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A Stochastic Difference Equation with Stationary Noise on Groups

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Abstract. We consider the stochastic difference equation $\eta_k = \xi_k \phi(\eta_{k-1})$, $k \in \mathbb{Z}$ on a locally compact group G, where ϕ is an automorphism of G, ξ_k are given G-valued random variables, and η_k are unknown G-valued random variables. This equation was considered by Tsirelson and Yor on a one-dimensional torus. We consider the case when ξ_k have a common law μ and prove that if G is a distal group and ϕ is a distal automorphism of G and if the equation has a solution, then extremal solutions of the equation are in one-to-one correspondence with points on the coset space $K\backslash G$ for some compact subgroup K of G such that μ is supported on $Kz = z\phi(K)$ for any z in the support of μ. We also provide a necessary and sufficient condition for the existence of solutions to the equation.

1 Introduction

Stochastic and random difference equations have been considered by many in different settings (see, for instance, [Ke73, Ts75, Yo92]). Here we consider the following type of equation.

Let G be a locally compact (Hausdorff) group. Consider the stochastic difference equation on G

$$\eta_k = \xi_k \phi(\eta_{k-1}), \quad k \in -\mathbb{N}, \tag{1.1}$$

where η_k and ξ_k are G-valued random variables and ϕ is an automorphism of G. The random variables (ξ_k) are given and are called the *noise process* of equation (1.1). We are interested in finding the law of the unknown process (η_k). We further assume that for any k, the random variable ξ_k is independent of η_j for $j < k$, and this assumption will be enforced whenever an equation of type (1.1) is considered.

B. Tsirelson [Ts75] considered the following stochastic difference equation on the real line:

$$\eta_k = \xi_k + \mathrm{frac}(\eta_{k-1}) \quad k \in -\mathbb{N} \tag{1.2}$$

to obtain his celebrated example of the stochastic differential equation

$$dX_t = dB_t + b^+(t, X)dt, \quad X(0) = 0 \tag{1.3}$$

that has a unique, but not strong, solution, where $\mathrm{frac}(x)$ is the fractional part of $x \in \mathbb{R}$, (ξ_k) is a given stationary Gaussian noise process, and (B_t) is the one-dimensional

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Brownian motion. It was also noted that under some conditions the solution of stochastic difference equation (1.2) determines the solution of Tsirelson's stochastic differential equation (1.3) (see [Ts75] for more details).

It is easy to see that the set of all solutions (η_k) of equation (1.1) is a convex set, hence by extremal solution we mean an extreme point of the convex set of all solutions.

M. Yor [Yo92] formulated equation (1.2) in the form of equation (1.1) on the one-dimensional torus $\mathbb{R}/\mathbb{Z}$ when ϕ is the identity automorphism and (ξ_k) is a general noise process. In particular, [Yo92] proved that extremal solutions of the equation (1.1) are in one-to-one correspondence with points on the coset space $(\mathbb{R}/\mathbb{Z})/M$, where M is a closed subgroup of $\mathbb{R}/\mathbb{Z}$. When ϕ is the identity automorphism, equation (1.1) was considered on general compact groups [AkUY08], and when the noise law (ξ_k) is stationary, equation (1.1) was considered on abelian groups [Ta09]. The main results of [AkUY08, HiY10, Ta09] extended the result of [Yo92] and proved that the extremal solutions can be identified with G/H where H is a certain compact subgroup of G if G is abelian or compact.

Assuming that the noise process (ξ_k) is stationary, we obtain the following extension of [Yo92] to all locally compact distal groups G (that is, e is not in the closure of $\{gxg^{-1} \mid g \in G\}$ for any $x \in G \setminus \{e\}$) when the automorphism ϕ is distal on G (that is, e is not in the closure of $\{\phi^n(x) \mid n \in \mathbb{Z}\}$ for any $x \in G \setminus \{e\}$).

Theorem 1.1 *Let G be a locally compact distal group and ϕ be a distal automorphism of G. Let $(\xi_k)_{k\in\mathbb{Z}}$ be G-valued random variables with common law μ. Suppose the equation*

$$\eta_k = \xi_k\phi(\eta_{k-1}), \quad k \in \mathbb{Z} \tag{1.4}$$

has a solution. Then there exists a compact subgroup K_μ such that for any z in the support of μ, μz^{-1} is supported on $K_\mu = z\phi(K_\mu)z^{-1}$ and there is a one-to-one correspondence between left K_μ-invariant probability measures λ on G and the laws (λ_k) of the solutions (η_k) of the equation (1.4)*, given by $\lambda_k = z_k\phi^k(\lambda)$ for all $k \in \mathbb{Z}$, where z_k are given by*

$$z_k = \begin{cases} z\phi(z)\cdots\phi^{k-1}(z) & k > 0,\\ e & k = 0,\\ \phi^{-1}(z^{-1})\cdots\phi^{k}(z^{-1}) & k < 0,\end{cases} \tag{1.5}$$

for any z in the support of μ. Moreover, extremal solutions (η_k) of the equation (1.4) *are in one-to-one correspondence with the elements of the coset space $K_\mu\backslash G$.*

It is easy to verify by induction that z_k $(k \in \mathbb{Z})$ given in (1.5) satisfy $z_{k+1} = z\phi(z_k)$ for all $k \in \mathbb{Z}$, which is often used here.

2 Preliminaries

Let G be a locally compact (Hausdorff) group and let $\mathrm{Aut}(G)$ be the group of all bi-continuous automorphisms of G. Let ϕ be an automorphism of G. For a (regular

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Borel) probability measure μ on G, we define probability measures $\check{\mu}$ and $\phi(\mu)$ on G by $\check{\mu}(E) = \mu(E^{-1})$ and $\phi(\mu)(E) = \mu(\phi^{-1}(E))$ for any Borel subset E of G.

For any two probability measures μ and λ, the convolution of μ and λ is denoted by $\mu * \lambda$ and is defined by

$$\mu * \lambda(E) = \int \mu(Ex^{-1})d\lambda(x)$$

for any Borel subset E of G. For $n \geq 1$ and for a probability measure μ on G, μ^n denotes the n-th convolution power of μ.

For $x \in G$ and a probability measure μ on G, $x\mu$ (resp., μx) denotes $\delta_x * \mu$ (resp., $\mu * \delta_x$).

For a compact subgroup K of G, ω_K denotes the normalized Haar measure on K and a probability measure λ on G is called left K-invariant if $x\lambda = \lambda$ for all $x \in K$ (which is equivalent to $\omega_K * \lambda = \lambda$, by [He77, Theorem 1.2.7]).

We say that a sequence (λ_n) of probability measures on G converges (in the weak topology) to a probability measure λ on G if $\int f d\lambda_n \to \int f d\lambda$ for all continuous bounded functions f on G.

A set $\mathcal{F}$ of probability measures on G is said to be tight if for each $\epsilon > 0$ there is a compact set C_ϵ of G such that $\rho(C_\epsilon) > 1 - \epsilon$ for all $\rho \in \mathcal{F}$. It follows from Prohorov's Theorem that $\mathcal{F}$ is tight if and only if $\mathcal{F}$ is relatively compact in the space of probability measures on G equipped with weak topology (*cf.* [He77, Theorem 1.1.11]).

Let $(\xi_k)_{k\in\mathbb{Z}}$ be G-valued random variables. We are interested in investigating the law of random variables (η_k) that satisfies the stochastic difference equation

$$\eta_k = \xi_k \phi(\eta_{k-1}), \quad k \in \mathbb{Z}, \tag{2.1}$$

where ξ_k is independent of η_j for all $j < k$. [AkUY08] and [Yo92] considered only negative k, we also could have considered $k \in -\mathbb{N}$, but that would not have made any difference. Since we are interested in the law of the solutions of equation (2.1), we will be studying the corresponding convolution equation

$$\lambda_k = \mu_k * \phi(\lambda_{k-1}) \tag{2.2}$$

for all $k \in \mathbb{Z}$, where μ_k and λ_k are the laws of ξ_k and η_k respectively. It may be noted that [HiY10, Lemma 4.3(ii)] asserts that for a solution $(\lambda_k)_{k\in\mathbb{Z}}$ of equation (2.2) there exists a solution $(\eta_k)_{k\in\mathbb{Z}}$ of equation (2.1) whose marginal laws are $(\lambda_k)_{k\in\mathbb{Z}}$.

We consider equation (2.1) when ξ_k is stationary on the following type of locally compact groups and automorphisms.

Definition 2.1 A group Γ of automorphisms of a locally compact group G is called *distal* on G if e is not in the closure of the orbit $\Gamma(x) = \{\phi(x) \mid \phi \in \Gamma\}$ for any $x \in G \setminus \{e\}$. An automorphism ϕ of a locally compact group G is called distal if the group generated by ϕ is distal on G. A locally compact group G is called distal if the group of inner-automorphisms is distal on G. A locally compact group G is called *pointwise distal* if each inner-automorphism is distal on G.

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Compact extension of nilpotent groups, connected groups of polynomial growth, and SIN-groups (that is, groups having small invariant neighborhoods) are distal groups, and distal groups are pointwise distal (*cf.* [Ro86]). The class of (pointwise) distal groups is closed under direct product resulting in a rich class of groups satisfying the distal conditions.

It is easy to see that if an automorphism ϕ preserves a metric on G, then ϕ is distal on G. All unipotent matrices on finite-dimensional vector spaces are distal. It follows from the definition that inner-automorphisms of distal groups are distal. In the final section we give examples of compact groups in which all automorphisms are distal.

Given an automorphism ϕ of a locally compact group G, the following type of group is useful in our approach to equation (2.1). The semidirect product of $\mathbb{Z}$ and G (with respect to ϕ) is denoted by $\mathbb{Z} \ltimes_\phi G$ and is defined by

$$(n, g)(m, h) = \big(n + m, g\phi^n(h)\big)$$

for all $n, m \in \mathbb{Z}$ and $g, h \in G$. Equipped with the product topology of $\mathbb{Z} \times G$, $\mathbb{Z} \ltimes_\phi G$ is a locally compact group. Since $\mathbb{Z}$ has discrete topology, $\{(0, x) \mid x \in G\}$ is an open subgroup of $\mathbb{Z} \ltimes_\phi G$ and G will be identified with the open subgroup $\{(0, x) \mid x \in G\}$ under the map $x \mapsto (0, x)$.

Given a probability measure μ on a locally compact group G and an automorphisms ϕ of G, we will also be studying probability measures $n \otimes \mu$ on $\mathbb{Z} \ltimes_\phi G$ defined by

$$n \otimes \mu(A \times B) = \delta_n(A)\mu(B)$$

for any subset A of $\mathbb{Z}$ and any Borel subset B of G. The measure $0 \otimes \mu$ will be simply written as μ.

3 Distal Groups and Distal Automorphisms

In this section we prove the following result, to be used in the proof of Theorem 1.1. G is assumed to be a metrizable group.

Proposition 3.1 *Let G be a locally compact group. Suppose $\phi \in \operatorname{Aut}(G)$ and Γ is a subgroup of $\operatorname{Aut}(G)$ such that Γ and ϕ are distal on G and ϕ normalizes Γ (that is, $\phi\Gamma\phi^{-1} = \Gamma$). Let $\widetilde{\Gamma}$ be the group generated by ϕ and Γ. Then $\widetilde{\Gamma}$ is distal on G.*

In particular, if a locally compact group G is distal and $\phi \in \operatorname{Aut}(G)$ is distal on G, then $\mathbb{Z} \ltimes_\phi G$ is distal.

We first note that the group generated by ϕ and Γ is $\bigcup_n \phi^n\Gamma = \bigcup_n \Gamma\phi^n$ if $\phi\Gamma\phi^{-1} = \Gamma$. We now prove Proposition 3.1 for connected Lie groups, for compact groups, and for totally disconnected groups separately and combine these to obtain the general case.

Proposition 3.2 *Let G be a totally disconnected locally compact group. If Γ, ϕ, and $\widetilde{\Gamma}$ are as in Proposition* 3.1*, then $\widetilde{\Gamma}$ is distal on G.*

Proof If e is in the closure of $\widetilde{\Gamma}(x)$ for some $x \in G$. By [JaR07], G has small ϕ-invariant compact open subgroups, hence e is in the closure of $\Gamma(x)$ as $\widetilde{\Gamma} = \bigcup_n \phi^n\Gamma$. Since Γ is distal on G, $x = e$. Thus, $\widetilde{\Gamma}$ is distal on G. ∎

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We next consider the connected Lie group case.

Proposition 3.3 *Let G be a connected Lie group. If Γ, ϕ, and $\widetilde{\Gamma}$ are as in Proposition* 3.1, *then $\widetilde{\Gamma}$ is distal on G.*

Proof Let $\mathcal{G}$ be the Lie algebra of G. By identifying each automorphism of G with its corresponding differential on the Lie algebra of G, we may view $\mathrm{Aut}(G)$ as a group of linear transformations on $\mathcal{G}$. Since Γ and ϕ are distal on G, [Ab81, Theorem 1.1] implies that Γ and ϕ are also distal on $\mathcal{G}$. In view of [Ab81, Theorem 1.1], it is sufficient to prove that $\widetilde{\Gamma}$ is distal on $\mathcal{G}$.

Let $V = \{v \in \mathcal{G} \mid \Gamma(v) \text{ is bounded}\}$. Then V is a non-trivial Γ-invariant subspace of $\mathcal{G}$ (*cf.* [CoG74] for non-trivialness of V). Since Γ is distal on $\mathcal{G}$, Γ is distal on V. Since $\Gamma(v)$ is bounded for any $v \in V$, Γ restricted to V is contained in a compact group of linear transformations of V. This implies that V has a basis of small Γ-invariant neighborhoods of 0. Since $\phi\Gamma\phi^{-1} = \Gamma$, V is ϕ-invariant. If 0 is in the closure of $\widetilde{\Gamma}(v) = \bigcup \Gamma\phi^n(v)$ for some $v \in V$, then 0 is in the closure of $\{\phi^n(v) \mid n \in \mathbb{Z}\}$ as V has small Γ-invariant neighborhoods of 0. Since ϕ is distal, $v = 0$. Thus, $\widetilde{\Gamma}$ is distal on V. Since Γ and ϕ are distal on $\mathcal{G}$, Γ and ϕ are also distal on $\mathcal{G}/V$. Since dimension of $\mathcal{G}/V$ is smaller than the dimension of $\mathcal{G}$, by induction we get that $\widetilde{\Gamma}$ is distal on $\mathcal{G}/V$. Hence $\widetilde{\Gamma}$ is distal on $\mathcal{G}$. ■

We now prove the final case, compact groups.

Proposition 3.4 *Let G be a compact group. If Γ, ϕ, and $\widetilde{\Gamma}$ are as in Proposition* 3.1, *then $\widetilde{\Gamma}$ is distal on G.*

Proof Since a factor of distal action is distal, using Proposition 3.2, we may assume that G is a compact connected group. Let $I(G)$ be the group of inner-automorphisms of G and let Δ be the group generated by Γ and $I(G)$. Since $I(G)$ is compact, Δ is distal on G. Since $I(G)$ is normal in $\mathrm{Aut}(G)$ and ϕ normalizes Γ, ϕ normalizes Δ. Let $\widetilde{\Delta}$ be the group generated by Δ and ϕ.

Let $x \in G$ be such that e is in the closure of $\widetilde{\Gamma}(x)$. Let T be a maximal, compact, connected, abelian subgroup of G containing x (*cf.* [HoM98, Theorem 9.32]) and $O_T = \{\alpha \in \mathrm{Aut}(G) \mid \alpha(T) = T\}$. Then $\mathrm{Aut}(G) = I(G)O_T$ (*cf.* [HoM98, Corollary 9.87]). Since Δ and $\widetilde{\Delta}$ contain $I(G)$, we get that $\Delta = I(G)\Delta_T$ and $\widetilde{\Delta} = I(G)\widetilde{\Delta}_T$, where $\Delta_T = \Delta \cap O_T$ and $\widetilde{\Delta}_T = \widetilde{\Delta} \cap O_T$.

Let $\phi_1 \in I(G)$ and $\phi_2 \in \widetilde{\Delta}_T$ be such that $\phi = \phi_1\phi_2$. Since G contains a basis of $I(G)$-invariant neighborhoods of e, $\phi_2 = \phi_1^{-1}\phi$ is distal on G. Since both ϕ and ϕ_1 normalize Δ, ϕ_2 also normalizes Δ. Since $\phi_2 \in O_T$, ϕ_2 normalizes Δ_T. Let Λ be the group generated by ϕ_2 and Δ_T. Then T is invariant under Λ.

Let $\alpha \in \widetilde{\Delta}_T$. Then $\alpha = \alpha_1\alpha_2\phi^n$ for some $\alpha_1 \in I(G)$, $\alpha_2 \in \Gamma$ and $n \in \mathbb{Z}$. Since $\phi = \phi_1\phi_2$ with $\phi_1 \in I(G)$ and $I(G)$ is normalized by all automorphisms of G, we get that $\alpha = \alpha_1'\alpha_2\phi_2^n$ for some $\alpha_1' \in I(G)$. Since $\Gamma \subset \Delta = I(G)\Delta_T$, $\alpha = \alpha_1''\alpha_2'\phi_2^n$ for some $\alpha_1'' \in I(G)$ and $\alpha_2' \in \Delta_T$. Thus, $\widetilde{\Delta}_T \subset I(G)\Lambda$, hence $\widetilde{\Delta} = I(G)\widetilde{\Delta}_T \subset I(G)\Lambda$.

Since ϕ_2 and Δ_T are distal on G and T is invariant under both ϕ_2 and Δ_T, ϕ_2 and Δ_T are distal on T. Since ϕ_2 normalizes Δ_T, [Ra09, Lemma 5.3] implies that Λ is distal on T. Since e is in the closure of $\widetilde{\Gamma}(x) \subset \widetilde{\Delta}(x) \subset I(G)\Lambda(x)$, e is in the closure of $I(G)\Lambda(x)$. Since $I(G)$ is compact, e is in the closure of $\Lambda(x)$. Since $x \in T$ and Λ is distal on T, we get that $x = e$. Thus, $\widetilde{\Gamma}$ is distal on G. ∎

Proof of Proposition 3.1 Let G^0 be the connected component of identity in G. Then G^0 is a $\widetilde{\Gamma}$-invariant closed normal subgroup of G. By [RaS10, Theorem 3.3], Γ and ϕ are distal on G/G^0. Proposition 3.2 implies that $\widetilde{\Gamma}$ is distal on G/G^0. Since G^0 is a connected locally compact group, G^0 contains a maximal compact normal subgroup M such that G^0/M is a Lie group. Then M is a characteristic subgroup of G^0. By [RaS10, Theorem 3.1], Γ and ϕ are distal on G^0/M. Proposition 3.3 implies that $\widetilde{\Gamma}$ is distal on G^0/M. Now applying Proposition 3.4 we get that $\widetilde{\Gamma}$ is distal on M. Thus we have shown that $\widetilde{\Gamma}$ is distal on M, G^0/M and G/G^0. Hence $\widetilde{\Gamma}$ is distal on G.

For the second part, let G be a distal group and let $\phi \in \mathrm{Aut}(G)$ be distal on G. Then the group of all inner-automorphisms of G is distal on G, and it can easily be seen that the group of all inner-automorphisms is normalized by any automorphism. Thus, the first part implies that the group generated by inner-automorphisms and ϕ is distal on G. Since $(\mathbb{Z} \ltimes_\phi G)/G$ is discrete, we get that $\mathbb{Z} \ltimes_\phi G$ is distal. ∎

4 Dissipating Measures

A probability measure μ on a locally compact group G is called *dissipating* if for any compact set C in G, $\sup_{x \in G} \mu^n(Cx) \to 0$.

In the study of dissipating measures the smallest closed normal subgroup a coset of which contains the support of μ plays a crucial role. Let N_μ denote this normal subgroup of G. Let G_μ be the closed subgroup generated by the support of μ. If G_μ is non-compact and G_μ/N_μ is compact, then [JaRW96] showed that μ is dissipating. Many sufficient conditions (on G_μ or on μ) for μ to be dissipating have been provided by [Ja99, Ja07], for instance if G_μ is not amenable, then μ is dissipating ([Ja07, Corollary 3.6]).

We will now provide a necessary and sufficient condition for equation (2.1) to have a solution.

Proposition 4.1 *Let G be a locally compact group and let $(\xi_k)_{k\in\mathbb{Z}}$ be G-valued random variables with common law μ. Let ϕ be an automorphism of G. Then there is a solution (η_k) of the equation*

$$\eta_k = \xi_k \phi(\eta_{k-1}), \quad k \in \mathbb{Z}$$

if and only if the probability measure $\rho = 1 \otimes \mu$ on $\widetilde{G} = \mathbb{Z} \ltimes_\phi G$ is not dissipating.

Remark 4.2 If ϕ is the identity in Proposition 4.1, then $\rho^n = n \otimes \mu^n$, hence for any compact set C of G, $\sup_{a \in \widetilde{G}} \rho^n(Ca) = \sup_{x \in G} \mu^n(Cx)$. Since G is open in $\widetilde{G}$ and ρ is dissipating if and only if $\sup_{x \in \widetilde{G}} \rho^n(Ex) \to 0$ for some compact neighborhood E of e, we get that ρ is dissipating if and only if μ is dissipating. Thus, when ϕ is trivial, the equation in Proposition 4.1 has a solution if and only if μ is not dissipating.

Proof We first define μ_j by $\mu_j = \mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)$ for $j \geq 1$ and let $\mu_0 = \delta_e$. If (η_k) is a sequence of G-valued random variables such that $\eta_k = \xi_k \phi(\eta_{k-1})$ for all

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$k \in \mathbb{Z}$, let λ_k be the law of η_k. Then the corresponding convolution equation is

$$\lambda_k = \mu * \phi(\lambda_{k-1})$$

for all $k \in \mathbb{Z}$. Iterating the convolution equation we get that

$$\lambda_k = \mu_j * \phi^j(\lambda_{k-j})$$

for all $k \in \mathbb{Z}$ and all $j \geq 1$. It follows from [He77, Theorems 1.2.21(iii)] that there is a sequence (g_j) in G such that the sequence $(\mu_j g_j^{-1})$ is tight.

Consider the group $\widetilde{G} = \mathbb{Z} \ltimes_\phi G$ and the measure $\rho = 1 \otimes \mu$ on $\widetilde{G}$. Then $\rho^j = j \otimes \mu_j$, hence we get that $\rho^j(j, g_j)^{-1} = (j \otimes \mu_j)(-j, \phi^{-j}(g_j^{-1})) = \mu_j g_j^{-1}$. This implies that $(\rho^j(j, g_j)^{-1})$ is tight. Then there is a compact set C of $\widetilde{G}$ such that $\rho^j(C(j, g_j)) > \frac{1}{2}$ for all $j \geq 1$. This proves that ρ is not dissipating.

For the converse, suppose that ρ on $\widetilde{G} = \mathbb{Z} \ltimes_\phi G$ is not dissipating. We first assume that G is separable. Then by [Cs66, Theorem 3.1], there is a sequence (n_j, g_j) in $\widetilde{G}$ such that $(\rho^j(n_j, g_j)^{-1})$ converges (see also [To65]). Now since $\rho^j = j \otimes \mu_j$, we get that $\rho^j(n_j, g_j)^{-1} = (j - n_j) \otimes \mu_j \phi^{j-n_j}(g_j^{-1})$, hence $(\mu_j x_j)$ converges for $x_j = \phi^{j-n_j}(g_j^{-1})$. Let $\gamma = \lim \mu_j x_j$. Then $\gamma = \mu * \phi(\lim(\mu_{j-1} x_{j-1})) * \phi(x_{j-1}^{-1}) x_j$, hence by [He77, Theorems 1.2.21(ii) and 1.1.11], we get that $(\phi(x_{j-1}^{-1}) x_j)$ is relatively compact. If z is the inverse of a limit point of $(\phi(x_{j-1}^{-1}) x_j)$, then $\gamma = \mu * \phi(\gamma) z^{-1}$. Now define z_k for $k \in \mathbb{Z}$ as in equation (1.5). Then it is easy to verify by induction that $z_{k+1} = z\phi(z_k)$ for all $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, let $\lambda_k = \gamma z_k$. Then

$$\lambda_{k+1} = \gamma z_{k+1} = \mu * \phi(\gamma) z^{-1} z_{k+1} = \mu * \phi(\gamma)\phi(z_k) = \mu * \phi(\lambda_k)$$

for all $k \in \mathbb{Z}$.

In general, suppose G is any locally compact group. Let G_1 be the closed subgroup generated by the support of the probability measure $\frac{1}{2}[1 \otimes \mu + 0 \otimes \mu]$. Then G_1 is a σ-compact closed subgroup of $\mathbb{Z} \ltimes_\phi G$ and contains the support of μ. Since G_1 contains the support of $1 \otimes \mu$, $(1, e) \in G_1$. This implies that $G \cap G_1$ is a closed, σ-compact, ϕ-invariant subgroup of G containing the support of μ. Now replacing G by the smallest ϕ-invariant closed subgroup of G containing the support of μ, we may assume that G is σ-compact. This implies that $\mathbb{Z} \ltimes_\phi G$ is σ-compact. Then by [HeR79, Theorem 8.7], there exists a compact normal subgroup K of $\mathbb{Z} \ltimes_\phi G$ such that $(\mathbb{Z} \ltimes_\phi G)/K$ is separable. Since $(\mathbb{Z} \ltimes_\phi G)/G$ has no compact subgroup, $K \subset G$. Since K is a normal subgroup of $\mathbb{Z} \ltimes_\phi G$, K is ϕ-invariant. This shows that G contains a ϕ-invariant compact normal subgroup K such that G/K is separable. Let μ' be the image of μ on G/K. Since K is compact, $1 \otimes \mu'$ is not dissipating on $(\mathbb{Z} \ltimes_\phi G)/K$. By the previous case, there exist probability measures λ_k' on G/K such that $\lambda_k' = \mu' * \phi(\lambda_{k-1}')$ for all $k \in \mathbb{Z}$. It follows from [He77, 1.2.15(iii)] that there exists probability measures λ_k on G such that $\lambda_k * \omega_K = \lambda_k$ and image of λ_k on G/K is λ_k'. Since K is ϕ-invariant, $\mu * \phi(\lambda_{k-1}) * \omega_K = \mu * \phi(\lambda_{k-1})$ for all $k \in \mathbb{Z}$. Since both λ_k and $\mu * \phi(\lambda_{k-1})$ are projected onto the same probability measure λ_k' on G/K, by [He77, Theorem 1.2.15(iii)] we get that $\lambda_k = \mu * \phi(\lambda_{k-1})$ for all $k \in \mathbb{Z}$. ∎

Using Proposition 4.1 we now provide an easy sufficient condition so that equation (1.4) has a solution. Proposition 4.3 of [Ja99] provides some further sufficient conditions for equation (2.1) to have a solution.

Corollary 4.3 *Let G be a locally compact group and let $(\xi_k)_{k\in\mathbb{Z}}$ be G-valued random variables with common law μ. Let ϕ be an automorphism of G. Suppose there is a compact subgroup K of G such that μ is supported on K and $\phi(K) \subset K$. Then there is a solution (η_k) of the equation $\eta_k = \xi_k\phi(\eta_{k-1})$, $\quad k \in \mathbb{Z}$.*

Proof In view of Proposition 4.1, it is sufficient to prove that $\rho = 1 \otimes \mu$ is not dissipating. If there is a compact subgroup K such that $\phi(K) \subset K$ and μ is supported on K, then $\rho^n * \delta_{(-n,e)} = \mu * \phi(\mu) * \cdots * \phi^{n-1}(\mu)$ is also supported on K and hence ρ is not dissipating. ∎

Remark 4.4 If G is compact, then it follows from Corollary 4.3 that there exists a solution to equation (2.1), but it is easy to verify that $\lambda_k = \omega_G$ for all $k \in \mathbb{Z}$ is a solution.

5 Shifted Convolution Property

A probability measure μ on a locally compact group G is said to have the shifted convolution property if either μ is dissipating or there is a compact subgroup K of G and a $g \in G$ such that $\mu^n g^{-n} \to \omega_K$ and $gKg^{-1} = K$. The shifted convolution property was studied in details in [RaS10], where it was shown that all probability measures on a locally compact group G have the shifted convolution property if and only if the group G is pointwise distal (see [RaS10, Theorem 6.1]). We first prove the following result, which provides a sufficient condition for the existence of large collection of solutions.

Proposition 5.1 *Let G be a locally compact group and let μ be a probability measure on G. Suppose there is a compact subgroup K of G such that for any z in the support of μ, μz^{-1} is supported on $K = z\phi(K)z^{-1}$. For any z in the support of μ and for any left K-invariant probability measure λ, define z_k as in equation* (1.5) *and λ_k by $\lambda_k = z_k\phi^k(\lambda)$. Then (λ_k) is a solution to the equation*

$$\lambda_k = \mu * \phi(\lambda_{k-1}), \quad k \in \mathbb{Z}. \tag{5.1}$$

Proof Assume that there is a compact subgroup K of G such that for any x in the support of μ, μx^{-1} is supported on $K = x\phi(K)x^{-1}$. Suppose z is in the support of μ and λ is a left K-invariant probability measure on G. For any $k \in \mathbb{Z}$, define z_k as in equation (1.5) and define λ_k by $\lambda_k = z_k\phi^k(\lambda)$. Then it is easy to see that $z_{k+1} = z\phi(z_k)$ for all $k \in \mathbb{Z}$ and $\lambda_{k+1} = z_{k+1}\phi^{k+1}(\lambda) = z\phi(\lambda_k)$ for all $k \in \mathbb{Z}$.

We first claim that λ_k is left K-invariant for all $k \geq 0$. Our claim is based on induction. For $k \geq 1$, if λ_{k-1} is left K-invariant, then $\phi(\lambda_{k-1})$ is left $\phi(K)$-invariant, hence for $x \in K$, $x\lambda_k = xz\phi(\lambda_{k-1}) = z\phi(\lambda_{k-1}) = \lambda_k$ as $\phi(K) = z^{-1}Kz$ implies $z^{-1}xz \in \phi(K)$. This proves that λ_k is left K-invariant if λ_{k-1} is left K-invariant. Since $\lambda_0 = \lambda$ is left K-invariant, induction implies that λ_k is left K-invariant for all $k \geq 0$. Similarly, we can prove that λ_k is left K-invariant for all $k \leq 0$.

Since μz^{-1} is supported on $K = z\phi(K)z^{-1}$, we get that $z^{-1}\mu$ is supported on $\phi(K)$. Since λ_{k-1} is left K-invariant, $\mu * \phi(\lambda_{k-1}) = z\phi(\lambda_{k-1}) = \lambda_k$ for all $k \in \mathbb{Z}$. Thus, (λ_k) is a solution to equation (5.1). ∎

Proposition 5.2 *Let G be a locally compact group and let ϕ be an automorphism of G. Let (ξ_k) be a sequence of G-valued random variables with common law μ. Suppose the measure $1 \otimes \mu$ is not dissipating and has the shifted convolution property on $\mathbb{Z} \ltimes_\phi G$. Then for any solution (η_k) of the equation*

$$\eta_k = \xi_k \phi(\eta_{k-1}), \quad k \in \mathbb{Z}, \tag{5.2}$$

and for any z in the support of μ, we have

(i) *a compact subgroup K_μ such that μ is supported on $K_\mu z = z\phi(K_\mu)$,*
(ii) *laws (λ_k) of solution (η_k) satisfy $\lambda_k = z\phi(\lambda_{k-1})$;*
(iii) *a one-to-one correspondence between left K_μ-invariant probability measures λ on G and the laws (λ_k) of the solutions (η_k) of the equation* (5.2) *given by $\lambda_k = z_k\phi^k(\lambda)$, where z_k is defined as in* (1.5).

Proof Let $\widetilde{G} = \mathbb{Z} \ltimes_\phi G$ and $\rho = 1 \otimes \mu$. Since ρ is not dissipating and has the shifted convolution property, there is a compact subgroup K of $\widetilde{G}$ and $g \in \widetilde{G}$ such that $\rho^k g^{-k} \to \omega_K$ and $gKg^{-1} = K$. Since $\widetilde{G}/G \simeq \mathbb{Z}$, we get that $K \subset G$. Let z be in the support of μ and $a = (1, z)$. Then a is in the support of ρ. By [Ei92, Theorem 4.3] we get that $\rho^k a^{-k} \to \omega_K$ and $aKa^{-1} = K$ (*cf.* [RaS10, Remark 1.2]). Then since $K \subset G$, $aKa^{-1} = K$ implies that $K = z\phi(K)z^{-1}$. Also, $\rho^k a^{-k} \to \omega_K$ implies that $\rho * \omega_K = \omega_K a$ and hence μ is supported on Kz. This proves (i).

We now prove (ii). Let (η_k) be a solution to equation (5.2) and let λ_k be the law of η_k, $k \in \mathbb{Z}$. We now claim that for $k \in \mathbb{Z}$, λ_k is left K-invariant. Define z_k as in (1.5) for any $k \in \mathbb{Z}$ and $\mu_j = \mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)$ for any $j \geq 1$. Then we get that $\rho^k a^{-k} = \mu_k z_k^{-1} \to \omega_K$. For $k \in \mathbb{Z}$,

$$\lambda_k = \mu * \phi(\lambda_{k-1}) = (\mu_i z_i^{-1}) * (z_i \phi^i(\lambda_{k-i})), \quad i \geq 1,$$

and hence by [He77, Theorems 1.2.21(ii) and 1.1.11], $(z_i\phi^i(\lambda_{k-i}))_{i\geq 1}$ is relatively compact. Thus, for any limit point ν of $(z_i\phi^i(\lambda_{k-i}))$, we get that $\lambda_k = \omega_K * \nu$. Thus, λ_k is left K-invariant. By (i), $z^{-1}\mu$ is supported on $\phi(K)$, hence for $k \in \mathbb{Z}$,

$$\lambda_k = \mu * \phi(\lambda_{k-1}) = zz^{-1}\mu * \phi(\lambda_{k-1}) = z\phi(\lambda_{k-1}),$$

as λ_{k-1} is left K-invariant. This proves (ii).

Let $\lambda = \lambda_0$. Then λ is left K-invariant, and for $k \geq 1$,

$$\lambda_k = z\phi(z)\cdots\phi^{k-1}(z)\phi^k(\lambda) = z_k\phi^k(\lambda).$$

For $k < 0$, $\lambda = z\phi(\lambda_{-1}) = \cdots = z\phi(z)\cdots\phi^{-k-1}(z)\phi^{-k}(\lambda_k)$, hence

$$\lambda_k = \phi^{-1}(z^{-1})\cdots\phi^k(z^{-1})\phi^k(\lambda) = z_k\phi^k(\lambda).$$

Thus, any solution of (5.2) is in the form given in (iii).

It follows from (i) that the conditions of Proposition 5.1 are satisfied. Thus, for a left K-invariant measure λ if we define (λ_k) as in the proposition, we get that λ_k satisfies $\lambda_k = \mu * \phi(\lambda_{k-1})$ for all $k \in \mathbb{Z}$. ∎

Corollary 5.3 *Let G, ϕ, and (ξ_k), μ be as in Proposition 5.2. Suppose $1 \otimes \mu$ has the shifted convolution property and $1 \otimes \mu$ is not dissipating. Then there exists a compact subgroup K such that extremal solutions (η_k) of the equation*

$$\eta_k = \xi_k \phi(\eta_{k-1}), \quad k \in \mathbb{Z} \tag{5.3}$$

are in one-to-one correspondence with the elements of the coset space $K \backslash G$.

Proof Let K be as in Proposition 5.2. Then it follows from Proposition 5.2 that left K-invariant measures and laws of solutions to equation (5.3) are in one-to-one correspondence.

Suppose λ is a left K-invariant measure and z is in the support of λ. Let $a \in K$ and U be a neighborhood of az. Then $\lambda(U) = \lambda(a^{-1}U) > 0$. This implies that az is in the support of λ. Thus, support of λ is a union of cosets of K. If the support of λ contains more than one coset, then the corresponding solution to the equation (5.3) is not extremal. This proves the corollary. ∎

We have the following converse to Proposition 5.2.

Proposition 5.4 *Let G be a locally compact group and let ϕ be an automorphism of G. Let (ξ_k) be G-valued random variables with common law μ. Suppose the laws of the solutions (η_k) of the equation*

$$\eta_k = \xi_k \phi(\eta_{k-1}), \quad k \in \mathbb{Z}$$

are left K-invariant for some compact subgroup K of G such that μz^{-1} is supported on $K = z\phi(K)z^{-1}$ for any z in the support of μ. Then $1 \otimes \mu$ on $\mathbb{Z} \ltimes_\phi G$ has the shifted convolution property.

Proof Let $\rho = 1 \otimes \mu$. We first assume that G is separable. Since the equation has solutions, by Proposition 4.1, $1 \otimes \mu$ on $\mathbb{Z} \ltimes_\phi G$ is not dissipating. As in Proposition 4.1, there are $x_j \in G$ and a probability measure γ on G such that $\mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)x_j \to \gamma$ and a solution (λ_k) with $\lambda_0 = \gamma$. The hypothesis implies that γ is left K-invariant.

Let z be in the support of μ and define z_k as in (1.5). Let $\mu_j = \mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)$ for $j \geq 1$. We now claim by induction that μ_j is supported on Kz_j for all $j \geq 1$. If μ_j is supported on Kz_j for some $j \geq 1$, then μ_{j+1} is supported on $Kz_j\phi^j(K)\phi^j(z)$. Since $Kz = z\phi(K)$, we get that $\phi^k(z)\phi^{k+1}(K) = \phi^k(K)\phi^k(z)$ for all $k \geq 0$. This shows that $z_k\phi^k(K) = Kz_k$, hence μ_{j+1} is supported on $Kz_j\phi^j(K)\phi^j(z) = Kz_j\phi^j(z) = Kz_{j+1}$. Since $\mu_1 = \mu$ is supported on Kz, the claim follows by induction.

For $j \geq 1$, let $\sigma_j = \mu_j x_j$. Then $\sigma_j * \check{\sigma}_j \to \gamma * \check{\gamma}$ and $\sigma_j x_j^{-1}$ is supported on Kz_j. This implies that $\sigma_j * \check{\sigma}_j$ is supported on K, hence $\gamma * \check{\gamma}$ is supported on K. Since γ is left K-invariant, $\gamma * \check{\gamma}$ is also left K-invariant and hence $\gamma * \check{\gamma} = \omega_K$. Now $\rho^j = j \otimes \sigma_{j-1}x_{j-1}^{-1}$, hence $\rho^j * \check{\rho}^j = \sigma_{j-1} * \check{\sigma}_{j-1}$ for all $j > 1$. This implies that $\rho^j * \check{\rho}^j \to \omega_K$. By [Ei92, Theorem 4.3], for any g in the support of ρ, $\rho^j g^{-j} \to \omega_K$. For any g in the support of $\rho = 1 \otimes \mu$, there is a z in the support of μ such that $g = (1, z)$ and hence $gKg^{-1} = z\phi(K)z^{-1} = K$. This proves that ρ has the shifted convolution property.

We now assume that G is any locally compact group. Then as in Proposition 4.1, replacing G by the smallest ϕ-invariant closed subgroup of G containing the support of μ, we may assume that G is σ-compact and hence $\mathbb{Z} \ltimes_\phi G$ is also σ-compact. Then by [HeR79, Theorem 8.7], each neighborhood U of e in G contains a compact normal subgroup K_U of $\mathbb{Z} \ltimes_\phi G$ such that $(\mathbb{Z} \ltimes_\phi G)/K_U$ is separable. Since K_U is a compact normal subgroup of $\mathbb{Z} \ltimes_\phi G$, K_U is ϕ-invariant. Thus, each neighborhood U of e in G contains a ϕ-invariant compact normal subgroup K_U such that G/K_U is separable. It can easily be verified that the assumptions in the proposition are valid for G/K_U with KK_U/K_U and for the image of μ on G/K_U. It follows from the previous case that image of ρ on $\mathbb{Z} \ltimes_\phi (G/K_U)$ has the shifted convolution property. By [RaS10, Proposition 2.3] we get that ρ itself has the shifted convolution property. ∎

We now extend the results of [AkUY08, Ta09, Yo92] when ξ_k is stationary on distal groups with distal automorphisms.

Proof of Theorem 1.1 Suppose G is a locally compact distal group and ϕ is a distal automorphism. Then as in Proposition 5.4, replacing G by the smallest ϕ-invariant closed subgroup of G containing the support μ, we may assume that each neighborhood U of e in G contains a ϕ-invariant compact normal subgroup K_U such that G/K_U is separable. By [RaS10, Theorem 3.1] we get that each G/K_U is distal and ϕ is distal on each G/K_U. Then Proposition 3.1 implies that $\mathbb{Z} \ltimes_\phi (G/K_U)$ is distal. By [RaS10, Theorem 6.1] we get that $1 \otimes \mu$ has the shifted convolution property on $\mathbb{Z} \ltimes_\phi (G/K_U)$. By [RaS10, Proposition 2.3] we get that $1 \otimes \mu$ has the shifted convolution property. Now the result follows from Proposition 5.2 and Corollary 5.3. ∎

Remark 5.5 Theorem 1.1 is true even if $\mathbb{Z} \ltimes_\phi G$ is a pointwise distal group. It is easy to construct examples of pointwise distal groups G with distal automorphisms ϕ so that $\mathbb{Z} \ltimes_\phi G$ is pointwise distal but not distal (see [Ra09, Example 1.1]).

Remark 5.6 We would like to remark that if $1 \otimes \mu$ does not have the shifted convolution property, then the conclusion on extreme points of the solutions in Theorem 1.1 may not be true even on compact groups. Consider the two dimensional torus $K = (\mathbb{R}/\mathbb{Z})^2$ and let ϕ be an automorphism of K such that

$$C(\phi) = \{x \in K \mid \phi^n(x) \to e \text{ as } n \to \infty\}$$

is dense in K, for instance if we take ϕ to be $\phi(x, y) = (x + y + \mathbb{Z}, x + 2y + \mathbb{Z})$ for all $x, y \in \mathbb{R}$, then $C(\phi) = \{(t + \mathbb{Z}, (\frac{1-\sqrt{5}}{2})t + \mathbb{Z}) \mid t \in \mathbb{R}\} \simeq \mathbb{R}$ is a vector (non-closed) subgroup of K and is dense in K. Take μ to be a probability measure on K such that support of μ is a compact subset contained in $C(\phi)$. Since ϕ on $C(\phi)$ is multiplication by $\frac{3-\sqrt{5}}{2}$, [Za96] implies that there is a probability measure ρ on $C(\phi)$ such that $\mu * \phi(\mu) * \cdots * \phi^i(\mu) \to \rho$ in the space of probability measures on $C(\phi)$. This implies that

$$\rho = \lim \mu * \phi(\mu) * \cdots * \phi^i(\mu) = \mu * \phi\big(\lim\big(\mu * \cdots * \phi^{i-1}(\mu)\big)\big) = \mu * \phi(\rho).$$

Taking $\lambda_k = \rho$ for all $k \in \mathbb{Z}$, we get a stationary solution to equation (2.1). Further, assume that $\mu \neq \delta_x$ for any $x \in K$. Then $\lambda_k = \rho$ are also not dirac measures. If $x\lambda_k =$

λ_k for some $x \in K$, then since $\lambda_k(C(\phi)) = 1$, we get that $\lambda_k(C(\phi)) = 1 = x\lambda_k(C(\phi))$, hence $x \in C(\phi)$ as $C(\phi)$ is a group. By [He77, Theorem 1.2.4], $\{g \in C(\phi) \mid g\rho = \rho\}$ is a compact subgroup of $C(\phi)$. Since $C(\phi)$ is a vector group, $C(\phi)$ has no non-trivial compact subgroup, and hence $x = e$. Thus, λ_k is not left invariant for any nontrivial compact subgroup of K. Hence, the conclusion on the extreme points of solutions in Theorem 1.1 does not hold.

We now present a situation where distality of ϕ is sufficient to guarantee the conclusion of Theorem 1.1; the proof is based on [Ja99, RaS10].

Theorem 5.7 *Let G be a locally compact group and let ϕ be a distal automorphism of G. Let (ξ_k) be a sequence of G-valued random variables with common law μ. If e is in the support of μ and the equation*

$$\eta_k = \xi_k\phi(\eta_{k-1}), \quad k \in \mathbb{Z} \tag{5.4}$$

has a solution, then the conclusions of Theorem 1.1 *hold.*

Proof Assume that e is in the support of μ and equation (5.4) has a solution. Let $\widetilde{G} = \mathbb{Z} \ltimes_\phi G$ and $\rho = 1 \otimes \mu$. We now prove that ρ has the shifted convolution property. Since equation (5.4) has a solution, ρ is not dissipating (*cf.* Proposition 4.1). Let N be the smallest closed normal subgroup of $\widetilde{G}$ such that a coset of N contains the support of μ. Since G is a closed normal subgroup of $\widetilde{G}$ and $\rho(G(1,e)) = 1$, $N \subset G$. Since N is normal in $\widetilde{G}$, N is a ϕ-invariant subgroup of G.

If $\rho(N(k,g)) = 1$ for some $g \in G$ and $k \in \mathbb{Z}$, then $\rho((k, Ng)) = 1$. This implies that $k = 1$ and $\mu(Ng) = 1$. Since e is in the support of μ, $e \in Ng$, hence $g \in N$. This implies that $\rho(N(1,e)) = 1$. By [Ja99, Theorem 3.9], ϕ restricted to N is contractive modulo a compact subgroup K (that is, $\phi^n(x)K \to K$ for all $x \in N$). Since ϕ is distal on G, ϕ is distal on N. By [RaS10, Corollary 3.2], $N = K$.

We denote the restriction of ϕ to N also by ϕ. Let $H = \mathbb{Z} \ltimes_\phi N$. Then ρ is supported on H. Since N is compact and ϕ is distal, we get that H is distal. By [RaS10, Theorem 6.1] we get that ρ has the shifted convolution property. Now the result follows from Proposition 5.2 and Corollary 5.3. ■

6 Examples

We first provide examples of groups for which the group of automorphisms is compact.

(i) *Compact p-adic Lie groups:* Let K be a compact p-adic Lie group. Then $\mathrm{Aut}(K)$ is a compact group (see [DidMS99, Corollary 8.35] or [Ra04]). The following are examples of compact p-adic Lie groups:

(a) If $\mathbb{Q}_p$ is the field of p-adic numbers with valuation $|\cdot|_p$, then $\mathbb{Z}_{p^n} = \{x \in \mathbb{Q}_p \mid |x|_p \leq p^{n-1}\}$ is a compact p-adic Lie group.
(b) The group $GL_k(\mathbb{Z}_p)$ of all invertible $k \times k$-matrices over $\mathbb{Z}_p$.
(c) Pro-p group of finite rank, that is a totally disconnected group of finite rank in which every open normal subgroup has index equal to some power of p.

(ii) *A Compact abelian group:* For $n \geq 1$, let A_n be the group of all n-th roots of unity and $A = \bigcup A_n$. Then A is a countable abelian group. Equip A with discrete topology. Let K be the dual of A. Then K is a compact (totally disconnected) metrizable group with dual A (see [HeR79, 24.15]).

Let K_n be the closed subgroup of K such that K/K_n is the dual of A_n. Since A_n is finite, K/K_n is finite. Then K_n is an open subgroup of K. Now, if $x \in \bigcap K_n$, then $x \in K_n$ for all $n \geq 1$. This implies that $a(x) = 1$ for all $a \in A_n$ and for all $n \geq 1$. Since $A = \bigcup A_n$, $a(x) = 1$ for all $a \in A$. Since A is the dual of K, $x = e$. Thus, $\bigcap K_n = e$.

Let α be an automorphism of K and let $\widehat{\alpha}$ be the corresponding dual automorphism on A. Then it is easy to see that $\widehat{\alpha}(A_n) = A_n$ for all $n \geq 1$. This implies that $\alpha(K_n) = K_n$. This proves that (K_n) is a sequence of arbitrarily small characteristic open subgroups, hence the group of automorphisms of K is compact.

(iii) *All automorphisms are distal but the group of automorphisms is not compact:* Let $\mathbb{R}/\mathbb{Z}$ be the one-dimensional torus and let K be the compact group in (i) or in (ii). Let $G = \mathbb{R}/\mathbb{Z} \times K$ be the direct product of $\mathbb{R}/\mathbb{Z}$ and K. Then G is a compact group.

Let τ be an automorphism of G. We now claim that there is an automorphism α of K and a character χ of K such that $\tau(z,x) = (z^{\pm 1}\chi(x), \alpha(x))$ for all $(z,x) \in G$. Since the connected component of identity in G is $\mathbb{R}/\mathbb{Z}$, we get that $\tau(\mathbb{R}/\mathbb{Z}) = \mathbb{R}/Z$, hence $\tau(z,e) = (z^{\pm 1}, e)$ for all $z \in \mathbb{R}/\mathbb{Z}$. Let $\alpha \colon K \to K$ be defined by $\alpha(x) = p(\tau(1,x))$ where $p \colon G \to K$ is the canonical projection of G onto K. It is easy see that α is a continuous homomorphism. If $\alpha(x) = e$, then $p(\tau(1,x)) = e$, and hence $\tau(1,x) = (z,e)$ for some $z \in \mathbb{R}/\mathbb{Z}$. But $\tau(z',e) = (z,e)$ for $z' = z$ or $z' = z^{-1}$. Since τ is an automorphism, $(z',e) = (1,x)$, hence $x = e$. This shows that α is injective. For $x \in K$, let $y \in K$ and $z \in \mathbb{R}/\mathbb{Z}$ be such that $\tau(z,y) = (1,x)$ as τ is onto. This implies that $\alpha(y) = p(\tau(1,y)) = p(\tau(z,y)) = x$. This proves that α is bijective. Continuity of α^{-1} follows from open mapping theorem, as K is a compact metrizable group (*cf.* [HeR79, 5.29]). Thus, α is an automorphism of K. Let $\chi \colon K \to \mathbb{R}/\mathbb{Z}$ be defined by $\chi(x) = q(\tau(1,x))$ where $q \colon G \to \mathbb{R}/\mathbb{Z}$ is the canonical projection of G onto $\mathbb{R}/\mathbb{Z}$. Then χ is a continuous homomorphism. For $z \in \mathbb{R}/\mathbb{Z}$ and $x \in K$, $\tau(z,x) = (z^{\pm 1}, e)\tau(1,x) = (z^{\pm 1}, e)(\chi(x), \alpha(x)) = (z^{\pm 1}\chi(x), \alpha(x))$. This proves the claim.

We now claim that τ is distal. Suppose that $(1,e)$ is in the closure of $\{\tau^n(z,x) \mid n \in \mathbb{Z}\}$. Then e is in the closure of $\{\alpha^n(x) \mid n \in \mathbb{Z}\}$. Since the group of automorphisms of K is compact, $x = e$. This implies that $(1,e)$ is in the closure of $\{\tau^n(z,e) \mid n \in \mathbb{Z}\} = \{(z^{\pm 1}, e)\}$, and hence $z = 1$. Thus, τ is distal. In fact, one can show that each τ generates a compact subgroup.

If K is not a finite group, then the group of automorphisms of G is not a compact group, as it is homeomorphic to $\{\pm 1\} \times \widehat{K} \times \mathrm{Aut}(K)$, where $\mathrm{Aut}(K)$ is the group of automorphisms of K.

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References

[Ab81] H. Abels, *Distal automorphism groups of Lie groups.* J. Reine Angew. Math. **329**(1981), 82–87. http://dx.doi.org/10.1515/crll.1981.329.82

[AkUY08] J. Akahori, C. Uenishi, and K. Yano, *Stochastic equations on compact groups in discrete negative time.* Probab. Theory Related Fields **140**(2008), no. 3–4, 569–593. http://dx.doi.org/10.1007/s00440-007-0076-z

[CoG74] J.-P. Conze and Y. Guivarc'h, *Remarques sur la distalité dans les espaces vectoriels.* C. R. Acad. Sci. Paris Sér. A **278**(1974), 1083–1086.

[Cs66] I. Csiszár, *On infinite products of random elements and infinite convolutions of probability distributions on locally compact groups.* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **5**(1966), 279–295. http://dx.doi.org/10.1007/BF00535358

[DidMS99] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups.* Second ed. Cambridge Studies in Advanced Mathematics, 61, Cambridge University Press, Cambridge, 1999.

[Ei92] P. Eisele, *On shifted convolution powers of a probability measure.* Math. Z. **211**(1992), no. 4, 557–574. http://dx.doi.org/10.1007/BF02571446

[He77] H. Heyer, *Probability measures on locally compact groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 94, Springer-Verlag, Berlin-New York, 1977.

[HeR79] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I. Structure of topological groups, integration theory, group representations.* Second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 115, Springer-Verlag, Berlin-New York, 1979.

[HiY10] T. Hirayama and K. Yano, *Extremal solutions for stochastic equations indexed by negative integers and taking values in compact groups.* Stochastic Process. Appl. **120**(2010), no. 8, 1404–1423. http://dx.doi.org/10.1016/j.spa.2010.04.003

[HoM98] H. Hofmann and S. A. Morris, *The structure of compact groups. A primer for the student—a handbook for the expert.* de Gruyter Studies in Mathematics, 25, Walter de Gruyter & Co., Berlin, 1998.

[JaRW96] W. Jaworski, J. Rosenblatt, and G. Willis, *Concentration functions in locally compact groups.* Math. Ann. **305**(1996), no. 4, 673–691. http://dx.doi.org/10.1007/BF01444244

[Ja99] ———, *On shifted convolution powers and concentration functions in locally compact groups.* In: Probability on algebraic structures (Gainesville, FL, 1999), Contemp. Math., 261, American Mathematical Society, Providence, RI, 2000, pp. 23–41.

[Ja07] ———, *Dissipation of convolution powers in a metric group.* J. Theoret. Probab. **20**(2007), no. 3, 487–503. http://dx.doi.org/10.1007/s10959-007-0072-3

[JaR07] W. Jaworksi and C. R. E. Raja, *The Choquet-Deny theorem and distal properties of totally disconnected locally compact groups of polynomial growth.* New York J. Math. **13**(2007), 159–174.

[Ke73] H. Kesten, *Random difference equations and renewal theory for products of random matrices.* Acta Math. **131**(1973), 207–248. http://dx.doi.org/10.1007/BF02392040

[Ra04] C. R. E. Raja, *A note on unitary representation problem with corrigenda to the articles: "Weak mixing and unitary representation problem" [Bull. Sci. Math. 124(2000), no. 7, 517–523] and "Identity excluding groups" [ibid. 126(2002), no. 9, 763–772].* Bull. Sci. Math. **128**(2004), no. 10, 803–809. http://dx.doi.org/10.1016/j.bulsci.2004.04.002

[Ra09] ———, *Distal actions and ergodic actions on compact groups.* New York J. Math. **15**(2009), 301–318.

[RaS10] C. R. E. Raja and R. Shah, *Distal actions and shifted convolution property.* Israel J. Math. **177**(2010), 391–411. http://dx.doi.org/10.1007/s11856-010-0052-7

[Ro86] J. Rosenblatt, *A distal property of groups and the growth of connected locally compact groups.* Mathematika **26**(1979), no. 1, 94–98. http://dx.doi.org/10.1112/S0025579300009669

[Ta09] Y. Takahashi, *Time evolution with and without remote past.* In: Advances in discrete dynamical systems, Adv. Stud. Pure Math., 53, Math. Soc. Japan, Tokyo, 2009, pp. 347–361.

[To65] A. Tortrat, *Lois de probabilité sur un espace topologique complètement régulier et produits infinis à termes indépendants dans un groupe topologique.* Ann. Inst. H. Poincaré Sect. B **1**(1964/1965), 217–237.

[Ts75] B. S. Tsirel'son, *An example of a stochastic differential equation that has no strong solution.* (Russian) Teor. Verojatnost. i Primenen. **20**(1975), no. 2, 427–430.

[Yo92] M. Yor, *Tsirel'son's equation in discrete time.* Probab. Theory Related Fields **91**(1992), no. 23, 135–152. http://dx.doi.org/10.1007/BF01291422

[Za96] O. K. Zakusilo, *Some properties of random vectors of the form* $\sum_{i=0}^{\infty} A^i \xi_i$. (Russian. English summary) Teor. Verojatnost. i Mat. Statist. Vyp. **13**(1975), 59–62, 162.

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