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On the Adjoint and the Closure of the Sum of Two Unbounded Operators

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Abstract. We prove, under some conditions on the domains, that the adjoint of the sum of two unbounded operators is the sum of their adjoints in both Hilbert and Banach space settings. A similar result about the closure of operators is also proved. Some interesting consequences and examples "spice up" the paper.

1 Introduction

Notations and Definitions Before we start we recall some known notions and results about unbounded operators. If A and B are two unbounded operators with domains $D(A)$ and $D(B)$ respectively, then B is said to be an extension of A, and we denote it by $A \subset B$, if $D(A) \subset D(B)$ and A and B coincide on each element of $D(A)$. An operator A is said to be densely defined if $D(A)$ is dense in the whole Hilbert space. The Hilbert adjoint of A is denoted by A^* and is known to be unique if A is densely defined.

An operator A is said to be closed if its graph is closed. It is called closable if it has a closed extension, and the smallest closed extension of it is called its closure and is denoted by $\overline{A}$ (a standard result is that A is closable if and only if T^* has a dense domain, in which case $\overline{A} = A^{**}$). One has A closed if and only if $A = \overline{A}$. An easy property that will be used in Section 3 is the following: if B is closable, then $A \subset B \Rightarrow \overline{A} \subset \overline{B}$. The operator A is said to be symmetric if $A \subset A^*$ and self-adjoint if $A = A^*$.

In this paper we will use unbounded operators defined on a Banach space, and we will need the adjoint of an operator. We will call it the Banach adjoint and denote it by A' (some authors call this the conjugate).

For more material background about unbounded operators in Hilbert and Banach spaces the reader may consult [9, 14, 17, 24].

When working with bounded operators (say on a Hilbert space), probably the most frequently used facts about adjoints are the following:

$$(AB)^* = B^*A^* \tag{1.1}$$

$$(A+B)^* = A^* + B^*. \tag{1.2}$$

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Their proofs are quite elementary. However, if one considers two unbounded operators, then these two equations need not hold unless some extra conditions are imposed. In general, with the assumptions that A, B and AB are all densely defined, one only has $B^*A^* \subset (AB)^*$ (of course AB is defined on the domain $D(AB) = \{\varphi \in D(B) : B\varphi \in D(A)\}$). One does have equality if A is bounded.

If $A + B$ is densely defined, then one only has $A^* + B^* \subset (A + B)^*$, and if either A or B is bounded (and the other densely defined), then "$\subset$" becomes "$=$".

Many authors have studied when equality (1.1) holds for unbounded operators. We summarize some known results in the literature. J. Von Neumann (see [14]) proved that if A is closed, then AA^* is self-adjoint, which means that equation (1.1) is true for the particular case $B = A^*$.

J. Diximer [7] defined a new product, which he denoted by "$\cdot$" and for which $(AB)^* = B^* \cdot A^*$ for closed A and B. If A and B are not closed, then only $(A \cdot B)^* = B^*A^*$ holds.

In late nineteeen-sixties and early nineteen-seventies several papers were published about equality (1.1). They all imposed some conditions in order to establish the equation for both Hilbert and Banach adjoints (see [3, 4, 10, 13, 26]).

In 1980, Albeverio, Høegh-Krohn, and Streit [1] also proved it and more recently K. Gustafson [11] simplified and generalized their result. It is worth noticing that the author has a paper about the closedness of the product of two closed operators. See [22]. See also [18].

The sum of two unbounded operators has many applications in physics. See *e.g.,* [25] (we digress a little bit to say that some authors have worked on the sum of two operators with non-dense domains, see *e.g.,* [16]). In this paper, we are interested in when equation (1.2) holds for two unbounded operators. Apart from being important in physics, this constitutes in itself an interesting mathematical question that has motivated us to try to prove yet another version of it.

When dealing with unbounded operators one has to be very careful with some manipulations especially with the domains. To illustrate some of these deficiencies consider the sum $A + B$ of two unbounded operators with the natural domain $D(A + B) = D(A) \cap D(B)$. This is in fact the real problem with the sum. It is quite likely that $D(A) \cap D(B)$ reduces to $\{0\}$. There are known examples of this particular question. It started with J. von Neumann [23] who proved that if A is an unbounded self-adjoint operator, then there exists a unitary operator U for which $D(A) \cap D(UAU^*) = \{0\}$. K. Schmüdgen [27] gave, in the spirit of von Neumann's paper (see [23]), operators with this property. T. Diagana [5,6], also gave an example of a potential Q (under some conditions) for which

$$D(-\triangle + Q) = D(-\triangle) \cap D(Q) = \{0\},$$

where $-\triangle$ denotes the time-independent Schrödinger operator on $L^2(\mathbb{R}^n)$.

Recently, H. Kosaki [15] gave many examples of couples of unbounded self-adjoint (and positive) operators having trivial intersections of domains. We give one of them. Consider A as the multiplication operator by the real valued-function $x \mapsto e^{x^2/2}$ on its domain $D(A) = \{f \in L^2(\mathbb{R}) : e^{x^2/2}f \in L^2(\mathbb{R})\}$ (relative to the Lebesgue measure). Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R})$. Now take $B = \mathcal{F}^{-1}A\mathcal{F}$.

Then $D(A) \cap D(B) = \{0\}$ (this example can be exploited further to work as a counter-example to equation (1.2) by taking A^* to be the operator A just mentioned and simlarly for B^* and B).

Equality (1.2) becomes tremendously important for self-adjoint operators. An important criterion that is used to establish it is the Kato–Rellich theorem (see *e.g.,* [25]). It states that if A is a self-adjoint operators with domain $D(A)$ and if B is self-adjoint (symmetric is enough in the original theorem) and A-bounded, *i.e.,* for some positive $a < 1$ (the infimum of all positive a is called the relative bound) and all positive b

$$\|Bf\| \leq a\|Af\| + b\|f\|$$

for all $f \in D(A)$, then $A + B$ is self-adjoint, *i.e.,*

$$(A + B)^* = A + B \quad (= A^* + B^*).$$

R. W. Beals [2] showed that if A is Fredholm, B is A-compact, and B' is A'-compact, then $(A + B)' = A' + B'$.

A similar result to that of Beals [2], with different hypotheses, was proved by Hess and Kato and appeared in [12]. They proved that if B (respectively B') is A-bounded (respectively A'-bounded) with both relative bounds smaller than one, then $(A + B)' = A' + B'$.

We will prove some closedness results in the last section, so we recall briefly some known results. B. Sz.-Nagy [28] showed the following.

Theorem 1.1 *If A is closed and B is A-bounded with relative bound $a < 1$, then $A+B$ is closed.*

A standard "$A = -B$" trick shows that the result is no longer true for $a = 1$. However, in applications dealing with partial differential operators then one can make the constant a as small as one wishes via a scaling argument almost every time, and hence $A + B$ will be closed (if A is the partial differential operator in question). See [25] for perturbed time-independent Schrödinger operator $-\triangle$, [20] for perturbed wave operator $\square$ and [21] for perturbed time-dependent Schrödinger operator $-i\frac{\partial}{\partial t} - \triangle_x$ (these references are actually about the self-adjointness of those perturbed operators, but we can show their closedness in the same way and by using Theorem 1.1).

There is also a paper by Dore and Venni [8] that treats the question of closedness of the sum of two closed operators on Banach spaces with applications to differential equations.

Finally, we summarize what will be done in this paper. We will prove equality (1.2) for two unbounded operators with hypotheses about the domains only. This will be Theorem 2.2 about Hilbert adjoints. Then we derive from it some direct consequences about the self-adjointness and the closedness of sums of operators. Then, we give the analog of Theorem 2.2 for Banach adjoints. In the end we prove a similar result about the closure of two operators.

2 Adjoint of the Sum

2.1 The Hilbert Space Version

In general, the operator $A + A^*$ (for a given unbounded densely defined operator A) need not be self-adjoint even if A is closed (unlike AA^*) as showed by the following example.[1] Consider the *closed* operator A defined by

$$Af(x) = xf'(x) \text{ with domain } D(A) = \{f \in L^2(\mathbb{R}) :\ xf' \in L^2(\mathbb{R})\},$$

where the derivative is a weak one. Then

$$A^* f(x) = -xf'(x) - f(x) \text{ with } D(A^*) = \{f \in L^2(\mathbb{R}) :\ xf' \in L^2(\mathbb{R})\}.$$

A similar operator (in fact $-i(|x|f)'$) appeared in [19], in a different setting, where the reader may find more details. Then

$$D(A) = D(A^*) \text{ and } (A + A^*)f(x) = -f(x) \text{ defined on } D(A) \cap D(A^*) = D(A).$$

Thus $A + A^*$ is not self-adjoint on $D(A)$, since real multiples of the identity are self-adjoint on the whole Hilbert space and not on a proper subset (*i.e.*, $D(A)$) of it. We observe that $D[(A + A^*)^*] = L^2(\mathbb{R}) \not\subset D(A^{**}) = D(\overline{A}) = D(A)$. We are led to the following.

Proposition 2.1 *Let A be an unbounded densely defined and closed operator satisfying $D(A) \subset D(A^*)$ and $D[(A + A^*)^*] \subset D(A)$. Then*

$$(A + A^*)^* = A + A^* \text{ , i.e., } A + A^* \text{ is self-adjoint.}$$

Proof First, observe that $A + A^*$ is densely defined as $D(A) \subset D(A^*)$. Now, since A is closed, we know that

$$A^{**} + A^* = \overline{A} + A^* = A + A^* \subset (A + A^*)^*.$$

But since by assumption one has

$$D[(A + A^*)^*] \subset D(A) \text{ and } D(A) \subset D(A^*),$$

then we easily get

$$(A + A^*)^* = A^* + A^{**} = A^* + \overline{A} = A^* + A = A + A^*. \qquad \blacksquare$$

The following theorem generalizes the previous proposition to general densely defined A and B.

Theorem 2.2 *Assume that A and B are densely defined. If*

$$D(A) \subset D(B) \quad \textit{and} \quad D[(A + B)^*] \subset D(B^*),$$

then $(A + B)^* = A^* + B^*$.

[1] In fact, for closed A, $A + A^*$ is "only" symmetric.

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Proof That $A+B$ is densely defined follows from the assumption $D(A) \subset D(B)$.

Now, we only need show that $(A+B)^* \subset A^* + B^*$. Obviously $(A+B) - B \subset A$. But by hypothesis we have $D(A) \subset D(B)$ and hence $D(A) \subset D(A) \cap D(B)$. This yields $A = (A+B) - B$. Taking the adjoint gives us

$$(A+B)^* - B^* \subset [(A+B) - B]^* = A^*,$$

which gives

$$D[(A+B)^*] \cap D(B^*) \subset D(A^*),$$

and since $D[(A+B)^*] \subset D(B^*)$, then $D[(A+B)^*] \subset D(A^*)$. Thus

$$D[(A+B)^*] \subset D(A^*) \cap D(B^*).$$

This completes the proof. ∎

Remark The condition $D[(A+B)^*] \subset D(B^*)$ cannot merely be dropped. Let A be the unbounded self-adjoint operator defined by $Af(x) = xf(x)$ on its domain $D(A) = \{f \in L^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\}$. Now take $A = -B$, then $D(A) = D(B)$ and

$$D[(A+B)^*] = L^2(\mathbb{R}) \not\subset D(B^*) \; (= D(B)).$$

Hence one sees that only $A^* + B^* \subset (A+B)^*$ holds.

Theorem 2.2 generalizes the known version of equality (1.2) for bounded B and densely defined A (see *e.g.,* [17]). We have the following.

Corollary 2.3 *If A is densely defined and B is bounded on the whole Hilbert space $\mathcal{H}$, then $(A+B)^* = A^* + B^*$.*

Proof The proof is obvious, as $D(B) = D(B^*) = \mathcal{H}$. ∎

Corollary 2.4 *Let A and B be two self-adjoint operators with domains $D(A)$ and $D(B)$, respectively. If $D(B)$ contains both $D(A)$ and $D[(A+B)^*]$, then $A+B$ is self-adjoint.*

Corollary 2.5 *Let A and B be two densely defined operators such that $D(A) \subset D(B)$. If B is symmetric and $D[(A+B)^*] = D(A+B)$, then $(A+B)^* = A^* + B^*$.*

Proof Since B is symmetric and using the other hypotheses one has

$$D[(A+B)^*] = D(A+B) = D(A) \cap D(B) \subset D(B) \subset D(B^*),$$

and since $D(A) \subset D(B)$, then Theorem 2.2 applies. ∎

We can derive another corollary from Theorem 2.2 concerning the closedness of the sum of two operators.

Proposition 2.6 *Let A, B be densely defined operators such that A^* and B^* are also densely defined. Let $D(A^*) \subset D(B^*)$ and $D[(A^* + B^*)^*] \subset D(B)$. If A and B are closed, then so is $A+B$.*

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Proof First, the operator $A^* + B^*$ is densely defined, since $D(A^*) \subset D(B^*)$.

Now, using $D(A^*) \subset D(B^*)$ and $D[(A^* + B^*)^*] \subset D(B) = D(\overline{B}) = D[(B^*)^*]$ (as B is closed), Theorem 2.2 yields

$$(A^* + B^*)^* = A^{**} + B^{**} = \overline{A} + \overline{B} = A + B$$

since A and B are both closed. But the adjoint of a densely defined operator is always closed and hence the theorem is proved. ■

2.2 The Banach Space Version

Now we give the analog of Theorem 2.2 for unbounded operators in a Banach space.

Theorem 2.7 *Assume that A and B are densely defined. If* $D(A) \subset D(B)$ *and* $D[(A + B)'] \subset D(B')$*, then* $(A + B)' = A' + B'$.

Proof The proof is quite similar to that of Theorem 2.2. It is already known that $A' + B' \subset (A + B)'$, and the method of showing $(A + B)' \subset A' + B'$ is exactly the same as in Theorem 2.2. ■

3 Closure of the Sum

The purpose of this section is to prove a result similar to that of equation (1.2) for the closure of two operators. The idea is somewhat the same as in Theorem 2.2, but we need the following lemma whose proof is standard and can be found in [14].

Lemma 3.1 *Let A and B be two closable operators. Let A be a B-bounded operator. Then* $\overline{A}$ *is also* $\overline{B}$*-bounded and* $\overline{A} + \overline{B} \subset \overline{A + B}$.

The sum of two closed operators need not be closed. To show that we take again the example in the remark just below Theorem 2.2. Let A be defined by $Af(x) = xf(x)$ on $D(A) = \{f \in L^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\}$. Then A is closed on this domain. Now take $A = -B$, then B is also closed. Hence

$$A + B = A - A = \mathbf{0} \text{ on } D(\mathbf{0}) = D(A) \cap D(B) = \{f \in L^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\},$$

and the operator $\mathbf{0}$ is not closed on this domain. However, one has the following.

Theorem 3.2 *Let A and B be two closable operators. Let A be B-bounded with relative bound "a" smaller than one. Then,* $\overline{A + B} = \overline{A} + \overline{B}$.

Remark The referee pointed out that this result is known [29, Thm. 5.5, p. 93]. However, the proof we give here is different Weidmann's.

Proof By virtue of the previous lemma we need only show that $\overline{A + B} \subset \overline{A} + \overline{B}$. The way of finishing the proof is similar to that of Theorem 2.2, but some steps require a little more care. One always has $A + B - B \subset A$ and hence $\overline{A + B - B} \subset \overline{A}$.

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We always have for any φ

$$| \, \|B\varphi\| - \|A\varphi\| \, | \leq \|(A+B)\varphi\|,$$

which implies

$$\|B\varphi\| \leq \|(A+B)\varphi\| + \|A\varphi\|.$$

Since A is B-bounded with relative bound $a < 1$ (and let b be the other constant), then a simple computation gives us

$$\|B\varphi\| \leq \frac{1}{1-a}\|(A+B)\varphi\| + \frac{b}{1-a}\|\varphi\|,$$

i.e., B is $(A+B)$-bounded. Lemma 3.1 guarantees that $\overline{B}$ is $(\overline{A+B})$-bounded and it also yields

$$\overline{A+B} - \overline{B} \subset \overline{A+B-B}\,(\subset \overline{A}).$$

The $(\overline{A+B})$-boundedness of $\overline{B}$ also gives us $D(\overline{A+B}) \subset D(\overline{B})$. Therefore $\overline{A+B} \subset \overline{A} + \overline{B}$. Thus $\overline{A+B} = \overline{A} + \overline{B}$, establishing the result. ■

Remark The usual "$A = -B$" trick shows that the result is no longer true for $a = 1$.

We get back a known result about the closedness of the sum of two operators.

Corollary 3.3 *Suppose that two closed operators A and B satisfy the hypotheses of the previous theorem, then A + B is closed.*

We can also recover the known version of Theorem 3.2 for bounded A and closed B. We have the following.

Corollary 3.4 *Assume that B is closed and that A is bounded on a Hilbert space $\mathcal{H}$. Then A + B is closed.*

Proof Since A is bounded, then it is B-bounded (with any relative bound especially one which is smaller than one). ■

Acknowledgment We finally say that what has motivated us to do this work is a paper by K. Gustafson [11], which was about the product of two unbounded operators.

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