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NÖRLUND OPERATORS ON l_p

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ABSTRACT. The Nörlund matrix N_a is the triangular matrix $\{a_{n-k}/A_n\}$, where $a_n \geq 0$ and $A_n := a_0 + a_1 + \cdots + a_n > 0$. It is proved that, subject to the existence of $\alpha := \lim na_n/A_n$, $N_a \in B(l_p)$ for $1 < p < \infty$ if and only if $\alpha < \infty$. It is also proved that it is possible to have $N_a \in B(l_p)$ for $1 < p < \infty$ when $\sup na_n/A_n = \infty$.

1. **Introduction.** Let $a := \{a_n\}$ be a sequence of non-negative numbers, and let $A_n := a_0 + a_1 + \cdots + a_n > 0$. The Nörlund matrix $N_a := \{a_{nk}\}$ is defined by

$$a_{nk} := \begin{cases} a_{n-k}/A_n & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k > n. \end{cases}$$

The N_a-transform $y = \{y_n\}$ of the sequence $x = \{x_n\}$ is given by

$$y_n := (N_a x)_n := \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k.$$

Suppose throughout that

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and define

$$\sigma_1(n) := \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} \Big(\frac{n+1}{k+1}\Big)^{1/p},$$

$$\sigma_2(k) := \sum_{n=k}^{\infty} \frac{a_{n-k}}{A_n} \Big(\frac{k+1}{n+1}\Big)^{1/q},$$

$$M_1 := \sup_{n\geq 0} \sigma_1(n), \quad M_2 := \sup_{k\geq 0} \sigma_2(k).$$

Let

$$||N_a||_p := \sup_{||x||_p \leq 1} ||N_a x||_p,$$

where

$$||x||_p := \Big(\sum_{n=0}^{\infty} |x_n|^p\Big)^{1/p},$$

so that $N_a \in B(l_p)$, the Banach algebra of bounded linear operators on l_p, exactly when $||N_a||_p$ is finite (in which case it is the norm of N_a).

The following theorem concerning sufficient conditions for $N_a \in B(l_p)$ is due to Borwein and Cass [1, Theorem 2].

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THEOREM A. *If*

$$\frac{na_n}{A_n} = O(1), \tag{1}$$

then $N_a \in B(l_p)$ *and* $\|N_a\|_p \le M_1^{1/q} M_2^{1/p} < \infty$.

Cass and Kratz [3] showed that (1) is in fact necessary and sufficient for $N_a \in B(l_p)$ when

$$a_n := f(n), \tag{2}$$

where $f(x)$ is a logarithmico-exponential function for all sufficiently large positive values of x. They showed that na_n/A_n tends to a finite or infinite limit when (2) is satisfied, and proved:

THEOREM B. *Suppose that a_n is given by (2), and that $na_n/A_n \longrightarrow \alpha$.*
(i) Then $N_a \in B(l_p)$ if and only if $\alpha < \infty$.
(ii) If $\alpha < \infty$, then

$$\lim_{n\to\infty} \sigma_1(n) = \frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)} \le \|N_a\|_p \le M_1^{1/q} M_2^{1/p} < \infty.$$

Condition (2) is redundant when $\alpha = 0$.

For $\alpha > -1$, the Cesàro matrix C_α is the Nörlund matrix N_a with

$$a_n := \binom{n+\alpha-1}{n}.$$

It follows from an inequality of Hardy's [4] that, for $\alpha \ge 0$, $C_\alpha \in B(l_p)$ and

$$\|C_\alpha\|_p = \frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)}.$$

This is thus the attained lower bound of the norms of all Nörlund matrices satisfying the conditions of Theorem B(ii).

The primary object of this paper is to show that the requirement that a_n be generated by a logarithmico-exponential function is redundant in Theorem B(i), and can be replaced by a far less restrictive monotonicity condition in Theorem B(ii). To this end we shall prove:

THEOREM 1. *Suppose that $na_n/A_n \longrightarrow \alpha$.*
(i) Then $N_a \in B(l_p)$ if and only if $\alpha < \infty$.
(ii) If $\alpha < \infty$, then

$$\|N_a\|_p \le M_1^{1/q} M_2^{1/p} < \infty,$$

and if, in addition, $\{n^c a_n\}$ is eventually monotonic for every constant $c \ne 1-\alpha$, then

$$\lim_{n\to\infty} \sigma_1(n) = \frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)} \le \|N_a\|_p.$$

Further, the monotonicity condition is redundant when $\alpha = 0$.

In § 4 we construct a sequence $\{a_n\}$ such that $\{na_n/A_n\}$ is unbounded but $N_a \in B(l_p)$ for all finite $p > 1$, thus showing that (1) is not a necessary condition for $N_a \in B(l_p)$.

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2. **Preliminary results.**

LEMMA 1. *Suppose that* $\{a_n/A_n\}$ *is positive and null. Let*

$$h := h(n) := \max_{k \le n} \left[\frac{A_k}{a_k}\right].$$

Then

$$A_n \ge \sum_{n-h \le k \le n} a_k \ge \left(1 - \frac{1}{e}\right) A_n.$$

Further, if $na_n/A_n \longrightarrow \infty$, *then* $h(n) = o(n)$.

PROOF. Suppose without loss that $n - h \ge 1$. Then

$$\log \frac{A_n}{A_{n-h-1}} = \int_{A_{n-h-1}}^{A_n} \frac{dx}{x} = \sum_{k=n-h}^{n} \int_{A_{k-1}}^{A_k} \frac{dx}{x} \ge \sum_{k=n-h}^{n} \frac{a_k}{A_k}$$
$$\ge (h+1) \min_{k \le n} \frac{a_k}{A_k} \ge 1,$$

and so

$$\frac{A_{n-h-1}}{A_n} \le \frac{1}{e}.$$

Thus

$$\frac{1}{A_n} \sum_{k=n-h}^{n} a_k = 1 - \frac{A_{n-h-1}}{A_n} \ge 1 - \frac{1}{e}.$$

Now suppose that $na_n/A_n \longrightarrow \infty$. Then, for $m \ge 1$,

$$0 \le \limsup_{n \to \infty} \frac{h(n)}{n} \le \lim_{n \to \infty} \frac{1}{n} \max_{k \le m} \frac{A_k}{a_k} + \limsup_{n \to \infty} \max_{m \le k \le n} \frac{k}{n} \frac{A_k}{ka_k}$$
$$\le \max_{k \ge m} \frac{A_k}{ka_k} \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

It follows that $h(n)/n \longrightarrow 0$. ∎

LEMMA 2. *Suppose that* $na_n/A_n \longrightarrow \alpha$ *where* $0 < \alpha < \infty$. *Then*

$$\lim_{n \to \infty} n^c a_n = \begin{cases} \infty & \text{if } c > 1 - \alpha, \\ 0 & \text{if } c < 1 - \alpha. \end{cases}$$

PROOF. Let

$$\alpha_n := \frac{na_n}{A_n}.$$

Then, as $n \longrightarrow \infty$,

$$\log A_n - \log A_{n-1} = -\log\left(1 - \frac{\alpha_n}{n}\right) \sim \frac{\alpha}{n},$$

and hence

$$\log A_n = \log A_0 - \sum_{k=1}^{n-1} \log\left(1 - \frac{\alpha_k}{k}\right) = \big(\alpha + o(1)\big)\log n.$$

Consequently $A_n \sim n^{\alpha+o(1)}$, and so $a_n \sim \alpha n^{\alpha-1+o(1)}$. The desired conclusion follows. ∎

LEMMA 3. *Suppose that* $0 < \alpha < \infty$, $\delta < 1$, *and that* $n^c a_n$ *is eventually positive and increasing when the constant* $c > 1-\alpha$, *and eventually decreasing when* $c < 1-\alpha$. *Then, as* $n \longrightarrow \infty$,

$$I(n) := \frac{1}{n}\sum_{k=0}^{n} \frac{a_k}{a_n}\left(1 - \frac{k}{n+1}\right)^{-\delta} \longrightarrow \frac{\Gamma(\alpha)\Gamma(1-\delta)}{\Gamma(\alpha+1-\delta)}.$$

PROOF. Let $1 > c_1 > 1-\alpha > c_2$. Then there is a positive integer N such that

$$\left(\frac{k}{n}\right)^{-c_2} \le \frac{a_k}{a_n} \le \left(\frac{k}{n}\right)^{-c_1} \text{ for } n \ge k \ge N.$$

Since

$$\lim_{n\to\infty} na_n = \infty,$$

we obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{N-1} \frac{a_k}{a_n}\left(1 - \frac{k}{n+1}\right)^{-\delta} = 0,$$

and hence

$$\begin{aligned}\limsup_{n\to\infty} I(n) &\le \lim_{n\to\infty} \frac{1}{n}\sum_{k=N}^{n} \left(\frac{k}{n}\right)^{-c_1}\left(1 - \frac{k}{n+1}\right)^{-\delta}\\ &= \int_0^1 x^{-c_1}(1-x)^{-\delta}\,dx = \frac{\Gamma(1-c_1)\Gamma(1-\delta)}{\Gamma(2-c_1-\delta)}\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to\infty} I(n) &\ge \lim_{n\to\infty} \frac{1}{n}\sum_{k=N}^{n} \left(\frac{k}{n}\right)^{-c_2}\left(1 - \frac{k}{n+1}\right)^{-\delta}\\ &= \int_0^1 x^{-c_2}(1-x)^{-\delta}\,dx = \frac{\Gamma(1-c_2)\Gamma(1-\delta)}{\Gamma(2-c_2-\delta)}.\end{aligned}$$

Letting $c_1 \longrightarrow 1-\alpha$ from the right and $c_2 \longrightarrow 1-\alpha$ from the left, we get the desired conclusion. ∎

The following lemma is a special case of a known result [2, Theorems 2].

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LEMMA 4. *If $\{b_n\}$ is a sequence of positive numbers, and if*

$$\mu_1 := \sup_{n\geq 0}\sum_{k=0}^{n}\frac{a_{n-k}}{A_n}\left(\frac{b_k}{b_n}\right)^{1/p} < \infty \text{ and } \mu_2 := \sup_{k\geq 0}\sum_{n=k}^{\infty}\frac{a_{n-k}}{A_n}\left(\frac{b_n}{b_k}\right)^{1/q} < \infty,$$

then $N_a \in B(l_p)$ and $\|N_a\|_p \leq \mu_1^{1/q}\mu_2^{1/p}$.

LEMMA 5. *If $\{b_n\}$ is a bounded sequence of positive numbers such that $\sum b_n = \infty$, and if, as $n \longrightarrow \infty$,*

$$\sum_{k=0}^{n}\frac{a_{n-k}}{A_n}\left(\frac{b_k}{b_n}\right)^{1/p} \longrightarrow \sigma \text{ (finite or infinite)},$$

then $\|N_a\|_p \geq \sigma$.

PROOF. Observe that if

$$D_n := \prod_{k=0}^{n}\left(1-\frac{b_k}{b}\right)^{-1} \text{ where } b > \sup_{k\geq 0} b_k,$$

and $d_n := D_n - D_{n-1}$ for $n \geq 1$, then $b_n = bd_n/D_n$ and $D_n \longrightarrow \infty$. The desired conclusion is now a consequence of a known result [2, Theorem 4]. ∎

3. **Proof of Theorem 1.**

PART (i). In view of Theorem A, it suffices to show that $N_a \notin B(l_p)$ when $\alpha = \infty$. Suppose therefore that $\alpha = \infty$.

If $\limsup a_n/A_n > 0$, then $\sum |a_n/A_n|^p = \infty$, but this implies that $N_a e^0 \notin l_p$, where $e^0 = (1, 0, 0, \ldots)$, so that $N_a \notin B(l_p)$.

Suppose that $\lim a_n/A_n = 0$. Since $\alpha = \infty$, we have, by Lemma 1 with $\delta := 1/p$, that, as $n \longrightarrow \infty$,

$$\begin{aligned}\sigma_1(n) &= \frac{(n+1)^\delta}{A_n}\sum_{k=0}^{n} a_k(n+1-k)^{-\delta} \geq \frac{(n+1)^\delta}{A_n}\sum_{n-h\leq k\leq n} a_k(n+1-k)^{-\delta}\\ &\geq \left(\frac{n+1}{h+1}\right)^\delta \frac{1}{A_n}\sum_{n-h\leq k\leq n} a_k \geq \left(1-\frac{1}{e}\right)\left(\frac{n+1}{h+1}\right)^\delta \longrightarrow \infty.\end{aligned}$$

It follows, by Lemma 5, that $N_a \notin B(l_p)$.

PART (ii). The case $\alpha = 0$ is part of Theorem B, so suppose that $0 < \alpha < \infty$. Since $na_n = O(A_n)$, it follows from Theorem A that $N_a \in B(l_p)$ and $\|N_a\|_p \leq M_1^{1/q}M_2^{1/p} < \infty$.

The monotonicity condition together with Lemma 2 ensures that a_n satisfies the monotonicity conditions of Lemma 3. Hence, by Lemma 3 with $\delta := 1/p$,

$$\sigma_1(n) = \frac{na_n}{A_n}I(n) \longrightarrow \frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)}$$

and so, by Lemma 5,

$$\|N_a\|_p \geq \frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)}.$$ ∎

4. **Construction.** We construct a sequence $\{a_n\}$ such that $\{na_n/A_n\}$ is unbounded but $N_a \in B(l_p)$ for all finite $p > 1$. Let

$$a_n := \begin{cases} 1 & \text{when } n = m^2, m = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $n = m^2$,

$$\frac{na_n}{A_n} = \frac{m^2}{m+1} \longrightarrow \infty \text{ as } m \longrightarrow \infty.$$

Thus $\{na_n/A_n\}$ is unbounded.

Now define

$$S_1(n) := \sum_{k=0}^{n} \frac{a_{n-k}}{A_n}\left(\frac{b_k}{b_n}\right)^{1/p}, \quad S_2(k) := \sum_{n=k}^{\infty} \frac{a_{n-k}}{A_n}\left(\frac{b_n}{b_k}\right)^{1/q},$$

where

$$b_n := \frac{1}{\sqrt{n+1}}.$$

In order to prove that $N_a \in B(l_p)$ it suffices, by Lemma 4, to demonstrate that $S_1(n) = O(1)$ and $S_2(k) = O(1)$.

For $1 \leq m^2 \leq n < (m+1)^2, \delta := 1/(2p)$, we have

$$\begin{aligned} S_1(n) &= \frac{(n+1)^\delta}{A_n} \sum_{k=0}^{n} a_k (n+1-k)^{-\delta} = \frac{(n+1)^\delta}{m+1} \sum_{k=0}^{m} (n+1-k^2)^{-\delta} \\ &\leq \frac{(n+1)^\delta (n+1-m^2)^{-\delta}}{m+1} + \frac{1}{m+1} + \frac{(n+1)^\delta}{m+1} \sum_{k=1}^{m-1} (m^2-k^2)^{-\delta} \\ &\leq (m+1)^{2\delta-1} + (m+1)^{-1} + (m+1)^{2\delta-1} m^{-\delta} \sum_{k=1}^{m-1} (m-k)^{-\delta} \\ &= (m+1)^{2\delta-1} + (m+1)^{-1} + (m+1)^{2\delta-1} m^{-\delta} \sum_{k=1}^{m-1} k^{-\delta} \\ &= O(m^{2\delta-1} + m^{-1} + m^{2\delta-1} \cdot m^{1-2\delta}) = O(1). \end{aligned}$$

Further, for $1 \leq m^2 \leq k+1 < (m+1)^2$, $\mu := 1/(2q)$, we have

$$\begin{aligned} S_2(k) &= (k+1)^\mu \sum_{n=k}^{\infty} \frac{a_{n-k}}{A_n (n+1)^\mu} \\ &\leq (k+1)^\mu \sum_{n=k}^{2k} \frac{a_{n-k}}{A_n (n+1)^\mu} + (k+1)^\mu \sum_{n=k+1}^{\infty} \frac{a_n}{A_n (n+1)^\mu} \\ &\leq 1 + (m+1)^{2\mu} \sum_{r=m}^{\infty} \frac{1}{r^{1+2\mu}} = 1 + O(m^{2\mu} \cdot m^{-2\mu}) = O(1). \end{aligned}$$

Hence $N_a \in B(l_p)$.

REFERENCES

1. D. Borwein and F. P. Cass, *Nörlund matrices as bounded operators on l_p,* Archiv der Mathematik **42**(1984), 464–469.

2. D. Borwein and A. Jakimovski, *Matrix operators on l_p,* Rocky Mountain J. Math. **9**(1979), 463–477.

3. F. P. Cass and W. Kratz, *Nörlund and weighted mean matrices as bounded operators on l_p,* Rocky Mountain J. Math. **20**(1990), 59–74.

4. G. H. Hardy, *An inequality for Hausdorff means,* J. London Math. Soc. **18**(1943), 46–50.

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