HAGEN, R., ROCH, S. and SILBERMANN, B. *Spectral theory of approximation methods for convolution equations* (Operator Theory: Advances and Applications, Vol. 74, Birkhäuser, Basel–Boston–Berlin, 1994), 392 pp., 3 7643 5112 8, (hardcover) £69.

Many problems in analysis reduce to solving an equation of the form $Au = f$, where u is an unknown function belonging to some Banach space X, f is a given function in some other Banach space Y and A is a continuous linear operator mapping X into Y. Such an equation will have a unique solution for each f in Y, given by $u = A^{-1}f$, precisely when A is invertible and then by a classical theorem of Banach A^{-1} is itself continuous. In practice, however, it is not an easy task to determine whether a concrete operator is invertible and, even if it is known to be invertible, to construct the inverse and thereby solve the equation.

Matters are at least superficially simplified a little if $X = Y$, because in that case the question has a more algebraic flavour: Is A invertible in the Banach algebra $L(X)$ of all bounded linear operators of X into itself? However, even here this is usually not an easy question to resolve. In some circumstances it is somewhat easier to tackle the question of *essential invertibility*. An operator A is said to be essentially invertible if it is invertible modulo the ideal $K(X)$ of compact operators on X, that is, if the coset $A + K(X)$ is invertible in the Calkin algebra $L(X)/K(X)$. In operator-theoretic terms this is equivalent to asking that A be a Fredholm operator. (A Fredholm operator is one with finite-dimensional kernel and finite-codimensional range.) Often the resolution of the question of essential invertibility involves a blend of Banach algebra techniques and classical analysis. Once some criterion for essential invertibility has been established, it is then sometimes possible to deduce genuine invertibility in more restricted circumstances.

By way of illustration consider an operator A on $L^2(\mathbb{R})$ of the form

$$Au(t) = a(t)u(t) + b(t)Su(t), \tag{1}$$

where a and b are continuous functions on the one-point compactification $\dot{\mathbb{R}}$ of $\mathbb{R}$ (that is, they have limits as $t \to \pm\infty$ and these limits are equal), and S is the singular integral operator

$$Su(t) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u(s)}{s-t}\,ds$$

(so that iS is the Hilbert transform). In this case A is essentially invertible if and only if the functions $a(t) \pm b(t)$ do not vanish on $\dot{\mathbb{R}}$, whilst A is actually invertible if and only if the winding number of the curve

$$t \to \frac{a(t)+b(t)}{a(t)-b(t)} \quad (t \in \dot{\mathbb{R}})$$

about the origin is zero.

Even when a criterion such as this for invertibility has been established, it is often not easy to calculate the inverse operator A^{-1} explicitly and approximation methods are needed instead to solve the original equation $Au = f$. A typical approximation scheme might be as follows. Suppose that $\{P_n\}$ and $\{R_n\}$ are sequences of finite-rank projections acting on X such that $P_n x \to x$ and $R_n x \to x$ for each $x \in X$ and that $\{A_n\}$ is a sequence of operators mapping $P_n X$ to $R_n X$ for each n. Consider the equations

$$A_n u_n = R_n f, \tag{2}$$

where the unknown u_n belongs to the range of P_n. If there is some n_0 such that these finite-dimensional equations have unique solutions for all $n \geq n_0$ and for all f and if the solutions converge to a solution of $Au = f$, then the scheme (2) applies to the operator A. The sequence $\{A_n\}$ usually has the property that $A_n P_n x \to Ax$ for all x and, if this is assumed along with the invertibility of A, the scheme applies to A precisely when the operators A_n are invertible for large enough n and the norms of the inverses are uniformly bounded.

The aim of the present book is to develop an abstract framework in which to study such approximation methods for various classes of convolution operators. In addition to operators of the form (1) but with the weaker assumption of piecewise continuity on the coefficient functions a and b, other operators related to singular integral operators are also studied such as Toeplitz and Wiener–Hopf operators, Hankel operators and Mellin convolutions. Singular integral operators associated with Lyapunov curves are also considered and as well as the classical Lebesgue L^p spaces of the real line certain weighted L^p spaces can also be treated by the authors' methods. Various different approximation methods are considered in the context of the more abstract theory and individual convergence criteria established for each.

The theory developed in the present book has its origins in work of I. B. Simonenko in the 1960s and A. Kozak in the 1970s, but their results applied to somewhat restricted classes of operators. However, a new approach was proposed by B. Silbermann (initially for a wider class of Toeplitz operators than it had previously been possible to handle) in 1981. Since then the subject has gone from strength to strength and the present work represents an up-to-date account of the current state of affairs, with many results appearing in print for the first time.

The book is clearly written and the authors have gone to great pains to present the material in as friendly a way as possible, so that long technical proofs are sometimes deferred in order not to interrupt the flow. There is also a useful introductory section on the necessary background from Banach algebra and operator theory. It has to be said, though, that the work is not for the faint-hearted! Someone new to the subject (the present reviewer classifies himself in this category) will find it tough going, but those who persevere will find much of interest.

T.A. GILLESPIE

PISIER, G., *Similarity problems and completely bounded maps* (Lecture Notes in Mathematics Vol. 1618, Springer-Verlag, Berlin–Heidelberg–New York–London–Paris–Tokyo–Hong Kong, 1996), vii + 156 pp., 3 540 60322 0, (softcover) £20.50.

This book is concerned with three similarity problems involving operators on Hilbert space H.

Q1. Given a uniformly bounded continuous representation of a locally compact group $\pi : G \to B(H)$, when can one find an invertible operator $S \in B(H)$ so that $g \mapsto S^{-1}\pi(g)S$ is a unitary representation?

Q2. Is every bounded unital algebra homomorphism u from a C^*-algebra A to $B(H)$ similar to a $*$-representation?

Q3. Let T be a polynomially bounded operator so that $\|p(T)\| \leq C\sup\{|p(z)| : |z| = 1\}$ for all polynomials p. Can one find an invertible S for which $\|S^{-1}TS\| \leq 1$?

These problems turn out to be related by the important general concepts of lacunarity, complete boundedness and Schur multipliers. The author introduces these ideas in turn, showing that several partial answers to these questions can be viewed in a unified framework. Each chapter reaches a main idea rapidly and discusses its background and significance in remarks at the end.

A classical result of Dixmier shows that Q1 always has a positive answer when G is a discrete amenable group. In Chapter 2 the author considers a combinatorial criterion for amenability due to Hulanicki and Kesten. He uses this to show that there are uniformly bounded representations of the free group F_∞ not similar to unitary representations. Switching to Q3, he then uses Fournier's algorithm to obtain $T \in B(H)$ with $\|T^n\| \leq C$ for $n \geq 1$ which is not polynomially bounded.

Completely bounded maps become involved in the theory as the answers to Q2 and Q3 are affirmative when the homomorphism involved is completely bounded. Paulsen showed that $T \in B(H)$ is similar to a contraction if and only if the maps $p \mapsto p(T) \in M_n \bar{\otimes} B(H)$ are bounded,

https://doi.org/10.1017/S0013091500023853 Published online by Cambridge University Press