SIN, COS, EXP AND LOG OF LIOUVILLE NUMBERS

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Abstract

For any Liouville number α, all of the following are transcendental numbers: e^{α}, $\log_e \alpha$, $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, $\sinh \alpha$, $\cosh \alpha$, $\tanh \alpha$, $\arcsin \alpha$ and the inverse functions evaluated at α of the listed trigonometric and hyperbolic functions, noting that wherever multiple values are involved, every such value is transcendental. This remains true if 'Liouville number' is replaced by 'U-number', where U is one of Mahler's classes of transcendental numbers.

1. Introduction

In 1844, Joseph Liouville proved the existence of transcendental numbers [1, 2]. He introduced the set $\mathcal{L}$ of real numbers, now known as Liouville numbers, and showed that they are all transcendental.

DEFINITION 1.1. A real number ξ is called a *Liouville number* if for every positive integer n, there exists a pair of integers (p, q) with $q > 1$, such that

$$0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Alan Baker in his classic work [2] on transcendental number theory said: 'A classification of the set of all transcendental numbers into three disjoint aggregates, termed S-, T-, and U-numbers was introduced by Mahler [8] in 1932, and it has proved to be of considerable value in the general development of the subject'.

In this paper, we demonstrate just how powerful and useful Mahler's classification of transcendental numbers is.

The following beautiful theorem, which is a corollary of the Lindemann–Weierstrass theorem, appears as Theorem 9.11 in Ivan Niven's book [12]. We prove the analogous result with 'algebraic number' replaced by 'Liouville number'.

THEOREM 1.2. *The following numbers are transcendental:*

(i) e^{α}, $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, $\sinh \alpha$, $\cosh \alpha$, $\tanh \alpha$;

(ii) $\log_e \alpha$, $\arcsin \alpha$ *and, in general, the inverse functions of all those listed in part (i),*

for any nonzero algebraic number α; wherever multiple values are involved, every such value is transcendental.

It is not widely known that e^{α} and $\log_e \alpha$ are transcendental numbers when α is a Liouville number, though the exp case is stated explicitly in [6, page 98] and the log case, as pointed out to the second author by Michel Waldschmidt, is implicit in [3, Section 3.5]. The proof in our paper for exp is different from that in [6].

However, the results for the trigonometric and hyperbolic functions do not appear in print. The proofs of all these results for Liouville numbers, and indeed a wider class of numbers, depend on properties of the Mahler classes of transcendental numbers.

REMARK 1.3. One might think that the results for trigonometric and hyperbolic functions might follow trivially from the result for exp since, for example, $\cosh(x) = \frac{1}{2}(\exp(x) + \exp(-x))$. However, the sum of two transcendental numbers is not necessarily transcendental. Indeed, in 1962, Erdős [5] proved that *every* real number is a sum of two Liouville numbers and it is proved in [4] that there are $2^{\mathfrak{c}}$ subsets W of the Liouville numbers such that every real number is the sum of two numbers in W.

2. Mahler classes

We follow the presentation in [3, Section 3]. While the definitions and results therein are stated and proved for real numbers, *mutatis mutandis*, they carry over to the case of complex numbers. Mahler's classification partitions the set $\mathbb{C}$ of all complex numbers into four sets (the fourth set in fact turns out to be the set of all algebraic numbers), characterised by the rate with which a nonzero polynomial with integer coefficients approaches zero when evaluated at a particular number.

Given a polynomial $P(X) \in \mathbb{C}[X]$, recall that the height of P, denoted by $H(P)$, is the maximum of the absolute values of the coefficients of P. Given a complex number ξ, a positive integer n and a real number $H \geq 1$, we define the quantity

$$w_n(\xi, H) = \min\{|P(\xi)| : P(X) \in \mathbb{Z}[X], H(P) \leq H, \deg(P) \leq n, P(\xi) \neq 0\}.$$

Furthermore, we set

$$w_n(\xi) = \limsup_{H \to \infty} \frac{-\log w_n(\xi, H)}{\log H}$$

and

$$w(\xi) = \limsup_{n \to \infty} \frac{w_n(\xi)}{n}.$$

With this notation in mind, Mahler partitions the complex numbers into four sets.

DEFINITION 2.1. Let ξ be a complex number. The number ξ is:

- an A-number if $w(\xi) = 0$;
- an S-number if $0 < w(\xi) < \infty$;
- a T-number if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any $n \geq 1$;
- a U-number if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for all $n \geq n_0$, for some positive integer n_0.

REMARK 2.2. Note that the A-numbers are the algebraic numbers.

The following theorem of Mahler records a fundamental property of the Mahler classes.

THEOREM 2.3 (see [3, Theorem 3.2]). *If $\xi, \eta \in \mathbb{C}$ are algebraically dependent, then they belong to the same Mahler class.*

The following theorem of Mahler is key to our main result.

THEOREM 2.4 ([8, 9]; see also [3, Section 3.5]). *If a is an algebraic number with $a \neq 0$, then $\exp(a) \in S$ and if $a \neq 0, 1$, then $\log(a) \in S \cup T$.*

THEOREM 2.5 [10]. *The number $\pi \in S \cup T$.*

REMARK 2.6. Note that the Liouville numbers are U-numbers. Furthermore, if ξ is a Liouville number, then $i\xi$ is a U-number by Theorem 2.3.

3. The main theorem

THEOREM 3.1. *For any U-number α, in particular for α any Liouville number, all of the following are transcendental numbers: e^α, $\log_e \alpha$, $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, $\sinh \alpha$, $\cosh \alpha$, $\tanh \alpha$ and the inverse functions evaluated at α of the listed trigonometric and hyperbolic functions, noting that wherever multiple values are involved, every such value is transcendental.*

PROOF. For ease of notation, we adopt the conventional notation that $\log$ denotes $\log_e$. We shall demonstrate the result for e^α, $\sin \alpha$, $\tan \alpha$, $\sinh \alpha$ and their inverses. The corresponding proofs for the remaining functions in each family are analogous.

Henceforth, let α be a U-number.

(1) $\exp \alpha$. Suppose that $\mu = e^\alpha$ is an algebraic number. Then, $\log \mu = \alpha \in S \cup T$ by Theorem 2.4. This is a contradiction since $\alpha \in U$. So e^α is a transcendental number.

(2) $\log \alpha$. Suppose μ is one of the values of $\log \alpha$ and is an algebraic number. Then $e^\mu = \alpha \in S$ by Theorem 2.4. This is a contradiction since $\alpha \in U$. So all values of $\log \alpha$ are transcendental.

(3) $\sin \alpha$. Suppose $\mu = \sin \alpha$ is an algebraic number. Then by Remark 2.6 and Theorem 2.4, $i\alpha \in U$. By item (1) proved above, $t = e^{i\alpha}$ is transcendental. Further, $2i \sin \alpha = e^{i\alpha} - e^{-i\alpha} = t - 1/t = 2i\mu = \beta$, where, by our supposition, β is an algebraic number. Now $t - 1/t = \beta$ implies that $t = \frac{1}{2}(\beta \pm \sqrt{4 - \beta^2})$. Since β is an algebraic

number, the right-hand side of the preceding equation is also an algebraic number, and this is a contradiction since t is a transcendental number. So $\sin \alpha$ is transcendental.

(4) $\arcsin \alpha$. Suppose first that one of the values of $\arcsin \alpha = \mu$ is 0. Then $\alpha = k\pi$, for some $k \in \mathbb{Z}$. If $k = 0$, then $\alpha = 0$ which contradicts the fact that $\alpha \in U$. If $\alpha = k\pi$, $k \neq 0$, then $k\pi \in S \cup T$ by Theorem 2.5, which contradicts the fact that $\alpha \in U$.

Next suppose that one of the values of $\arcsin \alpha = \mu$ is an algebraic number. Recall that $\arcsin \alpha = -i \log(i\alpha + \sqrt{1 - \alpha^2})$. Now, $i\mu = \log(i\alpha + \sqrt{1 - \alpha^2})$ implies $e^{i\mu} = \sqrt{1 - \alpha^2} + i\alpha$. By Theorem 2.4, $e^{i\mu} \in S$. However, $X = \alpha$ and $Y = \sqrt{1 - \alpha^2} + i\alpha$ satisfy the equation $P(X, Y) = Y^4 + 4Y^2X^2 - 2Y^2 + 1 = 0$ and hence X and Y are algebraically dependent. So by Theorem 2.3, X and Y are in the same Mahler class. As we are given $X \in U$, this implies $e^{i\mu} = Y \in U$ which is a contradiction as we saw that $e^{i\mu} \in S$. Thus, all values of $\arcsin \alpha$ are transcendental.

(5) $\tan \alpha$. Suppose that $\mu = \tan \alpha$ is algebraic. Then this implies that $i\mu = (t - 1/t)/(t + 1/t)$ is algebraic, where $t = e^{i\alpha}$, which is transcendental by Theorem 2.4. The former equation implies that $t^3 + t + i\beta t - i\beta = 0$, where $\beta = 1/\mu$. In the interest of brevity, we omit exhibiting the general solution for t, but we note that the polynomial has algebraic coefficients, and hence, each solution t is also algebraic, which is a contradiction.

(6) $\arctan \alpha$. Put $\mu = \arctan \alpha$, so $2i\mu = \log((i - \alpha)/(i + \alpha))$. As in item (4), $\mu \neq 0$. Suppose μ is an algebraic number. Then $e^{2i\mu} \in S$ by Theorem 2.4. Now $X = \alpha$ and $Y = (i - \alpha)/(i + \alpha)$ satisfy the equation $P(X, Y) = X^2Y^2 + 2X^2Y + X^2 - Y^2 + 2Y - 1 = 0$, and hence are algebraically dependent. Theorem 2.3 then implies that $(i - \alpha)/(i + \alpha) \in U$, which is a contradiction. So every value of $\arctan \alpha$ is transcendental.

(7) $\sinh \alpha$. Recall that $\sinh \alpha = \frac{1}{2}(e^\alpha - e^{-\alpha})$. By item (1), $t = e^\alpha$ is transcendental. Suppose $t - 1/t = 2\mu$ is an algebraic number. This implies that $t = \mu \pm \sqrt{1 - \mu^2}$ is also algebraic, which is a contradiction. So $\sinh \alpha$ is transcendental.

(8) $\text{arcsinh} \, \alpha$. We proceed as in item (4). Suppose $\mu = \text{arcsinh} \, \alpha = \log(\alpha + \sqrt{\alpha^2 + 1})$ is algebraic. By Theorem 2.4, $e^\mu \in S$. However, $X = \alpha$ and $Y = \alpha + \sqrt{1 + \alpha^2}$ satisfy the equation $P(X, Y) = Y^2 - 2XY - 1 = 0$. Hence, X and Y are algebraically dependent and therefore $\alpha + \sqrt{1 + \alpha^2} \in U$, which is a contradiction. So $\text{arcsinh} \, \alpha$ is transcendental. $\qquad \square$

REMARK 3.2. In fact, the above argument shows that if α is in the Mahler class T, then $\log \alpha$ is a transcendental number. Additionally, the theorem remains true for the composition of a trigonometric or hyperbolic function with the inverses of the other functions in the corresponding family. For instance, if α is a Liouville number, then $\sinh(\text{arccosh} \, \alpha)$ is transcendental.

REMARK 3.3. Noting Theorem 2.4, we see that in contrast with Theorem 3.1 and Remark 3.2, it is not true that $\log s$ is a transcendental number for all members s of the Mahler set S. However, it is of course true for all but a countably infinite number of $s \in S$, as the set of algebraic numbers is countably infinite.

REMARK 3.4. We conclude by recording that Corollary 6 of [7] implies that $\exp \alpha$ is a Liouville number for an uncountable number of Liouville numbers α. Recall Maillet's result, [11, Ch. 3], which says that if t is a Liouville number and $R(x)$ is a rational function with rational coefficients, then $R(t)$ is a Liouville number. In our case, we use $R(t) = \frac{1}{2}(t + 1/t)$. It follows from this and [7] that there exists an uncountable set of Liouville numbers α such that $\sinh \alpha$ is a Liouville number and there exists an uncountable set of Liouville numbers α such that $\cosh \alpha$ is a Liouville number.

OPEN QUESTION 3.5. If α is a Liouville number, is $\exp \alpha$ necessarily a Liouville number or a member of the Mahler set U?

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