

Therefore, $\overline{Q_{\mathfrak{qB}}(B)} = \overline{e\,{\rm Mat}_n(Q_{\mathfrak{qB}}(A))e}$ and

$$\overline{{\rm Mat}_n(Q_{\mathfrak{qB}}(A))e\,{\rm Mat}_n(Q_{\mathfrak{qB}}(A))} = \overline{{\rm Mat}_n(Q_{\mathfrak{qB}}(A))},$$

where $\overline{e\,{\rm Mat}_n(Q_{\mathfrak{qB}}(A))e}$ is the norm closure of $e\,{\rm Mat}_n(Q_{\mathfrak{qB}}(A))e$ in ${\rm Mat}_n(M_{\rm loc}(A))$, etc. We see that

$$\overline{e\,{\rm Mat}_n(Q_{\mathfrak{qB}}(A))e} = e\overline{{\rm Mat}_n(Q_{\mathfrak{qB}}(A))}e,$$

so $\overline{Q_{\mathfrak{qB}}(B)} = e\overline{{\rm Mat}_n(Q_{\mathfrak{qB}}(A))}e$. By [3, p. 40, Corollary 1.2.37(ii)], ${\rm Mat}_n(M(A)) = M({\rm Mat}_n(A))$. From the proof of [3, p. 79, Proposition 3.3.8], ${\rm Mat}_n(M_{\rm loc}(A)) = M_{\rm loc}({\rm Mat}_n(A))$ and ${\rm Mat}_n(M_{\rm loc}(B)) = M_{\rm loc}({\rm Mat}_n(B))$.

Since $Q_{\mathfrak{qB}}(B)$ is a $*$-subalgebra of $M_{\rm loc}(B)$, we can see that $\overline{{\rm Mat}_n(Q_{\mathfrak{qB}}(A))} = {\rm Mat}_n(\overline{Q_{\mathfrak{qB}}(A)})$. So $\overline{Q_{\mathfrak{qB}}(B)} = e\,{\rm Mat}_n(\overline{Q_{\mathfrak{qB}}(A)})e$. On the other hand,

$$\begin{aligned} 1 \in Q_{\mathfrak{qB}}(A) = {\rm Mat}_n(Q_{\mathfrak{qB}}(A))e\,{\rm Mat}_n(Q_{\mathfrak{qB}}(A)) &\subseteq \overline{{\rm Mat}_n(Q_{\mathfrak{qB}}(A))e\,{\rm Mat}_n(Q_{\mathfrak{qB}}(A))} \\ &= \overline{{\rm Mat}_n(Q_{\mathfrak{qB}}(A))}e\overline{{\rm Mat}_n(Q_{\mathfrak{qB}}(A))}. \end{aligned}$$

Therefore,

$$\overline{\mathrm{Mat}_n(Q_{\mathfrak{qB}}(A))} = \overline{\mathrm{Mat}_n(Q_{\mathfrak{qB}}(A))}e\overline{\mathrm{Mat}_n(Q_{\mathfrak{qB}}(A))} = \mathrm{Mat}_n(\overline{Q_{\mathfrak{qB}}(A)})e\,\mathrm{Mat}_n(\overline{Q_{\mathfrak{qB}}(A)}).$$

Now since $e \in \mathrm{Mat}_n(A) \subseteq \mathrm{Mat}_n(\overline{Q_{\mathfrak{qB}}(A)})$, it follows that $\overline{Q_{\mathfrak{qB}}(A)}$ and $\overline{Q_{\mathfrak{qB}}(B)}$ are Morita equivalent. Thus $\overline{A\,\mathrm{Cen}(Q_b(A))}$ and $\overline{B\,\mathrm{Cen}(Q_b(B))}$ are Morita equivalent since $\overline{Q_{\mathfrak{qB}}(A)} = \overline{A\,\mathrm{Cen}(Q_b(A))}$ and $\overline{Q_{\mathfrak{qB}}(B)} = \overline{A\,\mathrm{Cen}(Q_b(B))}$ from [18, Lemma 4.12(iii)].

(iii) The proof of this part follows from part (ii) and [4, Theorem, p. 253]. ■

From Corollary 12 and [19, Corollary 13], we obtain the following result. We use Mod-R to denote the category of right R-modules.

Corollary 13 *Let A be a unital C^*-algebra. Assume that P_A is a finitely generated projective generator for Mod-A. Then we have the following.*

(i) *$P \otimes_A Q_{\mathfrak{qB}}(A)$ is a finitely generated projective generator for Mod-$Q_{\mathfrak{qB}}(A)$.*

(ii) *$P \otimes_A \overline{A\,\mathrm{Cen}(Q_b(A))}$ is a finitely generated projective generator for Mod-$\overline{A\,\mathrm{Cen}(Q_b(A))}$.*

Lemma 14 *Let $\mathfrak{K}$ be a class of rings satisfying the following conditions, where n is a fixed positive integer:*

(i) *if $\Omega \in \mathfrak{K}$, then $T_n(\Omega) \in \mathfrak{K}$;*

(ii) *if the generalized triangular matrix ring $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right) \in \mathfrak{K}$, where M is an (A, B)-bimodule, then $A, B \in \mathfrak{K}$.*

Assume that S is a right essential overring of a ring R. Then $Q^S_{\mathfrak{K}}(R)$ (resp., $\widetilde{Q}_{\mathfrak{K}}(R)$) exists if and only if $Q^{T_m(S)}_{\mathfrak{K}}(T_m(R))$ (resp., $\widetilde{Q}_{\mathfrak{K}}(T_m(R))$ exists for all $m \leq n$. In this case, $Q^{T_m(S)}_{\mathfrak{K}}(T_m(R)) = T_m(Q^S_{\mathfrak{K}}(R))$ (resp., $\widetilde{Q}_{\mathfrak{K}}(T_m(R)) = T_m(\widetilde{Q}_{\mathfrak{K}}(R))$) for all $m \leq n$.

Proof We prove the result for the version involving $Q^S_{\mathfrak{K}}(R)$ and $Q^{T_m(S)}_{\mathfrak{K}}(T_m(R))$. The proof involving $\widetilde{Q}_{\mathfrak{K}}(R)$ and $\widetilde{Q}_{\mathfrak{K}}(T_m(R))$ is similar.

Assume that $Q^S_{\mathfrak{K}}(R)$ exists. The result is true for $n = 1$. Assume that the result is true for $m = k$. We show that the result is true for $m = k+1$. Let $W \in \mathfrak{K}$ be an intermediate ring between $T_{k+1}(R)$ and $T_{k+1}(S)$. Then $W = (W_{ij})$ is a $(k+1)$-by-$(k+1)$ generalized triangular matrix ring, where $R \subseteq W_{ij} \subseteq S$, each W_{ii} is an intermediate ring between R and S, and W_{ij} is a (W_{ii}, W_{jj})-bimodule for $i \neq j$. Now W can be blocked to matrix $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$, where $A = W_{11}$, $M = [W_{12}, \dots, W_{1k+1}]$, and B is the k-by-k generalized triangular matrix ring (W_{ij}) with $2 \leq i, j \leq k+1$. In this case, B is an intermediate ring between $T_k(R)$ and $T_k(S)$. By condition (ii), $Q^S_{\mathfrak{K}}(R)$ is a subring of A and $T_k(Q^S_{\mathfrak{K}}(R))$ is a subring of B. Since $1 \in W_{ij}$ for $1 \leq i, j \leq n$ and each W_{1j} is a left A-module, $T_{k+1}(Q^S_{\mathfrak{K}}(R))$ is a subring of W. Hence, by induction, $Q^{T_m(S)}_{\mathfrak{K}}(T_m(R)) = T_m(Q^S_{\mathfrak{K}}(R))$ for all $m \leq n$.

Conversely, assume that $Q^{T_n(S)}_{\mathfrak{K}}(T_n(R))$ exists. Then $Q^{T_n(S)}_{\mathfrak{K}}(T_n(R))$ is a generalized triangular matrix ring that is intermediate between $T_n(R)$ and $T_n(S)$. Let $V = e_{11}Q^{T_n(S)}_{\mathfrak{K}}(T_n(R))e_{11}$, where $e_{11} \in T_n(S)$ is the matrix with 1 in the $(1,1)$-position and zero elsewhere. Then V is an intermediate ring between R and S. By condition (ii), $V \in \mathfrak{K}$. If $U \in \mathfrak{K}$ is an intermediate ring between R and S, then $T_n(U) \in \mathfrak{K}$ and

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$T_n(U)$ is an intermediate ring between $T_n(R)$ and $T_n(S)$. Hence $Q_{\mathfrak{R}}^{T_n(S)}(T_n(R))$ is a subring of $T_n(U)$, so V is a subring of U. Therefore, $V = Q_{\mathfrak{R}}^{S}(R)$. ■

We get the following result immediately from Lemma 14 and [15, 32].

Proposition 15 *Let R be a ring and S a right essential overring of R. Then $Q^S_{\mathfrak{qB}}(R)$ (resp., $\widetilde{Q}_{\mathfrak{qB}}(R)$) exists if and only if $Q^{T_n(S)}_{\mathfrak{qB}}(T_n(R))$ (resp., $\widetilde{Q}_{\mathfrak{qB}}(T_n(R))$ exists for all n. In this case, $Q^{T_n(S)}_{\mathfrak{qB}}(T_n(R)) = T_n(Q^S_{\mathfrak{qB}}(R))$ (resp., $\widetilde{Q}_{\mathfrak{qB}}(T_n(R)) = T_n(\widetilde{Q}_{\mathfrak{qB}}(R))$) for all n.*

Proposition 16 *Let S be a right ring of quotients of R. Then the following conditions are equivalent.*

(i) $Q^S_{\mathfrak{FI}}(R)$ *exists.*

(ii) $Q^{T_n(S)}_{\mathfrak{FI}}(T_n(R))$ *exists for all positive integers n.*

(iii) $Q^{T_n(S)}_{\mathfrak{FI}}(T_n(R))$ *exists for a fixed positive integer n.*

In this case, $Q^{T_n(S)}_{\mathfrak{FI}}(T_n(R)) = T_n(Q^S_{\mathfrak{FI}}(R))$ for all positive integers n.

Proof (i)⇒(ii) Assume that $Q^S_{\mathfrak{FI}}(R)$ exists. We proceed by induction. The result is true for $n = 1$. Suppose that the result is true for $n = k$. We show the result for $n = k + 1$. Let $W \in \mathfrak{FI}$ be an intermediate ring between $T_{k+1}(R)$ and $T_{k+1}(S)$. Then $W = (W_{ij})$ is a $(k + 1)$-by-$(k + 1)$ generalized triangular matrix ring, where $R \subseteq W_{ij} \subseteq S$, each W_{ii} is an intermediate ring between R and S, and W_{ij} is a (W_{ii}, W_{jj})-bimodule for $i \neq j$. Also, W can be blocked to a matrix $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$, where $A = W_{11}$, $M = [W_{12}, \dots, W_{1k+1}]$, and B is the k-by-k generalized triangular matrix ring (W_{ij}) with $2 \leq i, j \leq k+1$. We see that B is an intermediate ring between $T_k(R)$ and $T_k(S)$. By [15, Corollary 1.6], $B \in \mathfrak{FI}$. Let $I \trianglelefteq A$. Since $1 \in W_{ij}$ for all $1 \leq i, j \leq k + 1$, it follows that $W_{jj} \subseteq W_{1j}$ for all $1 \leq j \leq n$. So $N = I[W_{12}, \dots, W_{1k+1}]$ is an (A, B)-bisubmodule of M. Again from [15, Corollary 1.6], there is $f = f^2 \in A$ with $N_B \leq^{\text{ess}} fM_B$. Since $1 \in W_{ij}$, $I \subseteq fA$. Let $0 \neq fa \in fA$. Then $[a, 0, \dots, 0] \in M$. Hence there is $b \in B$ satisfying $0 \neq fab \in N$. Thus there exists $x \in W_{2j}$ for some j with $2 \leq j \leq k + 1$, such that $0 \neq fax \in IW_{2j}$. Since $R \subseteq W_{2j} \subseteq S$ and S is a right ring of quotients of R, I_R is dense in IW_{2j_R}, so there is $r \in R$ with $0 \neq faxr$ and $xr \in I$. Thus $I_R \leq^{\text{ess}} fA_R$, hence $I_A \leq^{\text{ess}} fA_A$. So $A \in \mathfrak{FI}$. Therefore $Q^S_{\mathfrak{FI}}(R)$ is a subring of A. Since each W_{1j} is a left A-module and $1 \in W_{1j}$, $A \subseteq W_{1j}$ for all $1 \leq j \leq k + 1$. Consequently, $T_{k+1}(Q^S_{\mathfrak{FI}}(R))$ is a subring of W. Thus, by induction, $Q^{T_n(S)}_{\mathfrak{FI}}(T_n(R)) = T_n(Q^S_{\mathfrak{FI}}(R))$ for all n.

(ii)⇒(iii) It is obvious.

(iii)⇒(i) Assume that $Q^{T_n(S)}_{\mathfrak{FI}}(T_n(R))$ exists for some n. Then $Q^{T_n(S)}_{\mathfrak{FI}}(T_n(R))$ is an intermediate ring between $T_n(R)$ and $T_n(S)$. Let $V = e_{nn}Q^{T_n(S)}_{\mathfrak{FI}}(T_n(R))e_{nn}$, where $e_{nn} \in T_n(S)$ is the matrix with 1 in the (n, n)-position and zero elsewhere. Then V is an intermediate ring between R and S. Also by [15, Corollary 1.6], $V \in \mathfrak{FI}$. If $U \in \mathfrak{FI}$ is an intermediate ring between R and S, then $T_n(U) \in \mathfrak{FI}$ by [12, Corollary 2.5]. Hence $Q^{T_n(S)}_{\mathfrak{FI}}(T_n(R))$ is a subring of $T_n(U)$, so V is a subring of U. Therefore $V = Q^S_{\mathfrak{FI}}(R)$. ■

Lemma 17 (see [6, Lemma 3.10]) *Let R be a right FI-extending ring. Then $\mathbf{B}(T) \subseteq R$ for any right essential overring T of R.*

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From [12, Theorem 4.7], a semiprime ring is quasi-Baer if and only if it is right FI-extending. However, Lemma 17 and Example 3 show that even a left and right nonsingular Artinian quasi-Baer ring is not necessarily right FI-extending. Also, [16, Example 3.16] exhibits a nonsemiprime nonsingular ring R with $R = R\mathbf{B}(Q(R))$, but R is not right FI-extending. However, in contrast to Theorem 2, our next result provides a large class of nonsemiprime rings T for which

$$\widehat{Q}_{\mathfrak{qB}}(T) = \widehat{Q}_{\mathfrak{FI}}(T) = T\mathbf{B}(Q(T)).$$

Theorem 18 *Let R be a semiprime ring and n a positive integer. Then we have the following.*

(i) $\widehat{Q}_{\mathfrak{qB}}(T_n(R)) = T_n(\widehat{Q}_{\mathfrak{qB}}(R)) = T_n(R)\mathbf{B}(Q(T_n(R)))$.
(ii) $\widehat{Q}_{\mathfrak{FI}}(T_n(R)) = T_n(\widehat{Q}_{\mathfrak{FI}}(R)) = T_n(R)\mathbf{B}(Q(T_n(R)))$.

Proof (i) Let $T = T_n(R)$ and let S be a right ring of quotients of T. By [32, Proposition 9] and Theorem 2, $T_n(\widehat{Q}_{\mathfrak{qB}}(R)) \in \mathfrak{qB}$ and $T_n(\widehat{Q}_{\mathfrak{qB}}(R)) = T_n(R\mathbf{B}(Q(R)))$. Take $e \in \mathbf{B}(Q(R))$ and let $J = R \cap (1-e)Q(R)$. Then $r_{Q(R)}(J) = eQ(R)$ as in the proof of [18, Theorem 3.3].

Let K be the n-by-n matrix with J in the $(1,1)$-position and 0 elsewhere. Thus TKT is the n-by-n matrix with J throughout the top row and 0 elsewhere. Also, $Q(T)KQ(T)$ is the n-by-n matrix with $Q(R)JQ(R)$ in all positions because $Q(T) = \mathrm{Mat}_n(Q(R))$ from Lemma 8. Note that

$$TKT \subseteq SKS \subseteq Q(T)KQ(T), r_{Q(R)}(J) = eQ(R), \text{ and } r_{Q(R)}(Q(R)JQ(R)) = eQ(R).$$

Hence,

$$fQ(T) = r_{Q(T)}(Q(T)KQ(T)) \subseteq r_{Q(T)}(SKS) \subseteq r_{Q(T)}(TKT) = fQ(T),$$

where f is the diagonal matrix with e on the diagonal. Assume that $S \in \mathfrak{qB}$. Then there is $c = c^2 \in S$ with $cS = r_S(SKS) = S \cap fQ(T)$. Hence, $cQ(T) \subseteq fQ(T)$.

Suppose that there is a nonzero right ideal I of $Q(T)$ with $I \subseteq fQ(T)$ and $I \cap cQ(T) = 0$. Then there is $0 \neq x \in I \cap S$. Since $x \in fQ(T)$, $x \in r_S(SKS) = cS$, so $0 \neq x \in I \cap cQ(T)$, a contradiction. Therefore, $cQ(T)_{Q(T)} \leq^{\text{ess}} fQ(T)_{Q(T)}$. Consequently, $cQ(T) = fQ(T)$ by the modular law. Since f is central in $Q(T)$, it follows that $c = f$. Thus S contains all n-by-n constant diagonal matrices whose diagonal entries are from $\mathbf{B}(Q(R))$. Hence, by Theorem 2, $T_n(\widehat{Q}_{\mathfrak{qB}}(R)) \subseteq S$. Therefore, using Lemma 8, $\widehat{Q}_{\mathfrak{qB}}(T) = T_n(\widehat{Q}_{\mathfrak{qB}}(R)) = T_n(R)\mathbf{B}(Q(T_n(R)))$.

(ii) Since R is semiprime, $\widehat{Q}_{\mathfrak{FI}}(R) = R\mathbf{B}(Q(R)) = Q^{Q(R)}_{\mathfrak{FI}}(R)$ by Theorem 2. Hence from Proposition 16,

$$Q^{T_n(Q(R))}_{\mathfrak{FI}}(T_n(R)) \text{ exists,} \quad \text{and} \quad Q^{T_n(Q(R))}_{\mathfrak{FI}}(T_n(R)) = T_n(R\mathbf{B}(Q(R))),$$

so $T_n(R\mathbf{B}(Q(R)) \in \mathfrak{FI}$. Next let S be a right FI-extending intermediate ring between $T_n(R)$ and $Q(T_n(R))$. Then by Lemmas 8 and 17,

$$\mathbf{B}(Q(T_n(R)) = \mathbf{B}(Q(\mathrm{Mat}_n(R))) = \mathbf{B}(\mathrm{Mat}_n(Q(R))) \subseteq S.$$

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So $T_n(R\mathbf{B}(Q(R))) \subseteq S$. Therefore

$$\widehat{Q}_{\mathfrak{FI}}(T_n(R)) = T_n(\widehat{Q}_{\mathfrak{FI}}(R)) = T_n(R)\mathbf{B}(Q(T_n(R)))$$

by Lemma 8. ∎

For a ring R and a nonempty set Γ, $\mathrm{CFM}_\Gamma(R)$, $\mathrm{RFM}_\Gamma(R)$, and $\mathrm{CRFM}_\Gamma(R)$ denote the column finite, the row finite, and the column and row finite matrix rings over R indexed by Γ, respectively.

In [20, Theorem 1], it was shown that $\mathrm{CRFM}_\Gamma(R)$ is a Baer ring for all infinite index sets Γ if and only if R is semisimple Artinian. Our next result shows that the quasi-Baer property is always preserved by infinite matrix rings.

Theorem 19

(i) $R \in \mathfrak{qB}$ *if and only if* $\mathrm{CFM}_\Gamma(R)$ *(resp.,* $\mathrm{RFM}_\Gamma(R)$ *and* $\mathrm{CRFM}_\Gamma(R)) \in \mathfrak{qB}$.

(ii) *If* $R \in \mathfrak{FI}$, *then* $\mathrm{CFM}_\Gamma(R)$ *(resp.,* $\mathrm{CRFM}_\Gamma(R)) \in \mathfrak{FI}$.

(iii) *If R is semiprime, then we have that*

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R)) \subseteq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)),$$

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{RFM}_\Gamma(R)) \subseteq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)), \text{ and}$$

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{CRFM}_\Gamma(R)) \subseteq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)).$$

Proof (i) Let $S = \mathrm{CFM}_\Gamma(R)$ and let J be a right ideal of S. Then $J = \sum_{\lambda\in\Lambda} x_\lambda S$ for some $x_\lambda \in S$. Say $x_\lambda = (a^\lambda_{ij})_{(i,j)\in\Gamma\times\Gamma}$ for each $\lambda \in \Lambda$. Let J_k be the set of all entries of the k-th row of J. Then $J_k = \sum_{\lambda\in\Lambda, j\in\Gamma} a^\lambda_{kj}R$. Since $R \in \mathfrak{qB}$, there is $e = e^2 \in R$ with $r_R(\sum_{k\in\Gamma} J_k) = eR$. So $(e1_S)S \subseteq r_S(J)$, where 1_S is the identity matrix in S. Next to see that $r_S(J) \subseteq (e1_S)S$, let $(b_{ij})_{(i,j)\in\Gamma\times\Gamma} \in r_S(J)$. Then for each $k \in \Gamma$ and each $(i, j) \in \Gamma \times \Gamma$, so $J_k b_{ij} = 0$. Thus $b_{ij} \in r_R(\sum_{k\in\Gamma} J_k) = eR$. Note that $b_{ij} = eb_{ij}$ for each $(i, j) \in \Gamma \times \Gamma$. Hence $(b_{ij})_{(i,j)\in\Gamma\times\Gamma} \in (e1_S)S$. Therefore, $r_S(J) = (e1_S)S$, so $S \in \mathfrak{qB}$. By a similar method, we see that $\mathrm{RFM}_\Gamma(R)$ and $\mathrm{CRFM}_\Gamma(R)$ are in $\mathfrak{qB}$.

Conversely, if $\mathrm{CFM}_\Gamma(R)$ (resp., $\mathrm{RFM}_\Gamma(R)$ and $\mathrm{CRFM}_\Gamma(R)) \in \mathfrak{qB}$, then $R \in \mathfrak{qB}$ by [22, Lemma 2].

(ii) Let $M(R)$ denote either $\mathrm{CFM}_\Gamma(R)$ or $\mathrm{CRFM}_\Gamma(R)$. We use e_{ij} to denote the matrix with 1 in the (i, j)-position and 0 elsewhere. Assume that $0 \neq I \trianglelefteq M(R)$. Then there are $X \trianglelefteq R$ and $e = e^2 \in R$ with $X_R \leq^{\mathrm{ess}} eR_R$, $I \subseteq M(X)$, and $e_{ij}M(X)e_{hk} \subseteq I$ for all possible i, j, h, k, where $M(X)$ is either $\mathrm{CFM}_\Gamma(X)$ or $\mathrm{CRFM}_\Gamma(X)$. Let $f \in M(R)$ with e in all diagonal positions and 0 elsewhere. Take $0 \neq a \in fM(R)$. Then there is j_0 such that the j_0-th column of a is nonzero. Let $\{er_1, \dots, er_n\}$ be the finite set of all nonzero entries in the j_0-th column. Thus there exists $s \in R$ such that $\{er_1s, \dots, er_ns\} \subseteq X$ and $er_ks \neq 0$ for some $k \in \{1, \dots, n\}$. Let i_0 denote the row of a in which er_i appears. Take $c \in M(R)$ such that c has s in the (j_0, k_0)-position and 0 elsewhere. Then $ac = e_{1_01_0}ace_{k_0k_0} + e_{2_02_0}ace_{k_0k_0} + \cdots + e_{n_0n_0}ace_{k_0k_0}$. But $e_{i_0i_0}ace_{k_0k_0} \in I$ for $i_0 = 1_0, \dots, n_0$. So $ac \in I$. Hence, $I_{M(R)} \leq^{\mathrm{ess}} fM(R)_{M(R)}$. Therefore $M(R) \in \mathfrak{FI}$.

(iii) Since R is semiprime, so are $\mathrm{CFM}_\Gamma(R)$, $\mathrm{RFM}_\Gamma(R)$ and $\mathrm{CRFM}_\Gamma(R)$. Let $e \in \mathbf{B}(Q(R))$. Then $e \in \mathbf{B}(Q^m(R))$, so there is $J \trianglelefteq R$ with $\ell_R(J) = 0$ and $eJ \subseteq R$. Thus

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$\mathrm{CFM}_\Gamma(J) \trianglelefteq \mathrm{CFM}_\Gamma(R), \ell_{\mathrm{CFM}_\Gamma(R)}(\mathrm{CFM}_\Gamma(J)) = 0$, and $(e\mathbf{1})\,\mathrm{CFM}_\Gamma(J) \subseteq \mathrm{CFM}_\Gamma(R)$, where $\mathbf{1}$ is the identity matrix in $\mathrm{CFM}_\Gamma(R)$. Hence $e\mathbf{1} \in Q^m(\mathrm{CFM}_\Gamma(R))$, so $e\mathbf{1} \in \mathbf{B}(Q^m(\mathrm{CFM}_\Gamma(R)))$. Thus

$$\mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) = \mathrm{CFM}_\Gamma(R\mathbf{B}(Q(R))) \subseteq Q^m(\mathrm{CFM}_\Gamma(R)) \subseteq Q(\mathrm{CFM}_\Gamma(R)).$$

Now $\mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) \in \mathfrak{qB}$ by part (i). Hence, by Theorem 2,

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R)) \subseteq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)).$$

Similarly,

$$\mathrm{RFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) = \mathrm{RFM}_\Gamma(R\mathbf{B}(Q(R))) \subseteq Q^m(\mathrm{RFM}_\Gamma(R)) \subseteq Q(\mathrm{RFM}_\Gamma(R)),$$

and also

$$\mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) = \mathrm{CRFM}_\Gamma(R\mathbf{B}(Q(R))) \subseteq Q^m(\mathrm{CRFM}_\Gamma(R)) \subseteq Q(\mathrm{CRFM}_\Gamma(R)).$$

By a similar method as above,

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{RFM}_\Gamma(R)) \subseteq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) \text{ and } \widehat{Q}_{\mathfrak{qB}}(\mathrm{CRFM}_\Gamma(R)) \subseteq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)). \quad \blacksquare$$

We note that the $\mathrm{CFM}_\Gamma(R)$ case in Theorem 19(ii) was proved in [14, Corollary 4.7] by other methods. In [20, p. 445] it is shown that for any ring R, $\mathrm{CRFM}_\Gamma(R)$ is never right extending when Γ is countably infinite. For a semiprime ring R, Theorem 2, [12, Theorem 4.7], and Theorem 19(i) yield that $\mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R))$ exists and is right FI-extending. Hence with each semiprime ring, we can associate a right FI-extending ring which is not right extending. For a given nonempty set Γ, it was shown in [26] that $R \in \mathfrak{qB}$ if and only if the column finite $\Gamma \times \Gamma$ upper triangular matrix ring over R is quasi-Baer.

We might expect from Corollary 9 and Theorem 19 that either

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R)) = \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) \text{ or}$$
$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{RFM}_\Gamma(R)) = \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) \text{ or}$$
$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{CRFM}_\Gamma(R)) = \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)).$$

However, our next example shows that there are a commutative (von Neumann) regular ring R and a nonempty set Γ such that *none* of these equalities holds.

Example 20 There exist a commutative (von Neumann) regular ring R and a set Γ such that $\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R)) \subsetneq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R))$, $\widehat{Q}_{\mathfrak{qB}}(\mathrm{RFM}_\Gamma(R)) \subsetneq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R))$, and $\widehat{Q}_{\mathfrak{qB}}(\mathrm{CRFM}_\Gamma(R)) \subsetneq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R))$. Let F be a field. Take a set Λ such that $|\Lambda| = |F|\aleph_0$. Let $F_i = F$ for all $i \in \Lambda$, and let

$$R = \Big\{ (\gamma_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} F_i \mid \gamma_i \text{ is a constant for all but finitely many } i \Big\}.$$

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Hence R is a subring of $\prod_{i\in\Lambda} F_i$. Then $Q(R) = \prod_{i\in\Lambda} F_i$ and the ring R is a commutative (von Neumann) regular ring (hence $\widehat{Q}_{\mathfrak{qB}}(R) = Q_{\mathfrak{qB}}(R)$).

Take $\Gamma = \widehat{Q}_{\mathfrak{qB}}(R) = R\mathbf{B}(Q(R))$ as a set. Note that

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R)) \subseteq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) \subseteq Q(\mathrm{CFM}_\Gamma(R))$$

by Theorem 2 and the proof of Theorem 19(iii). Since $\mathrm{CFM}_\Gamma(R)$ is semiprime,

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R)) = \mathrm{CFM}_\Gamma(R)\mathbf{B}(Q(\mathrm{CFM}_\Gamma(R))),$$

so $\mathbf{B}(Q(\mathrm{CFM}_\Gamma(R))) \subseteq \widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R))$. Thus ,

$$\mathbf{B}(Q(\mathrm{CFM}_\Gamma(R))) \subseteq \mathbf{B}(\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R))).$$

Therefore

$$\mathbf{B}(Q(\mathrm{CFM}_\Gamma(R))) = \mathbf{B}(\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R))).$$

Assume to the contrary that $\mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) = \widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R))$. Then

$$\mathbf{B}(Q(\mathrm{CFM}_\Gamma(R))) = \mathbf{B}(\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R))) = \mathbf{B}(\mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R))).$$

We let $\mu \in \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R))$ be a diagonal matrix whose entries are all distinct elements of $\widehat{Q}_{\mathfrak{qB}}(R)$. Then $\mu \in \widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R))$ by assumption.

From $\mathbf{B}(Q(\mathrm{CFM}_\Gamma(R))) = \mathbf{B}(\mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)))$, it follows that

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R)) = \mathrm{CFM}_\Gamma(R) \cdot \mathbf{B}(\mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R))).$$

Let $\mathbf{1}$ be the identity matrix in $\mathrm{CFM}_\Gamma(R)$. Then there exist $\theta_1, \dots, \theta_n \in \mathrm{CFM}_\Gamma(R)$ and $f_1, \dots, f_n \in \widehat{Q}_{\mathfrak{qB}}(R)$ such that $f_1\mathbf{1}, \dots, f_n\mathbf{1} \in \mathbf{B}(\mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)))$ are mutually orthogonal by [18, Lemma 4.9] (note that $f_i \in \mathbf{B}(\widehat{Q}_{\mathfrak{qB}}(R))$ for all i and these are mutually orthogonal) and

$$\mu = \theta_1 f_1 \mathbf{1} + \cdots + \theta_n f_n \mathbf{1}.$$

Thus for each entry of the diagonal of μ (or equivalently, each element of $R\mathbf{B}(Q(R))$), say a, there exist diagonal entries $\theta_i(a)$ of θ_i for $i = 1, \dots, n$ such that $a = \theta_1(a) f_1 + \cdots + \theta_n(a) f_n$. Thus $R\mathbf{B}(Q(R)) \subseteq Rf_1 + \cdots + Rf_n \subseteq R\mathbf{B}(Q(R))$, so $R\mathbf{B}(Q(R)) = Rf_1 + \cdots + Rf_n$. Therefore, $|R\mathbf{B}(Q(R))| = |R|$. If $|F|$ is finite or countably infinite, then $|R| = \aleph_0$, but $|R\mathbf{B}(Q(R))| \geq |\mathbf{B}(Q(R))| = 2^{\aleph_0}$ because $|\Lambda| = \aleph_0$ and $Q(R) = \prod_{i\in\Lambda} F_i$. Thus we have a contradiction. If $|F|$ is uncountably infinite, then $|R| = |\Lambda|$. But $|R\mathbf{B}(Q(R))| \geq |\mathbf{B}(Q(R))| = 2^{|\Lambda|}$. Hence, we again get a contradiction. Therefore, $\widehat{Q}_{\mathfrak{qB}}(\mathrm{CFM}_\Gamma(R)) \subsetneq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R))$.

Similarly,

$$\widehat{Q}_{\mathfrak{qB}}(\mathrm{RFM}_\Gamma(R)) \subsetneq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)) \quad \text{and} \quad \widehat{Q}_{\mathfrak{qB}}(\mathrm{CRFM}_\Gamma(R)) \subsetneq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{qB}}(R)).$$

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