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A GENERAL CRITERION FOR THE EXISTENCE OF INFINITE SIDON SETS

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Abstract

Let G be any compact group, connected or disconnected, with dual object $\hat{G}$. We define a family of local Sidon subsets of $\hat{G}$ in terms of allowable images of the representations. Using this family we develop a straightforward criterion whereby the existence of infinite Sidon subsets of $\hat{G}$ may be decided.

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1. Introduction

It is well known [10] that a compact group which admits infinitely many continuous irreducible representations of the same degree (that is, a non-tall compact group) necessarily admits an infinite Sidon subset of its dual object. For connected compact groups Cartwright and McMullen [2] obtained a structural criterion for the existence of infinite Sidon sets. In the course of their exposition, they introduced a special Sidon set on a connected product group which, after [6], they christened a Figà-Talamanca-Rider (FTR) set. By recasting the membership criterion in terms of allowable images of representations we are able to define a family of FTR sets for any compact group, connected or not, and these are always at least local Sidon sets. Using these higher order FTR sets we are able (see §9 below) to extend the analysis of Cartwright and McMullen to yield a general criterion to determine whether or not an arbitrary compact group admits an infinite Sidon set.

The jots and tittles omitted from this account may be found in full in [16]. I wish to express my particular gratitude to J. R. McMullen for encouragement in pursuing this subject.

2. Notation and terminology

The dual object of a compact group G will be taken to be a maximal set of pairwise inequivalent continuous unitary irreducible representations of G, and will be denoted $\hat{G}$. For $\sigma \in \hat{G}$ we denote by $d(\sigma)$ its degree, and if σ is an irreducible component of the continuous unitary representation ρ of G we write $\sigma \le \rho$ and denote by $m(\sigma, \rho)$ the multiplicity of σ in ρ. For $P \subseteq \hat{G}$ we denote by $\mathcal{T}_P(G)$ the space of trigonometric polynomials f whose Fourier transforms $\hat{f}$ are supported by P. The norm $\|A\|_{\phi_1} = \operatorname{tr}|A|$ on the space of bounded operators on a finite-dimensional hilbert space is described in [8, Appendix D]. A subset $P \subseteq \hat{G}$ is said to be *Sidon* if it enjoys the remarkable property that every $f \in \mathcal{T}_P(G)$ has an absolutely convergent Fourier series; equivalently, P is Sidon if its Sidon constant

$$\kappa(P) = \sup\left\{ \|\hat{f}\|_1 = \sum_{\sigma \in P} d(\sigma)\|\hat{f}(\sigma)\|_{\phi_1} : f \in \mathcal{T}_P(G), \|f\|_\infty \le 1 \right\}$$

is finite. $P \subseteq \hat{G}$ is *local Sidon* if $\kappa_0(P) = \sup\{\kappa(\sigma) = \kappa(\{\sigma\}) : \sigma \in P\}$ is finite. A compact group G is defined to be *tall* if it admits only finitely many inequivalent representations of each (finite) degree.

We denote by $M_n(\mathbf{C})$ the space of $n \times n$ complex matrices; by $U(n) \subseteq M_n(\mathbf{C})$ the group of unitary elements; by $SU(n) \subseteq U(n)$ the subgroup of matrices of determinant 1; by $O(n) \subseteq U(n)$ the subgroup of real orthogonal matrices; by $SO(n) \subseteq O(n)$ the subgroup of matrices of determinant 1, and by $\mathrm{Spin}(n)$ its simply-connected covering group; and by $\mathrm{Sp}(n) \subseteq U(n)$ the group of symplectic matrices (n even). The circle group is denoted $\mathbf{T}$.

If G is a compact group and H is a closed subgroup of G we write $H \le G$; if H is also normal we write $H \trianglelefteq G$. If $G \le U(n)$ we denote by $N_U(G)$ the unitary normalizer of G. If G is a compact connected almost simple Lie group, we will frequently identify $\sigma \in \hat{G}$ with its highest weight; the ordering of the fundamental weights is that of [15]. The positive integers are denoted $\mathbf{N}$.

3. FTR sets

In this and the succeeding section we define the Figà-Talamanca-Rider sets associated with a compact group, and outline some of their salient properties. These sets are the warp on which our existence criterion for Sidon sets is embroidered.

(3.1) Definition. Let $n \in \mathbf{N}$, $n \geq 2$. For $2 \leq n \leq 5$ put $\mathcal{S}_n = \{SU(n)\}$, and put $\mathcal{S}_6 = \{SU(6), \mathrm{Sp}(6)\}$. For $n \geq 7$, n odd, put $\mathcal{S}_n = \{SU(n), SO(n)\}$. For $n \geq 8$, n even, put $\mathcal{S}_n = \{SU(n), \mathrm{Sp}(n), SO(n)\}$. Let $\mathcal{N}_n = \{N \leq U(n)\colon N$ is tall, and for some $S \in \mathcal{S}_n$, $S \trianglelefteq N\}$.

Let G be a compact group: the *Figà-Talamanca-Rider set* of G is

$$\mathrm{FTR}(G) = \{\sigma \in \hat{G}\colon d(\sigma) > 1,\ G/\ker\sigma \simeq N \in \mathcal{N}_{d(\sigma)}\}.$$

It is clear from the definition that $\mathrm{FTR}(G)$ is stable under the action of $\mathrm{Aut}(G)$ on $\hat{G}$.

It is not clear whether the trivial representation should also be included in $\mathrm{FTR}(G)$. Some theorems (for example [2, 5.5]) seem to require $1 \in \mathrm{FTR}(G)$, whereas others (for example, [17, 4.4]) suggest its exclusion. Since estimates for the Sidon constant of $\mathrm{FTR}(G)$ are more conveniently expressed if $1 \notin \mathrm{FTR}(G)$, we shall opt for the exclusion of 1.

With a measure of tedium an explicit description of the $\mathcal{N}_n$ is possible:

$$\begin{aligned}
\mathcal{N}_n &= \{\mathbf{Z}_{mn}SU(n)\colon m \geq 1\}, && 2 \leq n \leq 5,\\
&= \{\mathbf{Z}_{mn}SU(n), \mathbf{Z}_{2m}\mathrm{Sp}(n)\colon m \geq 1\}, && n = 6,\\
&= \{\mathbf{Z}_{mn}SU(n), \mathbf{Z}_{m}SO(n)\colon m \geq 1\}, && n \geq 7, n \text{ odd},\\
&= \{\mathbf{Z}_{mn}SU(n), \mathbf{Z}_{2m}\mathrm{Sp}(n), \mathbf{Z}_{2m}SO(n), \mathbf{Z}_{2m}O(n), \mathbf{W}_{2m}(n)SO(n)\colon m \geq 1\},\\
& && n \geq 8, n \text{ even},
\end{aligned}$$

where $\mathbf{Z}_p = \{e^{2j\pi i/p}\colon 0 \leq j < p\}$, and writing $J_n = \mathrm{diag}(-1, 1, \ldots, 1) \in O(n)$, $\mathbf{W}_p(n) = \{(e^{\pi i/p}J_n)^j\colon 0 \leq j < 2p\} \simeq \mathbf{Z}_{2p}$.

(3.2) We now demonstrate that our FTR sets generalize those of Cartwright and McMullen. Direct calculation using [15] yields

Lemma 1.

$$\begin{aligned}
\mathrm{FTR}(SU(n)) &= \{\lambda_1, \lambda_{n-1}\}, && n \geq 2,\\
\mathrm{FTR}(\mathrm{Sp}(n)) &= \{\lambda_1\}, && n \geq 6,\ n \textit{ even},\\
\mathrm{FTR}(SO(n)) &= \{\lambda_1\}, && n \geq 7,\\
\mathrm{FTR}(\mathrm{Spin}(n)) &= \{\lambda_1\}, && n = 7, n \geq 9,\\
\mathrm{FTR}(\mathrm{Spin}(8)) &= \{\lambda_1, \lambda_3, \lambda_4\}.
\end{aligned}$$

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Estimates for the local Sidon constants of the above sets are given in [2, 4.5.1, 4.5.2], as well as for the Sidon constant of all except FTR(Spin(8)); the estimate $\kappa(\mathrm{FTR}(\mathrm{Spin}(8))) \leq 8$ is obtained by a calculation modelled on [2, 4.5.2(b)]. Hence

LEMMA 2. *Let F be one of the sets listed in Lemma 1. Then $\kappa(F) \leq 8$ and $\kappa_0(F) \leq 4$.*

(3.3) LEMMA. *Let $\mathcal{G} = T \times \prod_{\alpha \in A} G_\alpha$ where T is a connected compact abelian group and each G_α is a connected compact almost simple Lie group. Denote by $\pi_\alpha \colon \mathcal{G} \to G_\alpha$ the canonical projection. Then $\mathrm{FTR}(\mathcal{G}) = \bigcup_{\alpha \in A_1} \mathrm{FTR}(G_\alpha) \circ \pi_\alpha$ where $A_1 = \{\alpha \in A\colon$ for some $S \in \mathcal{S}_n$, $n \geq 2$, $G_\alpha \simeq S$ or the simply-connected covering group of $S\}$. Hence $\kappa_0(\mathrm{FTR}(\mathcal{G})) \leq 4$ and $\kappa(\mathrm{FTR}(\mathcal{G})) \leq 32$.*

PROOF. Let $\sigma \in \mathrm{FTR}(\mathcal{G})$; then σ has the form $\chi \times (\sigma_\alpha)_{\alpha \in A}$ for some $\chi \in \hat{T}$, $\sigma_\alpha \in \hat{G}_\alpha$, with $\sigma_\alpha = 1$ for all but finitely many α. Noting that $\sigma(\mathcal{G})$ is connected, we see from the definition (3.1) that $\sigma(\mathcal{G})$ is almost simple. Were χ non-trivial, $\sigma(\mathcal{G})$ would be non-tall, so $\chi = 1$. If $\sigma_\alpha \neq 1$ and $\sigma_\beta \neq 1$ for $\alpha \neq \beta$ then $\sigma(G_\alpha)$ and $\sigma(G_\beta)$ are proper non-trivial connected normal subgroups of $\sigma(G)$; hence σ has the form $\sigma_\alpha \circ \pi_\alpha$ for some $\alpha \in A$. That in fact $\alpha \in A_1$ follows from the classification theorem for almost simple Lie groups, and it is clear that $\sigma_\alpha \in \mathrm{FTR}(G_\alpha)$. The estimates for κ and κ_0 follow from [2, 2.2.1 and 5.2] with (3.2.2).

This lemma shows that the FTR sets defined by Cartwright and McMullen [2, 5.1], except for their inclusion of 1, agree with our definition when the factors of the product are of types other than B_2 and D_4. Since the discrepant elements have absolutely bounded degree (in fact degree ≤ 8), all results of [2] remain valid with our FTR set ($\cup\{1\}$) used in place of theirs.

(3.4) PROPOSITION. *Let G be a compact connected group. Then $\kappa(\mathrm{FTR}(G)) \leq 32$ and $\kappa_0(\mathrm{FTR}(G)) \leq 4$.*

PROOF. By the extended structure theorem [14, 6.5.6] there is a group $\mathcal{G}$ having the form of (3.3) and an epimorphism $\pi\colon \mathcal{G} \to G$. Clearly $\mathrm{FTR}(G) \circ \pi \subseteq \mathrm{FTR}(\mathcal{G})$, so the result follows by (3.3) and [2, 2.2.1].

(3.5) DEFINITION. Let G be a compact group, $H \trianglelefteq G$ and $Q \subseteq \hat{H}$. Fore each $\tau \in Q$, choose $P_\tau \subseteq \hat{G}$ so that for every $\sigma \in P_\tau$, $\sigma|_H$ is equivalent to the direct sum of $m(\tau, \sigma|_H)$ copies of τ. Then $P = \bigcup_{t \in Q} P_\tau$ is called a *pre-Hutchinson lifting* of Q; if in addition $\eta_0(P) = \sup\{d(\tau)^{-2} \sup\{d(\sigma)^2\colon \sigma \in P_\tau\}\colon \tau \in Q\}$

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is finite P is called a *local Hutchinson lifting* of Q, and if moreover $\eta(P) = \sup\{d(\tau)^{-2}\sum_{\sigma\in P_\tau} d(\sigma)^2 \colon \tau \in Q\}$ is finite P is called a *Hutchinson lifting* of Q. The base set Q is said to be *canonical* if $P_\tau \neq \varnothing$ for every $\tau \in Q$.

(3.6) LEMMA [17, 3.3]. *Let G be a compact group, $H \trianglelefteq G$ and suppose $P \subseteq \hat{G}$ is a pre-Hutchinson lifting of $Q \subseteq \hat{H}$. Then $\kappa(P) \leq \eta(P)\kappa(Q)$.*

COROLLARY. $\kappa_0(P) \leq \eta_0(P)\kappa_0(Q)$.

(3.7) PROPOSITION. *Let G be a compact group. Then $\kappa_0(\mathrm{FTR}(G)) \leq 4$.*

PROOF. We show that every element of $\mathrm{FTR}(G)$ is irreducible when restricted to G_e (the connected component of the identity); this implies $\mathrm{FTR}(G)$ is a local Hutchinson lifting of a subset of $\mathrm{FTR}(G_e)$ with $\eta_0(\mathrm{FTR}(G)) = 1$ and the result follows from (3.4) and (3.6).

Let $\sigma \in \mathrm{FTR}(G)$. Then $\sigma(G_e) = [\sigma(G)]_e \simeq S \in \mathcal{S}_{d(\sigma)}$ (and $d(\sigma) \geq 2$), and since [15] S has no non-trivial representation of degree less than $d(\sigma)$, and $S \not\simeq \{I_{d(\sigma)}\}, \sigma|_{G_e}$ is indeed irreducible.

(3.8) COROLLARY. *Let G be a compact group and suppose G_e is open in G. Then $\kappa(\mathrm{FTR}(G)) \leq 32|G/G_e|$.*

PROOF. Using Frobenius reciprocity we can show that in this case $\mathrm{FTR}(G)$ is a Hutchinson lifting of a subset of $\mathrm{FTR}(G_e)$ (with $\eta(\mathrm{FTR}(G)) \leq |G/G_e|$).

(3.9) EXAMPLE. If G/G_e is infinite, $\mathrm{FTR}(G)$ need not be Sidon, for consider $H = \prod_{n\geq 1} \mathbf{Z}_2$ and $G = SO(2k) \times H$ $(k \geq 4)$. Then $\mathrm{FTR}(G) = (\lambda_1) \times \hat{H}$. It can be shown (by a generalization of [2, 5.4.1]) that $\kappa(\mathrm{FTR}(G)) \geq \kappa(\hat{H})$, which is infinite by [8, 37. 4].

4. Higher order FTR sets

(4.1) DEFINITION. Let $k \in \mathbf{N}$. Within the set $\{H \leq U(k) \colon H$ is connected, irreducible (that is, its centralizer is the scalars) and semisimple$\}$ choose a representative from each conjugacy class and denote by $\mathcal{H}_k$ this collection of class representatives. Let $n \in \mathbf{N}$, $n > k$, and put

$$\mathcal{N}_{k,n} = \{N \leq U(kn) \colon N \text{ is tall, for some } S \in \mathcal{S}_n,\ H \in \mathcal{H}_k,\ S \otimes H \trianglelefteq N\}.$$

Let G be a compact group. Then the FTR *set of G of order k* is

$$\mathrm{FTR}_k(G) = \{\sigma \in \hat{G} \colon d(\sigma) = nk,\ n > k, \text{ and } G/\ker\sigma \simeq N \in \mathcal{N}_{k,n}\}.$$

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Since conjugate subgroups of $U(k)$ have isomorphic normalizers, it is clear that the definition of $\mathrm{FTR}_k(G)$ is independent of the choice of $\mathcal{H}_k$. (The following example shows that isomorphism classes cannot be used in place of conjugacy classes.) It follows, essentially from semisimplicity and the structure theorem [14, 6.4.5], that $\mathcal{H}_k$ is finite. Also, the only connected semisimple subgroup of $U(1)$ is $\{1\}$, so $\mathcal{N}_{1,n} = \mathcal{N}_n$ and $\mathrm{FTR}_1(G) = \mathrm{FTR}(G)$.

(4.2) EXAMPLE. Let $G = SO(8)$, $H = \mathrm{Spin}(5)$, $K = SU(3)$ and $S = SO(8961)$. Let σ_1 denote the self-representation of G and $\sigma_2 \in \hat{G}$ the representation having highest weight $\lambda_1 + 2\lambda_3$. Let $\tau_1 \in \hat{H}$ have highest weight $2\mu_1 + \mu_2$, $\tau_2 \in \hat{H}$ have highest weight μ_2, $\gamma_1 \in \hat{K}$ have highest weight $6\nu_1$ and $\gamma_2 \in \hat{K}$ have highest weight $3\nu_1$. Denote by ρ the self-representation of S, and put

$$\mathcal{G}_j = (\rho \times \sigma_j \times \tau_j \times \gamma_j)(S \times G \times H \times K) \qquad (j = 1, 2);$$

then $\mathcal{G}_j \leq U(80290560 = 8961 \times 8 \times 40 \times 28 = 8961 \times 224 \times 4 \times 10)$ is irreducible and $\mathcal{G}_1 \simeq \mathcal{G}_2$. However, $N_U(\mathcal{G}_1)$ has two connected components whereas $N_U(\mathcal{G}_2)$ is connected.

(4.3) DEFINITION. We introduce a slight perversion of the Sidon constant and list some of the properties which follow readily. For $X \leq U(m)$ let

$$\kappa(X) = m \sup \left\{ \left\| \int_X (\operatorname{tr} Ax) x^{-1}\, dx \right\|_{\phi_1} : \sup\{|\operatorname{tr} Ax| : x \in X\} \leq 1,\ A \in M_m(\mathbf{C}) \right\}.$$

Suppose $Y \trianglelefteq X \leq U(m)$. Then

(i) $1 \leq \kappa(X) \leq m^2$,

(ii) $\kappa(uXu^{-1}) = \kappa(X)$, $u \in U(m)$,

(iii) $\kappa(X) \leq \kappa(Y)$, and

(iv) $\kappa(X \otimes I_k) \leq k^2 \kappa(X)$.

Suppose $Y \leq X \leq U(m)$ and Y is irreducible. Then

(v) $\kappa(X) = \sup\{\|A\|_{\phi_1} : \sup\{|\operatorname{tr} Ax| : x \in X\} \leq 1,\ A \in M_m(\mathbf{C})\}$,

(vi) $\kappa(\mathbf{T}X) = \kappa(X)$,

(vii) $\kappa(X) \leq \kappa(Y)$.

Let G be a compact group and $\sigma \in \hat{G}$. Then

(viii) $\kappa(\sigma(G)) = \kappa(\sigma)$.

(4.4) LEMMA. *Let G be a compact group. Then $\kappa_0(\mathrm{FTR}_k(G)) \leq 4k^2$.*

PROOF. For $k = 1$ this is (3.7). Suppose $k \geq 2$ and $\sigma \in \mathrm{FTR}_k(G)$. Then $d(\sigma) = nk$, $n > k$, and there are $S \in \mathcal{S}_n$, $H \in \mathcal{H}_k$ and $S \otimes H \trianglelefteq N \in \mathcal{N}_{k,n}$ so that $\sigma(G) \simeq N$. We have $\kappa(\sigma) = \kappa(\sigma(G))$, but example (4.2) shows that $\sigma(G)$ and N need not be conjugate, so it is not immediate (it may not even be true) that

$\kappa(\sigma) = \kappa(N)$. Instead, we have $\sigma(G_e) = [\sigma(G)]_e \simeq N_e = S \otimes H$, and by (4.3) $\kappa(\sigma) \le \kappa(\sigma(G_e))$. Direct calculation (using [15]) shows that either (i) $\sigma(G_e)$ is conjugate to $S \otimes H'$ for some (not necessarily irreducible) homomorphic image H' of H, or (ii) $n = k+1$. If the latter, $\kappa(\sigma) \le d(\sigma) = k(k+1) \le 4k^2$. If the former, $\kappa(\sigma(G_e)) = \kappa(S \otimes H') \le \kappa(S \otimes I_k) \le k^2\kappa(S) \le 4k^2$, using (4.3), the fact that $S \otimes I_k \trianglelefteq S \otimes H'$ and (3.2.2).

(4.5) EXAMPLE. Even for connected G, $\mathrm{FTR}_k(G)$ need not be Sidon. Consider $G_1 = \prod_{n\ge1} SU(3)$, $G_2 = \prod_{n\ge1} SU(4)$ and $G = G_1 \times G_2$. Then $\mathrm{FTR}_3(G) = \mathrm{FTR}(G_1) \times \mathrm{FTR}(G_2)$, which is not Sidon by [2, 5.4.2]. The Sidonicity of $\mathrm{FTR}_3(G)$ fails here precisely because G is not tall. For tall G, proceeding by analogy with (3.3, 3.4) and using the above calculations (4.4) and the fact that Sidonicity is preserved under finite unions, it is possible to prove the following

(4.6) PROPOSITION. *Let G be a tall connected compact group, and let $k \in \mathbf{N}$. Then $\mathrm{FTR}_k(G)$ is Sidon.*

(4.7) PROPOSITION. *Let G be compact and tall, and suppose G_e is open in G. Let $k \in \mathbf{N}$. Then $\mathrm{FTR}_k(G)$ is Sidon.*

PROOF. Let $\sigma \in \mathrm{FTR}_k(G)$; a corollary of the calculations required for the proof of (4.4) is that every irreducible component of $\sigma|_{G_e}$ lies in $\bigcup_{m=1}^{k} \mathrm{FTR}_m(G_e)$. Since G_e is also tall, this union is Sidon by (4.6), and it follows, from (8.1) below, that $\mathrm{FTR}_k(G)$ is Sidon.

(4.8) DEFINITION. Let G be a compact group and $k \in \mathbf{N}$. We identify a useful subset of $\mathrm{FTR}_k(G)$: put

$$\mathrm{FTR}'_k(G) = \{\sigma \in \hat{G} \colon d(\sigma) = nk, n > k \text{ and } \sigma(G) \text{ is unitarily conjugate to some } N \in \mathcal{N}_{k,n}\}.$$

It is easily seen that $\mathrm{FTR}'_k(G)$ is unbounded (that is, the degrees of its elements are unbounded) if and only if $\mathrm{FTR}_k(G)$ is also unbounded, and direct computation shows that $\mathrm{FTR}'_1(G) = \mathrm{FTR}(G)$. However, the following extension of example (4.2) above shows that for $k > 1$, $\mathrm{FTR}'_k(G)$ can be a proper subset of $\mathrm{FTR}_k(G)$.

(4.9) EXAMPLE. We retain the notation of (4.2) with the exceptions that S now denotes $SO(17921)$ and $\tau_1 \in \hat{H}$ has highest weight $3\mu_1 + \mu_2$. Let $N = O(8)$, denote by ω_1 the self-representation of N and let ω_2 denote the induced representation σ_2^N. Put

$$N_j = (\rho \times \omega_j \times \tau_j \times \gamma_j)(S \times N \times H \times K) \qquad (j = 1, 2);$$

then $N_j \leq U(321144320 = 17921 \times 8 \times 80 \times 28 = 17921 \times 448 \times 4 \times 10)$ is irreducible and $N_1 \simeq N_2$. Now $(N_1)_e = (\rho \times \sigma_1 \times \tau_1 \times \gamma_1)(S \times G \times H \times K)$ so N_1 is conjugate to an element of $\mathcal{N}_{17920,\,17921}$. However, $(N_2)_e$ is reducible (since $\omega_2|_G$ is reducible), so N_2 is conjugate to no element of $\mathcal{N}_{17920,\,17921}$.

(4.10) We give now the distinguishing property of $\mathrm{FTR}'_k(G)$.

LEMMA. *Let G be a compact group and $k \in \mathbf{N}$, and suppose $G_e \leq K \trianglelefteq G$. Then $\mathrm{FTR}'_k(G)$ is a local Hutchinson lifting of a subset of $\mathrm{FTR}'_k(K)$, with $\eta_0(\mathrm{FTR}'_k(G)) = 1$.*

PROOF. Let $\sigma \in \mathrm{FTR}'_k(G)$: then $d(\sigma) = nk$, $n > k$, and $\sigma(G)$ is conjugate to some $N \in \mathcal{N}_{k,n}$. Also, $\sigma(K)$ is conjugate to a closed subgroup N_1 of N containing N_e (so N_1 is also open and hence tall); since N_e is irreducible and $N_1 \in \mathcal{N}_{k,n}$, $\sigma|_K$ is irreducible and belongs to $\mathrm{FTR}'_k(K)$.

5. Interlude—the case of a connected compact group

A restatement, in terms of higher order FTR sets, of the known existence result [2, 6.4.2] for infinite Sidon sets on a connected compact group provides a pattern for the general solution.

(5.1) PROPOSITION. *Let G be a compact group. The following are equivalent.*

(i) *G admits an infinite Sidon set,*

(ii) *G admits an infinite local Sidon set,*

(iii) *either G is not tall, or G is tall and $\mathrm{FTR}_k(G)$ is infinite for some $k \geq 1$.*

PROOF. The equivalence of (i) and (ii) is contained in [2, 6.4.2]; that (iii) implies (i) follows from [10, 2.5] and (4.6).

Suppose G is tall and admits an infinite Sidon set. From [14, 6.5.6] there is an epimorphism $\pi\colon \mathcal{G} \to G$ where $\mathcal{G} = \prod_{\alpha \in A} G_\alpha$ is a product of almost simple simply-connected compact Lie groups. By [2, 6.4.2] $\ker \pi$ is "almost trivial", that is, A is infinite and there are $B, \Gamma \subseteq A$, B finite and Γ infinite, $B \cap \Gamma = \varnothing$, and there is $\tau \in \hat{\mathcal{G}}_B$ (where $\mathcal{G}_B = \prod_{\alpha \in B} G_\alpha$) such that for every $(x_\alpha)_{\alpha \in A} \in \ker\, \pi$ there is $\theta \in \mathbf{C}$ so that $\tau((x_\alpha)_{\alpha \in B}) = \theta I_{d(\tau)}$ and $\rho_\alpha(x_\alpha) = \overline{\theta} I_{d(\rho_\alpha)}$ for all $\alpha \in \Gamma$, where $\rho_\alpha \in \mathrm{FTR}(G_\alpha)$ has highest weight λ_1. Let $k = d(\tau)$ and $\alpha \in \Gamma$. Then $\tau \times \rho_\alpha \times 1$ is trivial on $\ker \pi$ and hence induces an irreducible representation $\sigma \in \hat{G}$; moreover $\sigma \in \mathrm{FTR}_k(G)$ provided $d(\rho_\alpha) > k$. Since $\mathcal{G}$ is also tall there are at most finitely many $\alpha \in \Gamma$ for which $d(\rho_\alpha) \leq k$ and it follows that the cardinality of $\mathrm{FTR}_k(G)$ is at least that of Γ.

(5.2) REMARK. An alternative proof of the foregoing is possible by means of a structure theorem along the lines of [17, 4.4]; see [16, 4.9.2].

6. Projective representations

In the context of Hutchinson liftings and Clifford's Theorem, projective representations arise naturally; we refer the reader to [1, 4, 13] for more detail. This section is devoted to their relationship with Sidonicity.

(6.1) DEFINITION. Let G be a group; a *projective representation* of G is a map γ from G into the group of unitary operators on some Hilbert space such that γ preserves the identity and almost preserves multiplication insofar as $\gamma(xy) = \alpha(x,y)\gamma(x)\gamma(y)$, where $\alpha\colon G \times G \to \mathbf{T}$ is the *factor system* of γ (and we call γ an *α-representation* of G). Alternatively, γ may be considered as a homomorphism of G into a projective unitary group.

Irreducibility and unitary equivalence are defined by analogy with ordinary representations, which are projective representations having $\alpha \equiv 1$. Direct sums, inner and outer tensor products and complex conjugation are all defined in the obvious way. The set of all factor systems for G forms an abelian group under pointwise multiplication, and the subset of factor systems having the form

$$(*) \qquad \alpha(x,y) = \rho(xy)/\rho(x)\rho(y)$$

for some $\rho\colon G \to \mathbf{T}$, constitutes a normal subgroup. The corresponding quotient group is called the Schur multiplicator and can be identified with the second cohomology group of cocycles of G. Multiplying a factor system by one of the form $(*)$ corresponds to multiplying a representation by arbitrary constants: if γ is an α-representation of G then γ' defined by $\gamma'(x) = \rho(x)\gamma(x)$ is an α'-representation of G, where $\alpha'(x,y) = \alpha(x,y)\rho(xy)/\rho(x)\rho(y)$.

(6.2) Some properties of projective representations. Let G be a compact group, $H \trianglelefteq G$, and suppose $\sigma \in \hat{G}$ is a pre-Hutchinson lifting of $\tau \in \hat{H}$, that is $\sigma|_H$ is equivalent to the direct sum of $m = m(\tau, \sigma|_H)$ copies of τ. Then there is an irreducible projective representation $\tilde{\tau}$ of G, unique up to multiplication by arbitrary constants, which agrees with τ on H. Moreover, there is an irreducible projective representation γ of the quotient G/H so that (regarding γ in the obvious way as a representation of G) σ is equivalent to $\tilde{\tau} \otimes \gamma$; we have also $\mathbf{T}\tilde{\tau}(G) \leq U(d(\tau))$ and $\mathbf{T}\gamma(G/H) \leq U(m)$.

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Somewhat remarkably, given that we have imposed no measurability criteria on $\tilde{\tau}$ or γ (contrast [13]), is the following

LEMMA. *With the foregoing notation, and using subscripts to denote coordinate functions, we have*

(i) $\tilde{\tau} \otimes (\tilde{\tau})^-$ *and* $\gamma \otimes \overline{\gamma}$ *are continuous representations of* G,

(ii) $\int_G \tilde{\tau}_{ij}(x)(\tilde{\tau}_{i'j'}(x))^- dx = d(\tau)^{-1}\delta_{ii'}\delta_{jj'}$ *and* $\int_G \gamma_{kl}(x)(\gamma_{k'l'}(x))^- dx = m^{-1}\delta_{kk'}\delta_{ll'}$.

The proof is achieved by observing that the integrands above are sums of coordinate functions of σ, and hence are continuous.

(6.3) DEFINITION. We introduce a "Sidon constant" for $\tilde{\tau}$ and γ: define

$$\kappa(\tilde{\tau}) = \sup\{\|A\|_{\phi_1} : A \in M_{d(\tau)}(\mathbf{C}), \sup\{|\operatorname{tr}(A\tilde{\tau}(x))| : x \in G\} \le 1\}$$

and similarly for γ. It is easily seen that $1 \le \kappa(\tilde{\tau}) \le \kappa(\tau) \le d(\tau)$, and it follows from Lemma (6.2) that $1 \le \kappa(\gamma) \le m^2$ and that $\kappa(\tilde{\tau}) = \kappa(\mathbf{T}\tilde{\tau}(G))$ and $\kappa(\gamma) = \kappa(\mathbf{T}\gamma(G/H))$. By an argument similar to [2, 5.4.1] we obtain the inequality $\kappa(\sigma) \ge \kappa(\tilde{\tau})\kappa(\gamma)$.

7. The key estimates

In this section we derive new inequalities and extend known inequalities for Sidon constants which will be essential to later arguments.

(7.1) LEMMA. *Let* G *be a compact group and suppose* $\sigma \in \hat{G}$ *is a pre-Hutchinson lifting of some* $\tau \in \hat{G}_e$. *Then we have* $\log m \le 9216\kappa(\sigma)^6$ *writing* m *for* $m(\tau, \sigma|_{G_e})$.

PROOF. Construct $\tilde{\tau} \otimes \gamma$ equivalent to σ as in (6.2); then $d(\gamma) = m$ and $\kappa(\gamma) \le \kappa(\sigma)$. Now $\Gamma = \gamma \otimes \overline{\gamma}$ is a continuous representation of G/G_e, and $\mathcal{G} = (G/G_e)/\ker \Gamma$ is a finite group since G/G_e is profinite; moreover, since γ is scalar on $\ker\Gamma$, γ defines a projective representation γ' of $\mathcal{G}$, and $\mathbf{T}\gamma(G/G_e) = \mathbf{T}\gamma'(\mathcal{G})$. By [4, 53.7] there are a finite group $\mathcal{G}^*$ and an (ordinary) representation $\gamma'' \in (\mathcal{G}^*)^\frown$ such that $\mathbf{T}\gamma''(\mathcal{G}^*)$ is conjugate to $\mathbf{T}\gamma'(\mathcal{G})$; thus $\kappa(\gamma) = \kappa(\gamma'')$.

We now apply [12, Theorem] to $\{\gamma''\}$, keeping in mind [8, 37.25(a)], to obtain $\log d(\gamma'') \le 144B^2(2\kappa(\gamma''))^2$; closer investigation of the proofs of [12, Lemma 4, Theorem] shows that $B \le (2\kappa(\gamma''))^2$ obtains, whence the result.

(7.2) LEMMA. *Let* $\tilde{G}$ *be a compact simply-connected almost simple Lie group and* $\sigma \in (\tilde{G})^\frown$, $\sigma \ne 1$. *Put* $G = \sigma(\tilde{G})$ *and* $N = N_U(G)$, *and denote by* ω *the* *self-representation of N. Then* $\log d(\sigma) \le 384\kappa(\omega)^4 \log k$, *where k is a constant which depends only on the rank of $\tilde{G}$.*

PROOF. Let $f = \operatorname{tr}\omega \in \mathcal{T}_\omega(N)$; then [8, 27.19] $\|f\|_2^2 = 1$. Noting that $|N/N_e|$ is at most the cardinality of the group of diagram automorphisms of $\tilde{G}$, we see from [15] that $|N/N_e| \le 6$, whence [5, 5.3(ii)] $\|f\|_4^4 \ge \frac{1}{6}\int_{\tilde{G}} |\operatorname{tr}\sigma(x)|^4\, dx \ge \frac{1}{6}\log d(\sigma)/\log k$ [3, Corollary 1], where k is the maximum of the degrees of the fundamental representations of $\tilde{G}$. (Cecchini's estimate is far from optimal—see for example [5, 3.8]—but suffices for our purposes.) From [8, 37.10] we have $2^4 2!(2^{1/4}\kappa(\omega))^4\|f\|_2^4 \ge \|f\|_4^4$, and the result follows.

(7.3) LEMMA (after [2, 4.1 ff]). *Retaining the notation of the previous lemma, suppose $G \trianglelefteq H \le N$; denote by ψ the self-representation of H. Then $r^{1/3} \le 480n\kappa(\psi)$ unless $\sigma \in \mathrm{FTR}(\tilde{G})$, where r is the rank of $\tilde{G}$ and $n = |H/G|$.*

PROOF. Fix a maximal torus T of $\tilde{G}$, and for each $\rho \in (\tilde{G})^\wedge$ choose an orthonormal basis for the representation space of ρ so that the matrix of $\rho(x)$ is diagonal for each $x \in T$. We know that ρ corresponds uniquely to an irreducible representation ϕ_ρ of the Lie algebra $\mathfrak{g}$ of $\tilde{G}$, and moreover the characters $\rho_{jj}(x)$, $x \in T$ $(1 \le j \le d(\rho))$ correspond to the weights of ϕ_ρ. Reordering the basis if necessary we may suppose ρ_{11} corresponds to the highest weight of ϕ_ρ.

Denote by r the rank of $\tilde{G}$ and for $1 \le j \le r$ put $v_j(x) = (\rho_j)_{11}(x)$, $x \in \tilde{G}$, where the highest weight of ϕ_{ρ_j} is the fundamental weight λ_j. Then for each $\rho \in (\tilde{G})^\wedge$ if the highest weight of ϕ_ρ is $\lambda_\rho = l_1\lambda_1 + \cdots + l_r\lambda_r$, Giulini and Travaglini [7] have proved that $\rho_{11} = v^{l_1}v^{l_2}\cdots v^{l_r}$ holds on $\tilde{G}$. Consequently, if $s\rho$ $(s \in \mathbf{N})$ denotes the representation of $\tilde{G}$ such that $\phi_{s\rho}$ has highest weight $s\lambda_\rho$, we have $(s\rho)_{11} = (\rho_{11})^s$.

Let τ denote the self-representation of G; note that σ, τ and ψ have a common representation space, and that $\tau \circ \sigma = \sigma$. Denote by ψ_{jk}, τ_{jk} the relevant coordinate functions relative to the basis for σ chosen as above. Define $f \in \mathcal{T}_\psi(H)$ by $f(x) = \psi_{11}(x)$; then the orthogonality relations yield $\|f\|_2^2 = d(\psi)^{-1} = d(\sigma)^{-1}$. Now for $s \in \mathbf{N}$ (see [5, 5.3(ii)]),

$$\|f\|_{2s}^{2s} \ge \frac{1}{n}\int_G |\tau_{11}(y)|^{2s}\, dy = \frac{1}{n}\int_{\tilde{G}} |(s\sigma)_{11}(z)|^2\, dz = \frac{1}{n} d(s\sigma)^{-1},$$

whence [8, 37.10] $d(s\sigma)^{-1} \le n2^{2s}(2\kappa(\psi))^{2s}s^s\|f\|_2^{2s}$. Since $n \ge n^{1/2s}$ and since [2, 4.2] $d(s\sigma) \le s^M\, d(\sigma)$, where M is the number of positive roots not orthogonal to the highest weight of ϕ_σ, we have

$$\kappa(\psi) \ge \frac{1}{4n}(d(\sigma)^{1/2-1/2s}/s^{1/2+M/2s}).$$

Let $1/2 > \varepsilon > 0$ be given. Then as in [2, 4.2] we have

$$\kappa(\psi) > \frac{\varepsilon}{4n}(d(\sigma)^{1/2-\varepsilon}/M^{1/2+\varepsilon}).$$

If the highest weight of ϕ_σ is not listed in [2, 4.4], $d(\sigma) \geq r^3/8$ and $M \leq 2r^2$ so with $\varepsilon = 1/30$ we obtain $\kappa(\psi) > r^{1/3}/480n$. If the highest weight of ϕ_σ is listed in [2, 4.4] but $\sigma \notin \mathrm{FTR}(\tilde{G})$, then $d(\sigma) \geq r^2/2$ and $M \leq 4r$, so with $\varepsilon = 1/18$ we obtain $\kappa(\psi) > r^{1/3}/288n$.

(7.4) LEMMA. *We retain the notation of Lemma* (7.2). *Then there is an absolute constant* C ($C = 768 \times 960^3 \times \log 2$ *will do*) *so that* $\log(\kappa(\sigma)) \leq C\kappa(\omega)^7$.

PROOF. (i) Suppose $\tilde{G}$ is of type $A_1, B_r, C_r, E_7, E_8, F_4$ or G_2. Then $\tilde{G}$ admits only trivial diagram automorphisms [15], so $N = N_e = \mathbf{T}G$ and thus $\kappa(\sigma) = \kappa(\omega)$.

(ii) Suppose $\tilde{G}$ is of type D_4 or E_6. Then (7.2) $\log(\kappa(\sigma)) \leq 384\kappa(\omega)^4 \log k$ where [15] $k = 28$ for $\tilde{G}$ of type D_4 and $k = 2925$ for $\tilde{G}$ of type E_6.

(iii) Suppose $\tilde{G}$ is of type A_r ($r > 1$) or D_r ($r > 4$). If N is connected, then as in (i), $\kappa(\sigma) = \kappa(\omega)$. If N is disconnected but $\sigma \in \mathrm{FTR}(\tilde{G})$ we have $\kappa(\sigma) \leq 4$ (3.4). Otherwise, since the group of diagram automorphisms of $\tilde{G}$ is $\mathbf{Z}_2$ [15], it follows that $|N/N_e| = 2$, and because $N_e = \mathbf{T}G$ we can find $J \in N \setminus N_e$ so that $J^2 \in G$. Put $H = G \cup JG$ so that $G \trianglelefteq H \leq N$, $N = \mathbf{T}H$ and $H/G \simeq \mathbf{Z}_2$. Denote by ψ the self-representation of H; clearly $\kappa(\omega) = \kappa(\psi)$. Now from (7.2) we have $\log(\kappa(\sigma)) \leq 384\kappa(\omega)^4 \log k$ where [15] $k = \binom{r+1}{q}$ (writing q for $[(r+1)/2]$) for A_r and $k = \binom{2r}{r-2}$ for D_r. We estimate k by 2^r and 2^{2r} respectively, and from (7.3) obtain $r^{1/3} \leq 960\kappa(\omega)$, whence the lemma.

(7.5) LEMMA (see [2, 5.4.2]). *Let* $G \leq U(mn)$ *be a compact Lie group, and suppose that with respect to the tensor product basis* $\{\xi_1 \otimes \eta_1, \ldots, \xi_1 \otimes \eta_n, \xi_2 \otimes \eta_1, \ldots, \xi_m \otimes \eta_n\}$ *of* $\mathbf{C}^{mn}$ *corresponding to the orthonormal bases* $\{\xi_1, \ldots, \xi_m\}$ *and* $\{\eta_1, \ldots, \eta_n\}$ *of* $\mathbf{C}^m$ *and* $\mathbf{C}^n$ *respectively, G is contained in the* (*set theoretic*) *product of S with* $U(m) \otimes U(n)$ *where each* $s \in S$ *satisfies* $\langle s(\xi_i \otimes \eta_j), \xi_1 \otimes \eta_1 \rangle = \delta_{1i}\delta_{1j}$. *Denote by ω the self-representation of G, and suppose further that* $\omega \in \hat{G}$. *Then* $\kappa(\omega) \geq \min(m, n)^{1/2}$.

PROOF. It is clear from the symmetry of the hypotheses that we may suppose $m \leq n$ without loss of generality. Let $A = (a_{kl}) \in U(m)$ and define $B \in M_{mn}(\mathbf{C})$ by $B(\xi_i \otimes \eta_j) = \delta_{1i}\delta_{1j} \sum_{k=1}^m \sum_{l=1}^m a_{kl}\xi_k \otimes \eta_l$. Then for each $s \in S$, $Bs = B$, and if $x \in U(m)$ and $y \in U(n)$ we have

$$\begin{aligned} |\operatorname{tr} B(x \otimes y)| &= \left| \sum_{k=1}^{m} \sum_{l=1}^{m} x_{1k} y_{1l} a_{kl} \right| \\ &= \left| \left\langle \sum_{k=1}^{m} \overline{x_{1k}} \xi_k, A \sum_{l=1}^{m} y_{1l} \xi_l \right\rangle \right| \\ &\leq \|A\|_{\mathrm{op}} \left(\sum_{k=1}^{m} |x_{1k}|^2 \right)^{1/2} \left(\sum_{l=1}^{m} |y_{1l}|^2 \right)^{1/2} \\ &\leq 1 \end{aligned}$$

since A, x and y are all unitary.

Consider $f \in \mathcal{T}_\omega(G)$ defined by $f(x) = \operatorname{tr} Bx$; then $\|f\|_\infty \leq 1$ so $\kappa(\omega) \geq \|\hat{f}\|_1 = \operatorname{tr}|B| = m^{1/2}$.

8. The finite index case

Before presenting the general existence theorem for Sidon sets we deal with the intermediate case where the connected component of the group is open. A lifting theorem (8.1) relevant to this case is already well established; we therefore investigate the behaviour of Sidonicity under restriction to connected subgroups.

(8.1) PROPOSITION (essentially [9, 5.16]). *Let G be a compact group, $H \trianglelefteq G$ and suppose H is open in G. Let $Q \subseteq \hat{H}$ and put $P = \{\sigma \in \hat{G} : \sigma \leq \tau^G$ for some $\tau \in Q\}$. Then P is Sidon if and only if Q is Sidon.*

(8.2) PROPOSITION. *Let G be a compact group and $P \subseteq \hat{G}$. Suppose that P is local Sidon and a pre-Hutchinson lifting of $Q \subseteq \hat{H}$, where $H \trianglelefteq G$ is connected and Q is the canonical base set. Then Q is local Sidon.*

PROOF. Let $\tau \in Q$; since Q is canonical there is $\sigma \in P$ so that $\sigma|_H$ is equivalent to $p\tau$ for some $p \in \mathbf{N}$, and since $\kappa(\tau) = 1$ whenever $d(\tau) = 1$, we may suppose $d(\tau) \geq 2$. Using the construction (6.2) we obtain a projective extension $\tilde{\tau}$ of τ and a projective representation γ of the quotient G/H such that σ is equivalent to $\tilde{\tau} \otimes \gamma$ and $\mathbf{T}\tilde{\tau}(G) \leq N_U(\tau(H))$; moreover (6.3) $\kappa(\sigma) \geq \kappa(\tilde{\tau})$.

The structure theorem [14, 6.5.6] guarantees the existence of a family $(H_\alpha)_{\alpha \in A}$ of compact simply-connected almost simple Lie groups, a connected compact abelian group T and an epimorphism $\pi\colon \mathcal{H} = T \times \prod_{\alpha \in A} H_\alpha \to H$; then $\tau \circ \pi \in \hat{\mathcal{H}}$ so $\tau \circ \pi$ is equivalent to $\chi \times (\tau_\alpha)_{\alpha \in A}$ where $\chi \in \hat{T}$ and $\tau_\alpha \in \hat{H}_\alpha$ is non-trivial for at most finitely many α. Let $\alpha_1, \ldots, \alpha_k$ be the indices for which $\tau_\alpha \neq 1$, write τ_j for τ_{α_j} and put $H_j = \tau_j(H_{\alpha_j})$; notice that $k \geq 1$ since $d(\tau) \geq 2$. It is

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straightforward (albeit tedious) to show that there are $u_j \in U_j = U(d(\tau_j))$ and a group $S \le U(d(\tau))$ consisting of permutations such that $N_U(\tau(H))$ is conjugate to $\mathcal{N} = S \bigotimes_{j=1}^k u_j N_{U_j}(H_j) u_j^{-1}$ (the elements of S are of the kind which map $x \otimes y$ to $y \otimes x$ under conjugation). Now if ω denotes the self-representation of the group $\mathcal{N}$, we must have $\kappa(\tilde{\tau}) \ge \kappa(\omega)$ (6.3, 4.3(vii)).

Let m and n be complementary subproducts of $\prod_{j=1}^k d(\tau_j)$; the elements of S are such that hypotheses of (7.5) are satisfied, and we deduce $\kappa(\omega) \ge \min(m,n)^{1/2}$, whence, since $d(\tau_j) \ge 2$, we have $\kappa(\omega) \ge 2^{[k/2]}$ and thus $k \le 1 + 4\log_2 \kappa(\sigma)$. Moreover, if $d(\tau_j) > \kappa(\sigma)^2$ for more than one index j, (7.5) forces a contradiction. We therefore distinguish two cases:

(i) Suppose $d(\tau_j) \le \kappa(\sigma)^2$ for all j. Then $\kappa(\tau) \le d(\tau) \le \kappa(\sigma)^{2+8\log_2 \kappa(\sigma)}$.

(ii) Suppose $d(\tau_j) > \kappa(\sigma)^2$ for some (unique) j. Without loss of generality we may suppose $j = 1$. Then S has the special form $I \otimes S'$, and it follows from [2, 5.4.1] that $\kappa(\omega) \ge \kappa(\omega')$ where ω' is the self-representation of $N_{U_1}(H_1)$. Applying (7.4) we see that $\log \kappa(\tau_1) \le C\kappa(\omega')^7 \le C\kappa(\omega)^7$ for some absolute constant C, and from [2, 5.4.3] that $\kappa(\tau) \le \kappa(\tau_1) \prod_{j=2}^k d(\tau_j)^2 \le \exp(C\kappa(\sigma)^7)\kappa(\sigma)^{16\log_2 \kappa(\sigma)}$; since P is local Sidon it now follows that $\kappa_0(Q) < \infty$.

(8.3) LEMMA. *Let G be a compact group and $H \le G$ be open. Let $P \subseteq \hat{G}$, and suppose that for each $\sigma \in P$ there is some $\tau_\sigma \in \hat{H}$ so that the induced representation τ_σ^G is equivalent to σ. Choose one such τ_σ for each $\sigma \in P$ and put $Q = \{\tau_\sigma : \sigma \in P\}$; then $\kappa(Q) \le \kappa(P)$ and $\kappa_0(Q) \le \kappa_0(P)$.*

PROOF. Let $n = [G : H]$ and choose a left transversal $\{x_1 = e, \dots, x_n\}$ to H in G. For each $\sigma \in P$ fix an orthonormal basis for the representation space of σ so that the coordinate functions of σ are

$$\sigma_{(i-1)d+k,(j-1)d+l}(x) = \begin{cases} (\tau_\sigma)_{kl}(x_i^{-1}xx_j) & \text{if } x_i^{-1}xx_j \in H, \\ 0 & \text{otherwise,} \end{cases}$$

where $1 \le i, j \le n$, $1 \le k, l \le d = d(\tau_\sigma)$. The result follows routinely by associating to any $f = \sum_{\sigma\in P} \operatorname{tr}(A_\sigma \tau_\sigma) \in \mathcal{T}_Q(H)$ the function $g = \sum_{\sigma\in P} \operatorname{tr}(B_\sigma \sigma) \in \mathcal{T}_P(G)$ where $(B_\sigma)_{rs} = (A_\sigma)_{rs}$, $1 \le r, s \le d(\tau_\sigma)$, and $(B_\sigma)_{rs} = 0$ otherwise.

REMARK. Since by Frobenius reciprocity there are at most n possible τ_σ for each σ, it follows that $Q' = \{\tau \in \hat{H} : \tau^G \text{ is equivalent to } \sigma \text{ for some } \sigma \in P\}$ is a finite union of Sidon (respectively local Sidon) sets if P is Sidon (respectively local Sidon), and hence is Sidon (respectively local Sidon).

(8.4) PROPOSITION. *Let G be a compact group and suppose that G_e is open. Let $P \subseteq \hat{G}$ be local Sidon and put $Q = \{\tau \in \hat{G}_e : \tau \le \sigma|_{G_e} \text{ for some } \sigma \in P\}$. Then Q is local Sidon.*

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PROOF. Let $\sigma \in P$: Clifford's Theorem guarantees an open subgroup $S_\sigma \le G$ and $\rho_\sigma \in \hat{S}_\sigma$ so that σ is equivalent to ρ_σ^G and $\rho_\sigma|_{G_e} = n_\sigma \tau_\sigma$, for some $n_\sigma \in \mathbf{N}$ and $\tau_\sigma \in Q$. Since G_e is open there are only finitely many open subgroups of G, say $S_1, \ldots, S_k$. For $1 \le j \le k$ let $P_j = \{\sigma \in P\colon S_\sigma = S_j\}$ and put $R_j = \{\rho_\sigma\colon \sigma \in P_j\}$. Then P is the disjoint union of the P_j and from (8.3) it follows that $\kappa_0(R_j) \le \kappa_0(P)$. Now R_j is pre-Hutchinson relative to G_e: write Q_j for its canonical base set. Choose a transversal $\{x_1^j = e, \ldots, x_{m_j}^j\}$ to S_j in G; then from Clifford's Theorem we see that $Q = \bigcup_{j=1}^k \bigcup_{i=1}^{m_j} Q_j^{x_i^j}$ (where τ^x denotes the irreducible representation given by $y \mapsto \tau(xyx^{-1})$). By (8.2) each Q_j is local Sidon, and since $\kappa_0(Q_j^x) = \kappa_0(Q_j)$ it follows that Q is local Sidon.

(8.5) THEOREM. *Let G be a compact group and suppose that G_e is open. Then the following are equivalent*

(i) *G admits an infinite Sidon set,*
(ii) *G admits an infinite local Sidon set,*
(iii) *G_e admits an infinite Sidon set,*
(iv) *G_e admits an infinite local Sidon set.*

PROOF. That (i) implies (ii) is trivial. That (ii) implies (iv) follows from (8.4) together with Frobenius reciprocity. That (iii) and (iv) are equivalent is due to [2, 6.2.1]. That (iii) implies (i) follows from (8.1) and Frobenius reciprocity.

COROLLARY. *Let G be a compact group, $G_e \trianglelefteq H \le G$, and suppose G_e is open. Then the following are equivalent:*

(i) *G admits an infinite Sidon set,*
(ii) *G admits an infinite local Sidon set,*
(iii) *H admits an infinite Sidon set,*
(iv) *H admits an infinite local Sidon set.*

9. The general case

Before weaving our general existence theorem for Sidon sets from the foregoing strands, we require one further technical lemma.

(9.1) LEMMA. *Let G be a connected compact group and suppose $P \subseteq \hat{G}$ is infinite and local Sidon. Then P contains an infinite Sidon subset, which furthermore can be chosen to be unbounded whenever P is also unbounded.*

PROOF. (i) Suppose P contains an infinite bounded subset. Then [10, 2.4] P contains an infinite Sidon subset.

(ii) Suppose every bounded subset of P is finite. Let $\pi\colon \mathcal{G} = T \times \prod_{\alpha\in A} G_\alpha \to G$ be the epimorphism and product group guaranteed by [14, 6.5.6]; then $P\circ\pi \subseteq \hat{\mathcal{G}}$ is local Sidon, with every bounded subset finite. From [2, 5.5] we see that $P \circ \pi \subseteq Q_1 \times Q_2$, where Q_1 is bounded and $Q_2 = \mathrm{FTR}(G_2) \cup \{1\}$, G_2 being a subproduct of $\mathcal{G}$. The projection Q_3 of $P \circ \pi$ on $\hat{G}_2$ must be unbounded, as must Q_2. For each $\sigma \in Q_3$ choose $\tau_\sigma \in Q_1$ so that $\tau_\sigma \times \sigma \in P \circ \pi$ and put $Q = \{\tau_\sigma \times \sigma \colon \sigma \in Q_3\}$. Since Q_1 is bounded, Q is a Hutchinson lifting of Q_3 and hence (3.3, 3.6) is Sidon; moreover, $Q = P_1 \circ \pi$ for some unbounded $P_1 \subseteq P$, and P_1 is Sidon [2, 2.2.1].

(iii) If P is unbounded, create an unbounded subset P' of P by choosing one element of P for every degree which occurs. Now apply (ii) to P' to obtain an unbounded Sidon subset of P.

(9.2) In view of [10] our interest centres mainly on tall groups.

PROPOSITION. *Let G be a tall compact group. Then for G to admit an infinite Sidon set it is sufficient that there are an open subgroup $H \leq G$ and $k \in \mathbf{N}$ such that $\mathrm{FTR}_k(H)$ is infinite.*

PROOF. Fix a (left) transversal $\{x_1 = e, \dots, x_n\}$ to H in G and let $K = \bigcap_{j=1}^n x_j^{-1} H x_j$; then $K \trianglelefteq G$, $|G/K| \leq n!$ and $K_e = H_e = G_e$ [8, 7.8]. Now H is tall so (4.8) $\mathrm{FTR}'_k(H)$ is unbounded and (4.10) $\mathrm{FTR}'_k(K)$ is also unbounded. Put $F = \{\sigma|_{G_e} \colon \sigma \in \mathrm{FTR}'_k(K)\}$; by (4.10) F is an unbounded subet of $\mathrm{FTR}'_k(G_e) \subseteq \mathrm{FTR}_k(G_e)$. It follows from (4.6, 9.1) that F contains an unbounded Sidon subset F'. For each $\tau \in F'$ choose $\sigma_\tau \in \mathrm{FTR}'_k(K)$ so that $\sigma_\tau|_{G_e}$ is equivalent to τ; then $P = \{\sigma_\tau \colon \tau \in F'\} \subseteq \hat{K}$ is a Hutchinson lifting of F' (with $\eta(P) = 1$) and is therefore Sidon (3.6) and infinite. The proposition now follows by applying (8.1) and Frobenius reciprocity to the open normal subgroup K of G.

(9.3) PROPOSITION. *Let G be a tall compact group and suppose that G admits an infinite local Sidon set. Then G_e admits an unbounded local Sidon set and there are $k \in \mathbf{N}$ and an open subgroup $H \leq G$ so that $\mathrm{FTR}_k(H)$ is unbounded.*

PROOF. Suppose $P \subseteq \hat{G}$ is infinite, hence unbounded, and local Sidon. Let $\sigma \in P$: Clifford's Theorem guarantees an open subgroup $S_\sigma \leq G$, $\rho_\sigma \in \hat{S}$, $\tau_\sigma \in \hat{G}_e$ and $m_\sigma \in \mathbf{N}$ so that σ is equivalent to ρ_σ^G and $\rho_\sigma|_{G_e} = m_\sigma \tau_\sigma$. Now [12, Lemma 4] ensures that $\sup\{[G : S_\sigma] \colon \sigma \in P\}$ is finite, and a straightforward extension of [11, 2.1(ii)] shows that G contains only finitely many closed subgroups of any given finite index. Hence there are an infinite subset $P_1 \subseteq P$ and an open subgroup $S \leq G$ so that $S_\sigma = S$ for all $\sigma \in P_1$. Put $R = \{\rho_\sigma : \sigma \in P_1\} \subseteq \hat{S}$; R is local Sidon by (8.3), unbounded by Frobenius reciprocity, and pre-Hutchinson relative to G_e. Put $Q = \{\tau_\sigma : \sigma \in P_1\}$, so R is a pre-Hutchinson lifting of Q; for $\tau \in Q$ let $R_\tau = \{\rho \in R : \tau \leq \rho|_{G_e}\}$, so that $R = \bigcup_{\tau \in Q} R_\tau$.

Now for each $\rho \in R_\tau$ we have $\rho|_{G_e}$ equivalent to the direct sum of m_ρ copies of τ for some $m_\rho \in \mathbf{N}$, and since $S_e = G_e$ it follows from (7.1) that $\log m_\rho \leq 9216\kappa_0(R)^6$. Therefore R is local Hutchinson, Q is unbounded, and by (8.2) Q is local Sidon. Since G_e is connected it follows from (9.1) that Q contains an unbounded Sidon subset Q_1 having at most one element of each degree. Consider $\tau \in Q$: there is $\rho \in R_\tau$ and since $\rho|_{G_e}$ is equivalent to a direct sum of copies of τ, the centre of $\tau(G_e)$ contains at most as many elements as the centre of $\rho(S)$; since S is tall it follows [10, 3.2] that $\rho(S)$, and hence $\tau(G_e)$, is semisimple. Considerations akin to those required for [17, 4.4] show [16, 3.4.7] that there are a bounded set $B \subseteq \hat{G}_e$ and $q \in \mathbf{N}$ so that $Q_1 \subseteq B \cup \bigcup_{k=1}^{q} \mathrm{FTR}_k(G_e)$. It follows that there is $k \in \mathbf{N}$ so that $Q_1 \cap \mathrm{FTR}_k(G_e)$, and hence $\mathrm{FTR}_k(G_e)$, is unbounded; without loss of generality we may suppose $Q_1 \subseteq \mathrm{FTR}_k(G_e)$.

For each $\tau \in Q_1$ choose $\rho_\tau \in R$ and put $R_1 = \{\rho_\tau : \tau \in Q_1\}$; then R_1 is a Hutchinson lifting of Q_1, and for each $\rho \in R_1$ we have $\rho|_{G_e}$ equivalent to the direct sum of m_ρ copies of τ_ρ for some $m_\rho \in \mathbf{N}$ and $\tau_\rho \in Q_1$. By (6.2) we may find a projective extension $\tilde{\tau}_\rho$ of τ_ρ to S and a projective representation γ_ρ of S so that ρ is equivalent to $\tilde{\tau}_\rho \otimes \gamma_\rho$, and $\Gamma_\rho = \gamma_\rho \otimes \overline{\gamma}_\rho$ is a continuous representation of S with $d(\Gamma_\rho) \leq \eta(R_1)$. Since S is tall there are only finitely many possible Γ_ρ (up to equivalence), hence there is an infinite subset $R_2 \subseteq R_1$ so that Γ_ρ is equivalent to Γ for all $\rho \in R_2$. Put $H = \ker \Gamma$; then $H \trianglelefteq S$ whence $H \leq G$. By construction G_e is contained in the kernel of γ_ρ for each ρ, so Γ is really a representation of S/G_e; it follows that $S/H \simeq \Gamma(S/G_e)$ is finite since S/G_e is profinite. Therefore H is open in G.

Let $\rho \in R_2$ and $x \in H$. Then for some $u \in U(d(\rho))$ we have

$$\rho(x) = u(\tilde{\tau}_\rho(x) \otimes \gamma_\rho(x))u^{-1} = u(\tilde{\tau}_\rho(x) \otimes \beta_\rho(x) I_{m_\rho})u^{-1}$$

where $\beta_\rho : H \to \mathbf{T}$, since γ_ρ must be scalar on $H = \ker(\gamma_\rho \otimes \overline{\gamma}_\rho)$. It follows that $\rho|_H$ is equivalent to m_ρ copies of ψ_ρ, where $\psi_\rho(x) = \beta_\rho(x)\tilde{\tau}_\rho(x)$. It is easily seen that ψ_ρ defines a representation of H which is continuous because ρ is; morover $\psi_\rho|_{G_e} = \tau_\rho$ so ψ_ρ is irreducible, and ψ_{ρ_1} equivalent to ψ_{ρ_2} implies τ_{ρ_1} equivalent to τ_{ρ_2}, which (by choice of R_1) in turn implies ρ_1 equivalent to ρ_2. Thus $V = \{\psi_\rho : \rho \in R_2\} \subseteq \hat{H}$ is a Hutchinson lifting, with $\eta(V) = 1$, of an unbounded subset of Q_1.

Finally, let $V_1 = \{\psi \in V : d(\psi) > k(k+1)\}$; V_1 is unbounded. Consider $\psi \in V_1 : \psi|_{G_e} = \tau \in \mathrm{FTR}_k(G_e)$. Since $\tau(G_e)$ is irreducible and $d(\tau) > k(k+1)$ it follows from the proof of (4.4) that in fact $\tau \in \mathrm{FTR}'_k(G_e)$; because $\psi(H)$ is tall and $\tau(G_e) \trianglelefteq \psi(H)$ we deduce that $\psi \in \mathrm{FTR}'_k(H)$ and the proposition is complete.

REMARK. A careful tracing of the constituent constructions in the above argument yields the additional fact that for G a tall compact group, every infinite local Sidon set in $\hat{G}$ contains an infinite Sidon subset.

(9.4) Combining (9.2, 9.3) and [10, 2.4] we achieve our general criterion for the existence of an infinite Sidon set in the dual object of a compact group.

THEOREM. *Let G be a compact group. The following are equivalent*

(i) *G admits an infinite Sidon set,*

(ii) *G admits an infinite local Sidon set,*

(iii) *either G is not tall, or G is tall and there are $k \in \mathbf{N}$ and an open subgroup $H \leq G$ such that $\mathrm{FTR}_k(H)$ is infinite.*

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