1

Fundamental Aspects of the Real Projective Plane

> Whereas Euclidean geometry describes objects as they are, projective geometry describes objects as they appear.
>
> Kristen R. Schreck (2016, p. 159)

Three-dimensional Euclidean space, $\mathbb{R}^3$, is perhaps the most familiar and natural geometry to the lay person. In this introductory section, we will show how we can build a 'new world out of nothing' (to use János Bolyai's asseveration) from the interplay between perpendicularity and parallelism, of lines and planes together. This interplay leads to the *real projective plane* and *duality*.

1.1 Parallelism

In three-dimensional space, 'parallelism' applies to two different types of object – to lines and to planes. A line can be parallel to a plane, a plane can be parallel to another plane, and a line can be parallel to a line (see Figure 1.1). In particular, a line ℓ is parallel to a line ℓ' if they are, firstly, lying in a common plane (i.e., *coplanar*) and, secondly, non-intersecting. Two non-intersecting lines that are not coplanar are *skew*. Two planes are parallel if they are non-intersecting. A line ℓ and a plane π are parallel if they are non-intersecting or ℓ lies within π.

1.2 Perpendicularity

Similarly, 'perpendicularity' is a relation that works for lines and planes alike. We really only need to understand perpendicularity of lines to understand what happens when we introduce planes. Two lines are perpendicular if they are coplanar and perpendicular in the common plane. A line ℓ is perpendicular to

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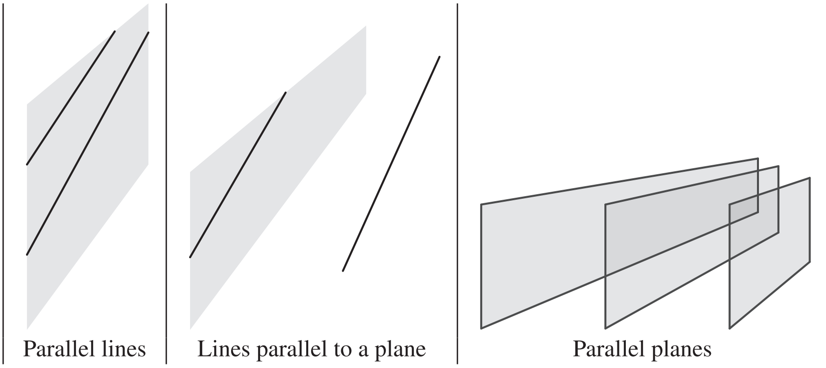

Figure 1.1 Parallel lines and planes.

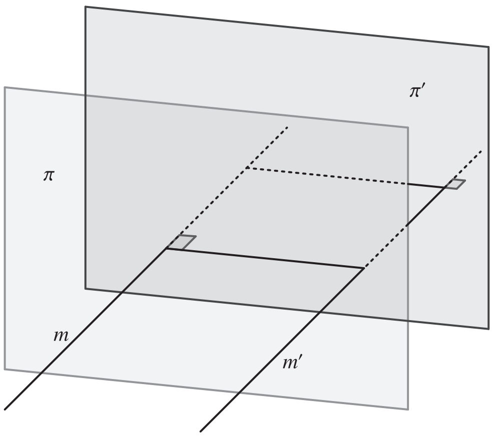

Figure 1.2 Interrelationship of perpendicularity and parallelism.

a plane π if ℓ is perpendicular to every line of π that it intersects. Two planes π and π' are perpendicular if they meet in a line ℓ, and π is perpendicular to some line of π'.

We will see soon that the most illuminating property of $\mathbb{R}^3$ is the interrelationship of perpendicularity and parallelism arising from the following property:

Parallel–Perpendicular Property: Let m, m' be lines and let π, π' be planes. If $m \parallel m'$, $\pi \parallel \pi'$, and $\pi \perp m$, then $\pi' \perp m'$ (see Figure 1.2).

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We can now elevate to the next level of abstraction. For each line ℓ, the **parallel class** $[\ell]$ of ℓ is the set of all lines parallel to ℓ (including ℓ itself). Similarly, write $[\pi]$ for the planes parallel to π. The first observation we make is the following:

Property 1.1
Let ℓ and π be a line and a plane (respectively). Then either

- *no element of* $[\ell]$ *is parallel to any element of* $[\pi]$*, or*
- *every element of* $[\ell]$ *is parallel to every element of* $[\pi]$*, and we say that* $[\ell]$ ***is parallel to*** $[\pi]$.

So it makes sense to write $[\ell] \parallel [\pi]$. This relation of parallelism between parallel classes of lines and planes is symmetric, and could abstractly be an *incidence relation* between two different types of objects. With this in mind, we make a second observation:

Property 1.2

- *For any two different parallel classes of lines* $[\ell]$ *and* $[\ell']$*, there is a unique parallel class of planes that is parallel to both* $[\ell]$ *and* $[\ell']$.
- *For any two different parallel classes of planes* $[\pi]$ *and* $[\pi']$*, there is a unique parallel class of lines that is parallel to both* $[\pi]$ *and* $[\pi']$.

We can manufacture a geometry $\mathcal{G}$ out of these parallel classes of lines and planes. The geometry we create will be *planar* in the sense that we have only two types of object, which we might as well temporarily call[1] *pistettä* (singular: *piste*) and *linjat* (singular: *linja*). This new geometry will consist only of objects and an incidence relation between them — no distance, no midpoints, no angles, no parallelism.

Pistettä	Parallel classes of lines of $\mathbb{R}^3$
Linjat	Parallel classes of planes of $\mathbb{R}^3$
Incidence	A parallel class of lines is ‘incident’ with a parallel class of planes if and only if they are parallel.

So from what we have discussed above, the incidence relation here is symmetric: two pistettä lie on a unique linja and two linjat have a unique piste in

[1] For some readers, the use of the words ‘points’ and ‘line’ will interfere with their understanding, so to make it clear that we are defining new points and new lines, we temporarily adopt another language for these new points and new lines.

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common. Therefore, there cannot be 'parallel' linjat in this geometry $\mathcal{G}$; two linjat are always concurrent. This geometry is an example of a non-Euclidean geometry – a **projective plane**.

1.3 Duality

Let $\mathcal{L}$ be the set of parallel classes of lines and let Π be the set of parallel classes of planes. There is a natural correspondence between $\mathcal{L}$ and Π: if $[\ell]$ is a parallel class of lines, then we take the set $\ell^\perp$ of all planes that are perpendicular to ℓ. By the parallel–perpendicular property, this set of planes is a parallel class of planes and did not depend on the representative we took from ℓ. Conversely, if $[\pi]$ is a parallel class of planes, we map to the set $\pi^\perp$ of all lines that are perpendicular to π. Thus we have the following correspondence:

$$[\ell] \longrightarrow [\ell]^\perp := [\ell^\perp]$$
$$[\pi] \longrightarrow [\pi]^\perp := [\pi^\perp]$$

Note that if we apply $\perp$ twice, then our objects are left invariant. For example, if we take all of the planes perpendicular to a line ℓ, and then take all of the lines perpendicular to those planes, it will result in the parallel class of ℓ. Moreover, this correspondence respects parallelism between elements of $\mathcal{L}$ and Π:

$$\left([\ell]^\perp\right)^\perp = [\ell], \qquad \left([\pi]^\perp\right)^\perp = [\pi],$$
$$[\ell] \parallel [\pi] \iff [\ell]^\perp \parallel [\pi]^\perp.$$

Finally, let's see what the perpendicularity correspondence $\perp$ does to $\mathcal{G}$. We saw above that it preserves incidence. So if P is a piste of $\mathcal{G}$ and m is a linja, then $P^\perp$ is incident with $m^\perp$ if and only if P is incident with m. We also saw that $\perp$ maps a piste to a linja, and then a linja to a piste, in such a way that if performed twice, it left the objects invariant. Such a map is called a **polarity**. This polarity also has the property that no piste P is incident with its image $P^\perp$; but we will return to this later once we have investigated projective planes more thoroughly.

1.4 Two Models of the Real Projective Plane

We have already seen that parallel classes of lines and planes of $\mathbb{R}^3$ form a projective plane – an incidence geometry of points and lines such that any pair of points determine a unique line, and two distinct lines always meet in a point. Each parallel class of lines has a representative passing through the origin O of $\mathbb{R}^3$. So we can replace each parallel class by a one-dimensional subspace

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Table 1.1 *The real projective plane.*

Points	1-dimensional subspaces of $\mathbb{R}^3$
Lines	2-dimensional subspaces of $\mathbb{R}^3$
Incidence	inclusion

of $\mathbb{R}^3$. Likewise, the parallel classes of planes can be simulated by taking two-dimensional subspaces of $\mathbb{R}^3$. Formally, the **real projective plane** $\mathsf{PG}(2,\mathbb{R})$ is the incidence structure defined in Table 1.1.

Theorem 1.3
(i) Any two points of $\mathsf{PG}(2,\mathbb{R})$ *are incident with a unique line.*
(ii) Any two lines $\mathsf{PG}(2,\mathbb{R})$ *are incident with a unique point.*

Proof The proof follows from elementary linear algebra. In particular, for (i), note that any two 1-dimensional subspaces of $\mathbb{R}^3$ span a unique 2-dimensional subspace. For (ii), we observe that any two 2-dimensional subspaces of $\mathbb{R}^3$ meet in a unique 1-dimensional subspace. □

We denote a point of $\mathsf{PG}(2,\mathbb{R})$ by homogeneous coordinates (x, y, z), $x, y, z \in \mathbb{R}$, not all zero. This means that we are simply dropping the angled brackets from the subspace $\langle(x, y, z)\rangle$ of $\mathbb{R}^3$; since

$$\langle(x, y, z)\rangle = \langle(cx, cy, cz)\rangle, \quad \text{for all } c \in \mathbb{R}\backslash\{0\},$$

we have $(x, y, z) = (cx, cy, cz)$. This is what we mean by saying that the coordinates are *homogeneous*, and it will be clear from the context that we are describing a point of $\mathsf{PG}(2,\mathbb{R})$ and not a vector of $\mathbb{R}^3$. Note that $(0, 0, 0)$ is not the homogeneous coordinates of any point of $\mathsf{PG}(2,\mathbb{R})$.

We denote a line with equation $ax + by + cz = 0$ by homogeneous line coordinates $[a, b, c]$, $a, b, c \in \mathbb{R}$, not all zero. (Again, note that $k(ax+by+cz) = 0$ for $k \in \mathbb{R}$ with $k \neq 0$ yields the same line, so these coordinates are indeed *homogeneous*.) A point (x, y, z) is incident with the line $[a, b, c]$ if and only if $ax + by + cz = 0$. Note that $[0, 0, 0]$ is not the homogeneous line coordinates of any line of $\mathsf{PG}(2,\mathbb{R})$.

Another way to define the real projective plane is to take the real Euclidean plane and enlarge it a little bit. Each line is equipped with an additional point – its *point at infinity* – which is simply the parallel class of that line. This ensures that two parallel lines now become two intersecting lines in the larger geometry. One extra line is introduced, and it is simply the set of all points at infinity – the *line at infinity*. Formally, the **extended Euclidean plane** is the incidence structure defined in Table 1.2.

Table 1.2 *The extended Euclidean plane.*

Points	points of $\mathbb{R}^2$ parallel classes of lines (*points at infinity*)
Lines	lines of $\mathbb{R}^2$ the line at infinity
Incidence	inherited from $\mathbb{R}^2$; a point at infinity is incident with every line of the corresponding parallel class and with the line at infinity

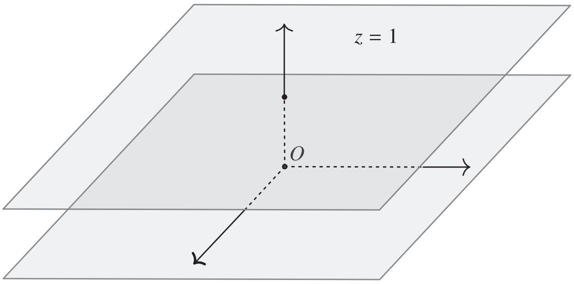

Figure 1.3 The Euclidean plane embedded in $\mathbb{R}^3$.

We have insinuated from the beginning that these models of incidence geometries are *the same*, and we have already said that they are models of *the* real projective plane. To make this mathematically correct, we define two incidence geometries (of points and lines) to be **isomorphic** if there is a bijection between their sets of points that respects their lines. In other words, there is a bijection ϕ mapping points of one incidence geometry onto the points of the other, such that if ℓ is a line of the first geometry, then the image of the points incident with ℓ (under ϕ) is precisely the set of points of a line of the second incidence geometry.

Theorem 1.4
The extended Euclidean plane and the real projective plane are isomorphic.

Proof Embed the Euclidean plane in $\mathbb{R}^3$ as the plane $z = 1$ (see Figure 1.3). Consider the projection via the origin O from a non-parallel plane π not on the origin to $z = 1$. The line ℓ that is the intersection of π and $z = 0$ will have no image and the line m that is the intersection of the plane parallel to π with $z = 1$ will have no preimage. The points of ℓ are in one-to-one correspondence with parallel classes of lines of $z = 1$: namely a point P corresponds to the parallel class of lines of $z = 1$ parallel to OP. Moreover, given a point Q of m, each

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Table 1.3 *From the real projective plane to the extended Euclidean plane.*

Real projective plane	Extended Euclidean plane
$\langle(a,b,c)\rangle$, $c \neq 0$	$\left(\frac{a}{c}, \frac{b}{c}, 1\right)$
$\langle(a,b,0)\rangle$	parallel class $\{bx - ay = c : c \in \mathbb{R}\}$ of the plane $z = 1$.

line ℓ on Q is the image of a line of π and these lines arising from Q form a parallel class in π. Moreover, the image of a line of π (other than ℓ) is a line of the plane $z = 1$.

Now, removing π from the picture, we have a one-to-one correspondence between the 1-dimensional subspaces of $\mathbb{R}^3$ of the points and parallel classes of the plane $z = 1$, and the 2-dimensional subspaces of $\mathbb{R}^3$ and the lines and the line at infinity of the plane $z = 1$ that preserves incidence. Therefore, the extended Euclidean plane and the real projective plane are isomorphic. It is worthwhile detailing this isomorphism in Table 1.3.

Composing this with the isomorphism $(a, b, 1) \mapsto (a, b)$ of the plane $z = 1$ with $\mathbb{R}^2$, we obtain:

- $\langle(x, y, z)\rangle, z \neq 0$, in the real projective plane corresponds to (X, Y) in $\mathbb{R}^2$ if and only if $X = \frac{x}{z}, Y = \frac{y}{z}$;
- $\langle(1, -m, 0)\rangle$ in the real projective plane corresponds to the parallel class of lines of slope m in $\mathbb{R}^2$;
- $\langle(0, 1, 0)\rangle$ in the real projective plane corresponds to the parallel class of vertical lines in $\mathbb{R}^2$. □

Moving between Cartesian (X, Y) coordinates and homogeneous (x, y, z) coordinates via the equations $X = x/z$, $Y = y/z$ is due to Hesse (1842). This was adopted by Cayley (1870) and generalised to n dimensions.

We say that a set of points is **collinear** if they are all incident with the same line. Likewise, a set of lines is **concurrent** if they are all incident with the same point. Using determinants, there is a simple test for when three points are collinear or three lines are concurrent.

Theorem 1.5
Three points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ *are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

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Proof $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ are collinear if and only if the matrix in the above determinant has rank two. □

Theorem 1.6
Three lines $[a_1, b_1, c_1], [a_2, b_2, c_2], [a_3, b_3, c_3]$ *are concurrent if and only if*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Proof $[a_1, b_1, c_1], [a_2, b_2, c_2], [a_3, b_3, c_3]$ are concurrent if and only if the matrix in the above determinant has nullity one. □

The **dual** of a statement about the plane is the statement that results after interchanging *point* and *line*, *collinear* and *concurrent*, *intersection* and *join* and making the necessary linguistic adjustments.

The Principle of Duality (Poncelet, 1822; Gergonne, 1825/6)
The dual of every theorem about $\mathsf{PG}(2, \mathbb{R})$ is also a theorem.

See also Poncelet (1995a, 1995b). For a proof of the *principle of duality*, note that the map taking a point (a, b, c) to a line $[a, b, c]$ preserves incidence. For an example of the principle of duality at play, note that Theorems 1.5 and 1.6 are dual.

1.5 Recap: The Real Projective Plane as Involving Points and Lines

When we think of the Euclidean space $\mathbb{R}^3$, we think of the incidence structure of points, lines, and planes. Now let us be more abstract and instead think of the incidence structure $\mathcal{J}$ whose 'points' are the parallel classes of lines in Euclidean space, and whose 'lines' are the parallel classes of planes in Euclidean space. The incidence relation would be derived from the natural incidence relation of class representatives.

new points	parallel classes of lines in Euclidean space
new lines	parallel classes of planes in Euclidean space

Now delete a parallel class Π of planes and all lines parallel to it to obtain a new incidence structure $\mathcal{A}$. Choose a plane π of Π, and a point P not on π.

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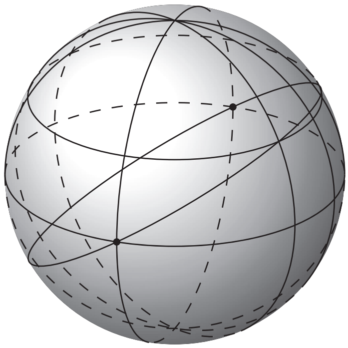

Figure 1.4 The real projective plane modelled on the sphere.

Consider the map ρ that takes a parallel class of lines to the intersection of its unique member on P with π, and a parallel class of planes to the intersection of its unique member on P with π. Since the parallel classes have parallel representatives if and only if the unique members on P are contained in one another, it follows that these parallel classes of lines and planes have parallel representatives precisely when the images under ρ are contained in one another. Therefore, $\mathcal{A}$ is isomorphic to the *affine plane* π. (The deleted objects thus naturally give the points at infinity of π as lines on P parallel to π and a line at infinity as the plane on P parallel to π.)

So we see here that the real projective plane can be realised as an extension of the incidence structure $\mathcal{A}$, and this incidence structure is just the parallel classes of lines and planes in Euclidean space, minus one class of planes.

Exercises

1.1 Consider the unit sphere S^2 in $\mathbb{R}^3$. Let $\mathcal{P}$ be the set of pairs of antipodal points of S^2, and let $\mathcal{L}$ be the set of great circles of S^2. Define incidence between elements of $\mathcal{P}$ and $\mathcal{L}$ by natural inclusion: a pair of antipodal points is incident with a great circle if both points lie on the great circle (compare with Figure 1.4). Show that this incidence geometry is isomorphic to $\mathsf{PG}(2,\mathbb{R})$.

1.2 Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two points of $\mathsf{PG}(2,\mathbb{R})$, written in homogeneous coordinates. Show that the unique line lying on these two points can be computed using the vector cross product.

1.3 A **quadrangle** is a set of four points, no three collinear, and a **quadrilateral** is the dual object of a quadrangle. What is a quadrilateral when expressed in terms of lines?

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