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CHANGES OF VARIABLES PRESERVING FOURIER-STIELTJES TRANSFORMS ON SIMPLY CONNECTED NILPOTENT LIE GROUPS

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1. Introduction. In [**1**] Beurling and Helson prove the following theorem.

THEOREM. *Let* φ: $\mathbf{R} \rightarrow \mathbf{R}$ *be a continuous map such that*

$$(1.1) \quad \sup_{u \in \mathbf{R}} \|e^{iu\varphi}\|_{B(\mathbf{R})} < \infty$$

Then φ *is affine.* (*Here* $B(\mathbf{R}) = \mathscr{F}M(\hat{\mathbf{R}})$ *denotes the space of Fourier-Stieltjes transforms on* $\mathbf{R}$.)

The hypothesis (1.1) of this theorem is equivalent to the statement that φ is a change of variables for Fourier-Stieltjes transforms. Explicitly this means that there is a constant $C(\varphi)$ such that

$$\|f \circ \varphi\|_{B(\mathbf{R})} \leqq C(\varphi)\|f\|_{B(\mathbf{R})} \quad \text{for all } f \in B(\mathbf{R}).$$

In this article we undertake the same question for simply connected nilpotent lie groups. The definition and elementary properties of Fourier-Stieltjes transforms on general locally compact groups may be found in [**2**] (Fourier-Stieltjes transforms are taken to be the coefficient functions of strongly continuous unitary representations of the group on Hilbert space.). In view of the translation invariance of Fourier-Stieltjes transforms we need only look at changes of variable which preserve identities.

Definition. Let G and H be SNL (simply-connected nilpotent lie) groups and φ: $G \rightarrow H$ a continuous map such that $\varphi(e) = e$. We say that φ is a *NCV* (*normalized change of variables*) *map* if there is a constant $C(\varphi)$ such that

$$\|f \circ \varphi\|_{B(G)} \leqq C(\varphi)\|f\|_{B(H)} \quad \text{for all } f \in B(H).$$

The goal of this article is the following result.

THEOREM. *Let G and H be SNL groups and φ: $G \rightarrow H$ a NCV map. Then either φ is a group homomorphism or φ is a group anti-homomorphism.*

Since group homomorphisms and anti-homomorphisms are NCV maps this theorem characterizes NCV maps.

We shall use the notation $\bar{G}$ for the lie algebra of a lie group G and $\bar{\psi}$ for the lie algebra homomorphism $\bar{\psi}$: $\bar{G} \rightarrow \bar{H}$ induced from a group homomorphism ψ: $G \rightarrow H$.

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2. Preliminary remarks and results. We start by indicating the main tools we shall need. All groups should be assumed locally compact.

(2.1) If φ_1 and φ_2 are NCV maps

$$H_1 \xrightarrow{\varphi_1} H_2 \xrightarrow{\varphi_2} H_3$$

so is $\varphi_2 \circ \varphi_1$.

We shall apply this often when one of φ_1 and φ_2 is a group homomorphism or anti-homomorphism. One special case is where H_2 is a direct product and φ_2 a canonical projection.

(2.2) Lemma. *Let*

$$G \xrightarrow{\varphi} H_0 \xrightarrow{j:\subseteq} H$$

where G, H_0 and H are SNL groups, and j is the inclusion of the subgroup H_0 into H. Then if $j \circ \varphi$ is a NCV map, so is φ.

Proof. Let $f \in B(H_0)$ with $||f||_B \leqq 1$. By the amenability of H_0 there exist $f_n \in A(H_0)$ with $||f_n||_B \leqq 1$ and $f_n \to f$ uniformly on compacta. Now the A-functions (unlike the B-functions) extend (Note: $A(G) = L^2(G) * L^2(G)$). Thus there are functions $\tilde{f}_n \in A(H)$ with $||\tilde{f}_n||_B \leqq 1$ and $\tilde{f}_n|_{H_0} = f_n$. Then by hypothesis $||f_n \circ \varphi||_B = ||\tilde{f}_n \circ j \circ \varphi||_B \leqq C(\varphi)$ for all n and $f_n \circ \varphi \to f \circ \varphi$ uniformly on compacta. It follows that $||f \circ \varphi||_B \leqq C(\varphi)$ as desired.

(2.3) Lemma. *Let*

$$G \xrightarrow{\pi} Q \xrightarrow{\varphi} H$$

where Q is some quotient group of G and π the canonical projection. Then if $\varphi \circ \pi$ is a NCV map, so is φ.

This follows easily from [**2**, (2.26)].

(2.4) Lemma. *Let G be a SNL group and let φ: $\mathbf{R} \to G$ be a NCV map. Then φ is a group homomorphism.*

Proof. We use induction on $\dim(G)$. If $\dim(G) = 1$, the result follows from the work of Beurling and Helson [**1**]. Also observe that if G is abelian it is a finite direct product of copies of $\mathbf{R}$ so that the result will follow by (2.1).

Now assume that $\dim(G) > 1$ and that the result holds true of all groups of lower dimension. Let Z be the (non-trivial) centre of G and π: $G \to G/Z$ the canonical projection. By (2.1) and the induction hypothesis there exists $Q \in \bar{G}/\bar{Z}$ with

$$\pi \circ \varphi(t) = \exp\,(tQ).$$

Now select $X \in \bar{G}$ with $\bar{\pi}(X) = Q$. Then one sees that there is a map ϑ: $\mathbf{R} \to Z$ such that

$$\varphi(t) = \vartheta(t) \exp\,(tX).$$

Thus φ takes values in the abelian subgroup H of G such that $\bar{H} = \mathbf{R}X + \bar{Z}$. The result now follows by (2.2) and the above treatment of the abelian case.

(2.5) Let G and H be SNL groups and let $\varphi: G \to H$ be a NCV map. Then by (2.1) and (2.4) the restriction of φ to a 1-parameter subgroup of G yields a group homomorphism. Thus we have a map $u: \bar{G} \to \bar{H}$ such that

$$\varphi(\exp(tX)) = \exp(tu(X)) \quad \text{for all } t \in \mathbf{R}, X \in \bar{G}.$$

We claim that u is a linear map.

Proof. It is clear that u is continuous and homogeneous (i.e., $u(tX) = tu(X)$ for all $t \in \mathbf{R}$ and for all $X \in \bar{G}$). Fix $X \in \bar{G}$ and define $\psi_X : G \to H$ by

$$\psi_X(x) = (\varphi(\exp(X)))^{-1}\varphi(\exp(X)x) \quad \text{for all } x \in G.$$

By the translation invariance of Fourier-Stieltjes transforms, we see that ψ_X is again a NCV map. Let $v(X, \cdot): \bar{G} \to \bar{H}$ be the mapping associated to ψ_X. Then we have

$$\varphi(\exp(X)\exp(Y)) = \exp(u(X))\exp(v(X, Y)) \quad \text{for all } X, Y \in \bar{G}$$

or

$$u(B(X, Y)) = B(u(X), v(X, Y)) \quad \text{for all } X, Y \in \bar{G}$$

where B is the Baker-Campbell-Hausdorff function (satisfying $\exp(B(X, Y)) = \exp(X)\exp(Y)$) which is actually a polynomial in our case. Now replacing X and Y by tX and tY and using the fact that v is homogeneous in the second variable, we have

$$u(B(tX, tY)) = B(tu(X), tv(tX, Y)) \quad \text{for all } t \in \mathbf{R}, \text{ and for all } X, Y \in \bar{G}.$$

Now v is clearly continuous on $\bar{G} \times \bar{G}$ so dividing by t (for $t \neq 0$) and letting $t \to 0$ we find

$$u(X + Y) = u(X) + v(0, Y).$$

But $v(0, Y) = u(Y)$. Hence u is linear.

For later use we observe that φ is a group homomorphism if and only if

$$u[X, Y] = [u(X), u(Y)] \quad \text{for all } X, Y \in \bar{G}$$

and that φ is a group anti-homomorphism if and only if

$$u[X, Y] = -[u(X), u(Y)] \quad \text{for all } X, Y \in \bar{G}.$$

(2.6) Lemma. *Let G be a SNL group and let $\varphi; G \to \mathbf{R}$ be a NCV map. Then φ is a group homomorphism.*

Proof. We identify $\bar{\mathbf{R}}$ to $\mathbf{R}$ and define a map $w: \bar{G} \times \bar{G} \to \mathbf{R}$ by

$$\varphi(\exp(X)\exp(Y)) = u(X) + u(Y) + w(X, Y) \quad \text{for all } X, Y \in \bar{G}.$$

Now $v(X, \cdot)$ is linear and $v(X, \cdot) = u + w(X, \cdot)$ so w is linear in the second variable. By symmetry, w is a bilinear map. Now $e^{i\varphi}$ lies in $B(G)$ and hence is uniformly continuous on G (for both left and right uniform structures). Thus for all $\epsilon > 0$ there exists a neighbourhood V of 0 in $\bar{G}$ such that

$$|\varphi(\exp(X)\exp(Y)) - \varphi(\exp(X))| < \epsilon \quad \text{for all } X \in \bar{G}, \text{ and for all } Y \in V.$$

Thus

$$|u(Y) + w(X, Y)| < \epsilon \quad \text{for all } X \in \bar{G}, \text{ and for all } Y \in V,$$

and by virtue of the bilinearity of w this is only possible if w is identically zero. This completes the proof.

We can now use (2.1) to replace $\mathbf{R}$ above by $\mathbf{R}^n$ and have the same conclusion.

(2.7) *Free lie algebras.* We shall denote by $\mathfrak{g}_\infty$ the free lie algebra on two generators P and Q. The universal property enjoyed by $\mathfrak{g}_\infty$ is the following. Whenever $\mathfrak{h}$ is a lie algebra and whenever $X, Y \in \mathfrak{h}$ there is a unique lie algebra homomorphism α, $\alpha: \mathfrak{g}_\infty \to \mathfrak{h}$ such that $\alpha(P) = X$ and $\alpha(Q) = Y$. We refer the reader to [**3**, p. 167 et seq.] for a discussion of this topic.

There is a natural grading of

$$\mathfrak{g}_\infty = \bigoplus_{n=1}^{\infty} \mathfrak{a}_n$$

where in Jacobson's terminology, $\mathfrak{a}_n$ consists of those elements which are homogeneous of degree n. One way of viewing this is to let $\bar{\alpha}_t$ be the unique lie algebra homomorphism

$$\bar{\alpha}_t : \mathfrak{g}_\infty \to \mathfrak{g}_\infty$$

such that $\bar{\alpha}_t(P) = tP$ and $\bar{\alpha}_t(Q) = tQ$. Then $\bar{\alpha}_t$ acts on $\mathfrak{a}_n$ by multiplication by t^n. It is easy to see that $[\mathfrak{a}_1, \mathfrak{a}_n] = \mathfrak{a}_{n+1}$ a fact we shall often need to use. In fact $\mathfrak{a}_1$ is the 2-dimensional subspace spanned by P and Q, $\mathfrak{a}_2$ is 1-dimensional being spanned by $[P, Q]$, $\mathfrak{a}_3$ is 2-dimensional being spanned by $[P, [P, Q]]$ and $[Q, [P, Q]]$, $\mathfrak{a}_4$ is 3-dimensional being spanned by $[P, [P, [P, Q]]]$, $[Q, [P, [P, Q]]] = [P, [Q, [P, Q]]]$ and $[Q, [Q, [P, Q]]]$ and so forth.

The ideals

$$\mathfrak{b}_n = \bigoplus_{m=n}^{\infty} \mathfrak{a}_m$$

form the (infinite) descending central series of $\mathfrak{g}_\infty$. The quotient lie algebra $\mathfrak{g}_N = \mathfrak{g}_\infty/\mathfrak{b}_{N+1}$, which we shall identify (as a vector space) in the obvious way as

$$\mathfrak{g}_N = \bigoplus_{n=1}^{N} \mathfrak{a}_n$$

is finite dimensional for all N and may be termed the *free nilpotent lie algebra of nilpotent length* $N + 1$ *on two generators* P and Q. It enjoys the universal property that whenever $\mathfrak{h}$ is a nilpotent lie algebra of nilpotent length $N + 1$ or less and whenever $X, Y \in \mathfrak{h}$ there is a unique lie algebra homomorphism α, $\alpha: \mathfrak{g}_N \to \mathfrak{h}$ such that $\alpha(P) = X$ and $\alpha(Q) = Y$. Thus $\mathfrak{g}_1$ is simply the abelian lie algebra spanned by P and Q while $\mathfrak{g}_2$ is the Heisenberg lie algebra spanned by P, Q and $[P, Q]$ with all other commutators zero.

We shall use the same notation $\bar{\alpha}_t$ for the corresponding lie algebra homomorphism

$$\bar{\alpha}_t: \mathfrak{g}_N \to \mathfrak{g}_N$$

and also when N is fixed we shall write

$$\mathfrak{b}_n = \bigoplus_{m=n}^{N} \mathfrak{a}_m$$

for the corresponding ideals in $\mathfrak{g}_N$ in the hope that no confusion will arise. We define also G_N to be the SNL group with $\bar{G}_N = \mathfrak{g}_N$ and let α_t be the corresponding group homomorphism $\alpha_t: G_N \to G_N$.

(2.8) We shall need to consider the following proposition $P(N, H)$ for N an integer $N \geqq 1$ and H a SNL group.

Definition. We say that $P(N, H)$ holds if whenever φ is a NCV map $\varphi: G_N \to H$ and $u: \mathfrak{g}_N \to \bar{H}$ is the mapping associated to φ as in (2.5), then

$$u[P, Q] = \epsilon(\varphi)[uP, uQ]$$

where $\epsilon(\varphi) = \pm 1$ is a choice of sign (depending on φ).

LEMMA. *Assume that $N \geqq 1$ and H is a SNL group such that $P(N, H)$ holds. Let G be a SNL group with nilpotent length $N + 1$ or less and suppose that $\varphi: G \to H$ is a NCV map. Then φ is either a group homomorphism or a group anti-homomorphism.*

Proof. Let u be the map of (2.5). We show first that there is a choice of sign $\epsilon: \bar{G} \times \bar{G} \to \{+1, -1\}$ such that

$$u[X, Y] = \epsilon(X, Y)[uX, uY] \quad X, Y \in \bar{G}.$$

Indeed let $X, Y \in \bar{G}$ be fixed and let $\bar{\alpha}$ be the unique lie algebra homomorphism $\bar{\alpha}: \mathfrak{g}_N \to \bar{G}$ such that $\bar{\alpha}(P) = X$ and $\bar{\alpha}(Q) = Y$. Let α be the corresponding lie group homomorphism $\alpha: G_N \to G$. By (2.1), $\varphi \circ \alpha$ is a NCV map and thus by hypothesis there is a choice of sign $\epsilon(X, Y)$ such that

$$u \circ \bar{\alpha}[P, Q] = \epsilon(X, Y)[u \circ \bar{\alpha}(P), u \circ \bar{\alpha}(Q)]$$

which is precisely our claim.

Now let $f \in \bar{H}'$. Then $f(u[X, Y])$ and $f[u(X), u(Y)]$ are polynomial functions on $\bar{G} \times \bar{G}$ with the same square. Thus there is a choice of sign $\epsilon(f)$ such that

$$f(u[X, Y]) = \epsilon(f) f[u(X), u(Y)] \quad X, Y \in \bar{G}.$$

Finally, define subsets A_+ and A_- of $\bar{H}'$ by

$$A_{\pm} = \{f; f \in \bar{H}', f(u[X, Y]) = \pm f[u(X), u(Y)], X, Y \in \bar{G}\}.$$

Then A_+ and A_- are vector subspaces of $\bar{H}'$ such that $A_+ \cup A_- = \bar{H}'$. It follows that either $A_+ = \bar{H}'$ or $A_- = \bar{H}'$ which yields the desired conclusion.

3. Proof of the theorem. By virtue of (2.8) we need only prove that statement $P(N, H)$ holds for each $N \geqq 1$ and each SNL group H. The proof is by a two loop induction, the inner loop working over the dimension of H and the outer loop working over N. Each inner loop starts trivially since if dim $(H) = 1$, then $H = \mathbf{R}$ so that (2.6) applies. The outer loop has to be started by special arguments when $N = 1$ and $N = 2$. We will indicate these arguments later and now concentrate on the general induction step.

(3.1) *General induction step.* Let $N \geqq 3$, let H be a SNL group and let φ: $G_N \rightarrow H$ be a NCV map. The induction hypothesis and (2.8) allow us to assume that the theorem is true whenever the domain group has nilpotent length less than $N + 1$ and whenever the domain group has nilpotent length $N + 1$ and the image group has lower dimension than H.

(3.2) Let Z be the (non-trivial) centre of H and let π: $H \rightarrow H/Z$ be the canonical projection. By inductive hypothesis $\pi \circ \varphi$ is either a group homomorphism or anti-homomorphism. We will assume it is a group homomorphism. (In the other case it suffices to compose φ with inversion on G_N). Let u: $\mathfrak{g}_N \rightarrow \bar{H}$ be the mapping of (2.5) and let v_∞: $\mathfrak{g}_\infty \rightarrow \bar{H}$ be the unique lie algebra homomorphism such that $v_\infty(P) = u(P)$, $v_\infty(Q), = u(Q)$. Although $\mathfrak{g}_N$ occurs naturally as a quotient of $\mathfrak{g}_\infty$, it will be convenient to identify $\mathfrak{g}_N$ to the subspace $\oplus_{n=1}^{N} \mathfrak{a}_n$ of $\mathfrak{g}_\infty = \oplus_{n=1}^{\infty} \mathfrak{a}_n$ in the obvious way (The bracket of two elements of $\mathfrak{g}_N$ taken in $\mathfrak{g}_N$ need not coincide with the bracket taken in $\mathfrak{g}_\infty$. In cases of possible confusion the bracket is suffixed with the lie algebra in which it is taken.). It is not obvious (though in fact true) that the restriction of v_∞ to $\mathfrak{g}_N$ is a lie algebra homomorphism. What is clear is that if $X \in \mathfrak{a}_k$, $Y \in \mathfrak{a}_l$ and $k + l \leqq N$ then $v_\infty[X, Y]\mathfrak{g}_N = [v_\infty X, v_\infty Y]$. Using this together with $[\mathfrak{a}_1, \mathfrak{a}_n] = \mathfrak{a}_{n+1}$ and the fact that $\bar{\pi} \circ u$ is a lie algebra homomorphism one easily establishes by induction that

$$u(X) = v_\infty(X) + w(X), \quad X \in \mathfrak{g}_N$$

where w: $\mathfrak{g}_N \rightarrow \mathfrak{z}(\bar{H})$ is a linear map taking values in the centre $\mathfrak{z}(\bar{H})$ of $\bar{H}$.

(3.3) Next let $X \in \mathfrak{a}_1$ and $Y \in \mathfrak{a}_N$ and let $G_N(X)$ be the closed subgroup of G_N for which $\overline{G_N(X)} = \mathbf{R}X \oplus \mathfrak{b}_2$. Then $G_N(X)$ has nilpotent length N and we may apply the induction hypothesis to the restriction of φ to $G_N(X)$. Hence

$$[v_\infty X, v_\infty Y] = [v_\infty X + wX, v_\infty Y + wY] = [uX, uY] = \pm u[X, Y]\mathfrak{g}_N = 0.$$

This shows that

$$v_\infty[X, Y]\mathfrak{g}_\infty = 0 \quad \text{for all } X \in \mathfrak{a}_1 \text{ and for all } Y \in \mathfrak{a}_N$$

or equivalently that v_∞ vanishes on $\mathfrak{a}_{N+1}$. Since the ideal generated by $\mathfrak{a}_{N+1}$ in $\mathfrak{g}_\infty$ is $\mathfrak{b}_{N+1}$ we see that v_∞ vanishes on $\mathfrak{b}_{N+1}$ and in consequence the restriction v of v_∞ to $\mathfrak{g}_N$ is a lie algebra homomorphism.

To recapitulate, we now have $v: \mathfrak{g}_N \to \bar{H}$ a lie algebra homomorphism and $w: \mathfrak{g}_N \to \mathfrak{z}(\bar{H})$ a linear map such that $u = v + w$.

(3.4) We denote by H_0 the closed subgroup of H such that $\bar{H}_0 = \operatorname{im}(v)$. Let $\mathfrak{z}_1$ be a subspace of $\mathfrak{z}(\bar{H})$ such that $\bar{H}_0 + \mathfrak{z}(\bar{H}) = \bar{H}_0 \oplus \mathfrak{z}_1$ and let K be the closed subgroup of H for which

$$\bar{K} = \bar{H}_0 \oplus \mathfrak{z}_1.$$

Then φ takes values in K and K is the direct product of H_0 with an abelian group. Thus by (2.1) the result will follow from the induction hypothesis unless $\mathfrak{z}_1 = 0$ and $H = H_0$. Henceforth we will assume that $\bar{H} = \operatorname{im}(v)$.

(3.5) Define the element Z' of $\mathfrak{z}(\bar{H})$ by $Z' = w[P, Q]$. Our objective is to show that $Z' = 0$. Suppose now that $Z' \neq 0$. We may assume without loss of generality that $\mathfrak{z}(\bar{H}) = \mathbf{R}Z'$, for if not we may find a non-trivial central ideal $\mathfrak{z}_2$ for which $Z' \notin \mathfrak{z}_2$ and obtain a contradiction with the induction hypothesis by considering the map $\pi \circ \varphi$, where $\pi: H \to L$ is the canonical projection on the SNL quotient group L of H defined by $\bar{L} = \bar{H}/\mathfrak{z}_2$.

Thus there is a map $\vartheta \in \mathfrak{g}_N'$ which annihilates $\mathfrak{a}_1$ such that

$$w(X) = \vartheta(X)Z' \quad \text{for all } X \in \mathfrak{g}_N.$$

This notation will be useful at a later stage.

At this point we split the proof into two cases, depending on whether $v(\mathfrak{a}_1 \oplus \mathfrak{a}_2) \cap \mathfrak{z}(\bar{H}) = \{0\}$ or not.

(3.6) *Case* $v(\mathfrak{a}_1 \oplus \mathfrak{a}_2) \cap \mathfrak{z}(\bar{H}) \neq \{0\}$. Then $(\mathfrak{a}_1 \oplus \mathfrak{a}_2) \cap v^{-1}(\mathfrak{z}(\bar{H})) \neq \{0\}$, and in any case $[P, Q] \in v^{-1}(\mathfrak{z}(\bar{H}))$. Since $[P, Q]$ generates the ideal $\mathfrak{b}_2$ we have $\mathfrak{b}_2 \subseteq v^{-1}(\mathfrak{z}(\bar{H}))$ and hence $\mathfrak{b}_3 \subseteq \ker(v)$. We aim to show that u vanishes on $\mathfrak{b}_3$. Indeed let $X \in \mathfrak{a}_1$ and $Y \in \mathfrak{b}_2$. Let $G_N(X)$ be the closed subgroup of G such that

$$\overline{G_N(X)} = \mathbf{R}X \oplus \mathfrak{b}_2.$$

Then $G_N(X)$ has nilpotent length N and we may apply the induction hypothesis to the restriction of φ to $G_N(X)$. This leads to

$$u[X, Y] = \pm[uX, uY] = \pm[vX, vY + wY] = \pm v[X, Y] = 0$$

since $[X, Y] \in \mathfrak{b}_3$. Thus the claim is proved. But now $\varphi = \chi \circ \psi$ where

$$G_N \xrightarrow{\psi} G_2 \xrightarrow{\chi} H$$

and $\bar{\psi}$ is the canonical projection $\bar{\psi}: \mathfrak{g}_N \to \mathfrak{g}_N/\mathfrak{b}_3$, where $\mathfrak{g}_N/\mathfrak{b}_3$ has been identified to $\mathfrak{g}_2$. The result now follows from the induction hypothesis and (2.3).

(3.7) *Case* $v(\mathfrak{a}_1 \oplus \mathfrak{a}_2) \cap \mathfrak{z}(\bar{H}) = \{0\}$. Let $Z' = w[P, Q] \in \mathfrak{z}(\bar{H})$. We wish to show that $Z' = 0$ and assume the contrary. Let $\mathfrak{a}$ be a vector subspace of $\bar{H}$ such that

$$\bar{H} = \mathfrak{a} \oplus \mathbf{R}Z' \quad \text{and} \quad v(\mathfrak{a}_1 \oplus \mathfrak{a}_2) \subseteq \mathfrak{a}$$

and let $A: \bar{H} \rightarrow \mathfrak{a}$ and $s: \bar{H} \rightarrow \mathbf{R}$ be the linear maps such that

$$Y = A(Y) + s(Y)Z' \quad \text{for all } Y \in \bar{H}.$$

Let Z be the centre of H so that $\bar{Z} = \mathfrak{z}(\bar{H}) = \mathbf{R}Z'$ and let π be the canonical projection $\pi: H \rightarrow H/Z$. Then the mapping

$$\pi \circ \exp: \mathfrak{a} \rightarrow H/Z$$

preserves haar measures and this enables us to realise the unitary representation π_λ of H on the space $L^2(\mathfrak{a})$ induced from the representation

$$\exp(sZ') \rightarrow e^{is\lambda}$$

of the subgroup Z in the form

$$\pi_\lambda(\exp(A + sZ'))\xi(U) = e^{i\lambda(s+\sigma(U,A))}\xi(\mathscr{A}(U, A))$$

where A, $U \in \mathfrak{a}$, $s \in \mathbf{R}$, $\xi \in L^2(\mathfrak{a})$ and where we have used the notations

$$\mathscr{A}(U, A) = A(B(U, A)), \quad \sigma(U, A) = s(B(U, A)),$$

B standing as usual for the Baker-Campbell-Hausdorff function. Now fix functions ξ, $\eta \in C_c(\mathfrak{a})$ of unit $L^2(\mathfrak{a})$ norm and define

$$\xi_t(U) = t^{-d/2}\xi(t^{-1}U), \quad \eta_t(U) = t^{-d/2}\eta(t^{-1}U) \quad (t > 0),$$

the dilated functions which also have unit $L^2(\mathfrak{a})$ norm (here, $d = \dim(\mathfrak{a})$). Next define functions f_t on G_N for $t > 0$ by

$$f_t(\exp(X)) = \int \pi_{t^{-2}}(\varphi \circ \alpha_t \exp(X))\xi_t(U)\overline{\eta_t(U)}dU.$$

Since α_t is a group homomorphism on G_N and φ is a NCV map we have

$$||f_t||_B \leqq C(\varphi).$$

Now using the fact that

$$\varphi \circ \alpha_t(\exp(X)) = \exp(u \circ \bar{\alpha}_t(X)) = \exp(A \circ v \circ \bar{\alpha}_t(X) + (s \circ v + \vartheta) \circ \bar{\alpha}_t(X)Z')$$

and replacing U by tU we have

$$f_t(\exp(X)) = \int e^{it^{-2}((s\circ v+\vartheta)\circ\bar{\alpha}_t(X)+\sigma(tU,A\circ v\circ\bar{\alpha}_t(X))}\xi(t^{-1}\mathscr{A}(tU, A \circ v \circ \bar{\alpha}_t(X))\overline{\eta(U)}dU.$$

Next we write $X = \sum_{n=1}^{N} X_n$ for the decomposition of an element X of $\mathfrak{g}_N$

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according to the direct sum $\mathfrak{g}_N = \oplus_{n=1}^N \mathfrak{a}_n$. By virtue of the hypothesis $v(\mathfrak{a}_1 \oplus \mathfrak{a}_2) \subseteq \mathfrak{a}$ and using well known facts about the Baker-Campbell-Hausdorff formula we find

$$\begin{aligned} s \circ v \circ \bar{\alpha}_t(X) &= t^3 p_1(t, X), \\ \vartheta \circ \bar{\alpha}_t(X) &= t^2 \vartheta(X_2) + t^3 p_2(t, X), \\ \sigma(tU, A \circ v \circ \bar{\alpha}_t(X)) &= \tfrac{1}{2} t^2 s[U, v(X_1)] + t^3 p_3(t, U, X), \\ \mathscr{A}(tU, A \circ v \circ \bar{\alpha}_t(X)) &= t(U + v(X_1)) + t^2 p_4(t, U, X), \end{aligned}$$

where p_1 and p_2 are polynomials on $\mathbf{R} \times \mathfrak{g}_N$ and p_3 and p_4 are polynomials on $\mathbf{R} \times \mathfrak{a} \times \mathfrak{g}_N$. It follows that as $t \to 0$, $f_t \to f$ uniformly on the compacta of G_N where

$$f(\exp(X)) = \int e^{i(\vartheta(X_2) + 1/2 s[U, v(X_1)])} \xi(U + v(X_1)) \overline{\eta(U)} dU$$

and where we still have $||f||_B \leqq C(\varphi)$. (See [**2**, Chapitre 2].)

To understand this function better we choose a vector subspace $\mathfrak{b}$ of $\mathfrak{a}$ such that

$$\mathfrak{a} = v(\mathfrak{a}_1) \oplus \mathfrak{b}$$

and use the notation $U = V + W = (V, W)$ for the decomposition of $U \in \mathfrak{a}$ according to this direct sum (with $V \in v(\mathfrak{a}_1)$ and $W \in \mathfrak{b}$). Then

$$f(\exp(X)) = \int e^{i(\vartheta(X_2) + 1/2 s[V+W, v(X_1)])} \xi(V + v(X_1), W) \overline{\eta(V, W)} dV dW$$

and we see that $s[V, v(X_1)] \in s \circ v(\mathfrak{a}_2) = \{0\}$. To simplify we set

$$\xi'(V, W) = e^{1/2 is[W, V]} \xi(V, W), \quad \eta'(V, W) = e^{1/2 is[W, V]} \eta(V, W)$$

and find the formula

$$f(\exp(X)) = e^{i\vartheta(X_2)} \int \xi'(U + v(X_1)) \overline{\eta'(U)} dU.$$

To recapitulate, we now know that whenever $\xi', \eta' \in C_c(\mathfrak{a})$ with unit $L^2(\mathfrak{a})$ norm we have $||f||_B \leqq C(\varphi)$. But now because of the amenability of $\mathfrak{a}$ we may express the function

$$g(\exp(X)) = e^{i\vartheta(X_2)}$$

as a uniform on compacta limit of f's and still have $||g||_B \leqq C(\varphi)$. But the function g is only uniformly continuous on G_N if ϑ vanishes on $\mathfrak{a}_2$, and this is exactly what we needed to show.

(3.8) *Induction step $N = 1$.* In this section we prove the proposition $P(1, H)$ for H a SNL group. The induction hypothesis and (2.8) allow us to assume that the theorem is true whenever the domain group is abelian and the image group has lower dimension than H. Let $\varphi: G_1(= \mathbf{R}^2) \to H$ be a NCV map.

We proceed as in (3.2). The argument of (3.3) does not apply for $N = 1$. It is easy to see however that we may assume that $\bar{H}$ is the linear span of $u(P)$, $u(Q)$ and the necessarily central element $[uP, uQ] = Z'$. If $\dim(H) \leqq 2$ then H is abelian and the result follows from (2.6). If $\dim(H) = 3$ then u is the inclusion map

$$u: \mathfrak{g}_1 = \mathfrak{a}_1 \rightarrow \mathfrak{g}_2 = \mathfrak{a}_1 \oplus \mathfrak{a}_2$$

and we will show that the corresponding φ is not a NCV map by explicit calculation. Assuming the contrary, and composing φ with an arbitrary coefficient function of the unitary representation of the Heisenberg group G_2 induced from the representation

$$\exp\,(sZ') \rightarrow e^{is}$$

of its centre, yields a bounded linear operator $M: L^2(\mathbf{R}) \hat{\otimes} L^2(\mathbf{R}) \rightarrow M(\mathbf{R}^2)$ of norm $C(\varphi)$ such that

$$\int e^{i(1/2xy+ty)}\xi(x+t)\eta(t)dt = \int e^{i(ux+vy)}d\mu(u,v) \quad \text{for all } x, y \in \mathbf{R}$$

where $\xi, \eta \in L^2(\mathbf{R})$ and $\mu = M(\xi \otimes \eta)$. Substituting $v = t + \frac{1}{2}x$ we find, by the uniqueness of the Fourier transform, that

$$\int f(x,v)\xi(v+\tfrac{1}{2}x)\eta(v-\tfrac{1}{2}x)dv = \int f(x,v)e^{iux}d\mu(u,v)$$
$$\text{for all } x \in \mathbf{R} \text{ and for all } f \in C(\mathbf{R}^2).$$

Now we choose a sequence of such f's suitably approximating linear measure on $\{(x,v); x = \gamma v\}$ and after integration with respect to x we find

$$\left| \int \xi(v(1+\tfrac{1}{2}\gamma))\eta(v(1-\tfrac{1}{2}\gamma))dv \right| \leqq C(\varphi)||\xi||_2||\eta||_2 \quad \text{for all } \gamma \in \mathbf{R}$$

for ξ and η in $C_c(\mathbf{R})$. This easily yields the desired contradiction.

(3.9) *Induction step* $N = 2$. In this section we prove proposition $P(2, H)$ for H a SNL group. The induction hypothesis and (2.8) allow us to assume that the theorem is true whenever the domain group is abelian and whenever the domain group has nilpotent length 3 and the image group has lower dimension than H. Let $\varphi: G_2 \rightarrow H$ be a NCV map.

We proceed as in (3.2), (3.3), (3.4) and (3.5). We may assume that the corresponding u can be decomposed as

$$u = v + w$$

where $v: \mathfrak{g}_2 \rightarrow \bar{H}$ is a surjective lie algebra homomorphism and $w: \mathfrak{g}_2 \rightarrow \mathfrak{z}(\bar{H})$ is a linear map vanishing on $\mathfrak{a}_1$. If $\dim(H) \leqq 2$ then H is abelian and the result follows from (2.6). In the remaining case ($\dim(H) = 3$) we have

$$u: \mathfrak{g}_2 = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \rightarrow \mathfrak{g}_2 = \mathfrak{a}_1 \oplus \mathfrak{a}_2$$

given by the identity on $\mathfrak{a}_1$ and by multiplication by some real number β on $\mathfrak{a}_2$. We need to show that $\beta = \pm 1$. The case $\beta = 0$ reduces to the abelian case (3.8) just treated by applying (2.3). Henceforth we shall assume that $\beta \neq 0$.

Composing φ with an arbitrary coefficient function of a unitary representation of the Heisenberg group G_2 induced from a non-trivial representation of its centre yields a bounded linear operator

$$K: L^2(\mathbf{R}) \hat{\otimes} L^2(\mathbf{R}) \to B(G_2)$$

of norm $C(\varphi)$ such that

$$f(\exp(xP + yQ + z[P, Q])) = \int e^{i(\beta z + 1/2xy + ty)} \xi(x + t)\eta(t)dt \quad \text{for all } x, y, z \in \mathbf{R}$$

where $\xi, \eta \in L^2(\mathbf{R})$ and $f = K(\xi \otimes \eta)$. But f satisfies

$$f(\exp(s[P, Q])g) = e^{i\beta s} f(g) \quad \text{for all } s \in \mathbf{R} \text{ and for all } g \in G_2$$

and so must be a coefficient function of the representation of G_2 induced from the non-trivial representation

$$\exp(s[P, Q]) \to e^{i\beta s}$$

of its centre. Thus there is a bounded linear operator

$$L: L^2(\mathbf{R}) \hat{\otimes} L^2(\mathbf{R}) \to L^2(\mathbf{R}) \hat{\otimes} L^2(\mathbf{R})$$

of norm $C(\varphi)$ such that

$$\int e^{i\beta(1/2xy + ty)} F(x + t, t)dt = \int e^{i(1/2xy + ty)} \xi(x + t)\eta(t)dt \quad \text{for all } x, y \in \mathbf{R}$$

where $F = L(\xi \otimes \eta)$. From the above it is easy to deduce by means of the uniqueness of fourier transforms that

$$F(t, s) = \beta\xi(\tfrac{1}{2}(1 + \beta)t - \tfrac{1}{2}(1 - \beta)s)\eta(-\tfrac{1}{2}(1 - \beta)t + \tfrac{1}{2}(1 + \beta)s).$$

Now using the fact that

$$\left| \int F(\gamma t, \pm\gamma^{-1}t)dt \right| \leqq C(\varphi)||\xi||_2||\eta||_2 \quad \text{for all } \gamma > 0$$

we obtain

$$4|\beta|^2 \leqq (C(\varphi))^2|(1 - \beta^2)(\gamma^2 + \gamma^{-2}) \mp 2(1 + \beta^2)| \quad \text{for all } \gamma > 0$$

which is only true if $|\beta| = 1$. This completes the proof.

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