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COFINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract

Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Let s be a nonnegative integer and let M and N be two R-modules such that $\mathrm{Ext}^j_R(M/\mathfrak{a}M, \mathrm{H}^i_{\mathfrak{a}}(N))$ is finite for all $i < s$ and all $j \geq 0$. We show that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is finite provided $\mathrm{Ext}^s_R(M/\mathfrak{a}M, N)$ is a finite R-module. In addition, for finite R-modules M and N, we prove that if $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is minimax for all $i < s$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is finite. These are two generalizations of the result of Brodmann and Lashgari ['A finiteness result for associated primes of local cohomology modules', *Proc. Amer. Math. Soc.* **128** (2000), 2851–2853] and a recent result due to Chu ['Cofiniteness and finiteness of generalized local cohomology modules', *Bull. Aust. Math. Soc.* **80** (2009), 244–250]. We also introduce a generalization of the concept of cofiniteness and recover some results for it.

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1. Introduction

Throughout this paper R is a commutative Noetherian ring, $\mathfrak{a}$ is an ideal and M and N are two R-modules. Let $\mathrm{H}^i_{\mathfrak{a}}(M)$ be the ith local cohomology module of M with respect to $\mathfrak{a}$. In [10] Grothendieck conjectured that 'for any finite R-module M, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ is finite for all i'. Although Hartshorne disproved Grothendieck's conjecture (see [11]), there are some partial positive answers to it. For example, in [3, Theorem 4.1] the authors showed that for a nonnegative integer s if M is a finite R-module such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is finite for all $i < s$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is finite. Recall that an R-module M is called $\mathfrak{a}$-cofinite if $\mathrm{Supp}(M) \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is a finite R-module for each i. In [9, Theorem 2.1] the authors improved [3, Theorem 4.1] by showing that for a nonnegative integer s, if M is a finite R-module such that $\mathrm{H}^j_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $j \leq s$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is finite.

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Let M and N be two R-modules. The generalized local cohomology module

$$\mathrm{H}^i_{\mathfrak{a}}(M, N) = \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N)$$

was introduced by Herzog in [12] and studied further in [17, 18]. Note that $\mathrm{H}^i_{\mathfrak{a}}(R, N) = \mathrm{H}^i_{\mathfrak{a}}(N)$. Now it is natural to think about Grothendieck's conjecture for the generalized local cohomology modules.

QUESTION 1.1. Let M and N be two finite R-modules. When is

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M, N))$$

finite?

In [2, Theorem 1.2] it is shown that, for a nonnegative integer s, if M and N are two finite R-modules such that $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is finite for all $i < s$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is finite. Recently, in [8] Chu has shown that for a nonnegative integer s, if M and N are two finite R-modules such that $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is Artinian for all $i < s$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is finite.

Let M and N be two R-modules. Then we say that N is $(\mathfrak{a}, M)$-cofinite if $\mathrm{Supp}(N) \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathrm{Ext}^i_R(M/\mathfrak{a}M, N)$ is a finite R-module for all i. Note that $(\mathfrak{a}, R)$-cofinite is the classical $\mathfrak{a}$-cofinite.

In this paper we give some answers to Question 1.1 which improve on some previous results. Our first main result shows that for two R-modules M and N such that $\mathrm{H}^i_{\mathfrak{a}}(N)$ is $(\mathfrak{a}, M)$-cofinite for all $i < s$, the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M, N))$ is finite provided $\mathrm{Ext}^s_R(M/\mathfrak{a}M, N)$ is finite.

Recall that an R-module M is called minimax if there is a finite submodule N of M such that M/N is Artinian; see [19]. The class of minimax modules includes all finite and all Artinian modules. We show that for finite R-modules M and N if $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is a minimax module for all $i < s$, then for all finite R-submodules L of $\mathrm{H}^s_{\mathfrak{a}}(M, N)$, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N)/L)$ is finite and in particular, $\mathrm{H}^s_{\mathfrak{a}}(M, N)$ has finitely many associated primes. This result improves on some previous ones, for example [6, Theorem 2.2], [3, Theorem 4.1], [9, Theorem 2.1], [13, Theorem 2.1], [2, Theorem 1.2] and [8, Theorem 2.5]. We use the terminology and notation of [7].

2. $(\mathfrak{a}, M)$-cofiniteness

DEFINITION 2.1. Let M and N be two R-modules. We say that N is $(\mathfrak{a}, M)$-cofinite if $\mathrm{Supp}(N) \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathrm{Ext}^i_R(M/\mathfrak{a}M, N)$ is a finite R-module for all i.

Note that $(\mathfrak{a}, R)$-cofinite is the classical $\mathfrak{a}$-cofinite. In the following we establish some results on $(\mathfrak{a}, M)$-cofinite modules. The proofs of the following two results are classical.

LEMMA 2.2. *In a short exact sequence, if two modules in the sequence are $(\mathfrak{a}, N)$-cofinite, then so is the third.*

COROLLARY 2.3. *If $f : H \to K$ is a homomorphism of $(\mathfrak{a}, N)$-cofinite modules and one of* $\ker f$, $\mathrm{im}\ f$ *and* $\mathrm{coker}\ f$ *is $(\mathfrak{a}, N)$-cofinite, then all three are $(\mathfrak{a}, N)$-cofinite.*

THEOREM 2.4. *Let* $(R, \mathfrak{m}, k)$ *be a local ring. If* M *is* $(\mathfrak{m}, N)$*-cofinite, then* $\operatorname{Hom}_R(N, M)$ *is Artinian.*

PROOF. The result follows clearly if $\operatorname{Hom}_R(N, M) = 0$. It is well known that a nonzero module M is Artinian if and only if $\operatorname{Supp}_R(M) = \{\mathfrak{m}\}$ and $\operatorname{Hom}(k, M)$ is finite. Note that

$$\operatorname{Hom}_R(N/\mathfrak{m}N, M) \cong \operatorname{Hom}_R(R/\mathfrak{m} \otimes_R N, M) \cong \operatorname{Hom}(R/\mathfrak{m}, \operatorname{Hom}_R(N, M)).$$

Since $\operatorname{Hom}(N/\mathfrak{m}N, M)$ is finite, $\operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Hom}_R(N, M))$ is finite too. Further $\operatorname{Supp}_R(\operatorname{Hom}_R(N, M)) \subseteq \operatorname{Supp}_R\{M\} \subseteq \{\mathfrak{m}\}$ and $\operatorname{Hom}_R(k, \operatorname{Hom}_R(N, M))$ is finite. □

LEMMA 2.5. *Let* M *be a minimax module and* N *be* $\mathfrak{a}$*-cofinite. Then* $\operatorname{Ext}^i_R(N/\mathfrak{a}N, M)$ *is minimax for all* i.

PROOF. It is well known that in an exact sequence $A \to B \to C$ of R-modules and R-homomorphisms, if A and C are minimax, then B is minimax too; see [4, Lemma 2.1]. Then one can deduce that $\operatorname{Hom}_R(M, N)$ is minimax whenever M is finite and N is minimax. Hence for such M and N we have that $\operatorname{Ext}^i_R(M, N)$ is minimax, as it can be seen using a projective resolution for M. Now let $\mathfrak{a} = (x_1, \ldots, x_n)$. Then $N/\mathfrak{a}N \cong \mathrm{H}^n(x_1, \ldots, x_n, N)$. As N is $\mathfrak{a}$-cofinite, all its Koszul cohomology modules are finite. In particular, $N/\mathfrak{a}N$ is finite. Now apply the argument for the finite case. □

One can replace 'minimax' with 'finite' in Lemma 2.5 to deduce the following.

COROLLARY 2.6. *Let* M *be a finite module and let* N *be* $\mathfrak{a}$*-cofinite. Then* $\operatorname{Ext}^i_R(N/\mathfrak{a}N, M)$ *is finite for all* i. *In particular, if* $\operatorname{Supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$, *then* M *is* $(\mathfrak{a}, N)$*-cofinite.*

THEOREM 2.7. *Let* M *be a* $(\mathfrak{a}, N)$*-cofinite* R*-module. Then* $\operatorname{Ass}_R(\operatorname{Hom}_R(N, M))$ *is finite.*

PROOF. Set $P := \operatorname{Hom}_R(N, M)$. Then $\operatorname{Hom}_R(N/\mathfrak{a}N, M) \cong \operatorname{Hom}_R(R/\mathfrak{a}, P) \cong 0 :_P \mathfrak{a}$ is finite. The essence of [14, Proposition 1.3] is that $0 :_P \mathfrak{a}$ is an essential submodule of P. For this let $0 \neq x \in P$. Since $\operatorname{Supp}_R(P) \subseteq \mathrm{V}(\mathfrak{a})$, there is a natural number n such that $\mathfrak{a}^n x = 0$ but $\mathfrak{a}^{n-1}x \neq 0$. Thus $0 \neq \mathfrak{a}^{n-1}x \subseteq Ax \cap 0 :_P \mathfrak{a}$. Hence each submodule of P has a nonzero intersection with $0 :_P \mathfrak{a}$. That is, $0 :_P \mathfrak{a}$ is an essential submodule of P. In other words, P has finite Goldie dimension. Hence $\operatorname{Ass}_R(P)$ is finite. □

The following result is the counterpart for the change of rings principle for $(\mathfrak{a}, P)$-cofinite modules, where P is a finite flat module; see [14, Proposition 1.5].

THEOREM 2.8. *Let* $f : A \to B$ *be a homomorphism of Noetherian rings. Let* $\mathfrak{a}$ *be an ideal of* A, M *an* A*-module and* P *a finite flat* A*-module.*

(a) *If* f *is flat, then* $M \otimes_A B$ *is* $(\mathfrak{a}B, P \otimes_A B)$*-cofinite whenever* M *is* $(\mathfrak{a}, P)$*-cofinite.*
(b) *If* f *is faithfully flat, the converse of (a) holds as well.*

PROOF. Note that $\operatorname{Ext}^i_A(P/\mathfrak{a}P, M) \otimes_A B \cong \operatorname{Ext}^i_B(P \otimes_A B/\mathfrak{a}B, M \otimes_A B)$; see [16, Proposition 7.39]. Since P is a flat A-module, $P \otimes_A B/\mathfrak{a}B \cong P \otimes_A B/P \otimes_A \mathfrak{a}B \cong (P \otimes_A B)/\mathfrak{a}(P \otimes_A B)$. Hence, $\operatorname{Ext}^i_A(P/\mathfrak{a}P, M) \otimes_A B \cong \operatorname{Ext}^i_B((P \otimes_A B)/\mathfrak{a}(P \otimes_A B), M \otimes_A B)$. □

3. Cofiniteness and minimaxness

There are several papers devoted to partially answering Question 1.1 in more general situations, for example [3, 6, 8, 9, 13]. The following theorem is the first main result of this paper is in this vein.

THEOREM 3.1. *Let s be a nonnegative integer. Let M and N be two R-modules such that $\mathrm{H}^i_{\mathfrak{a}}(N)$ is $(\mathfrak{a}, M)$-cofinite for all $i < s$. If $\operatorname{Ext}^s_R(M/\mathfrak{a}M, N)$ is a finite (respectively minimax) R-module, then $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is finite (respectively minimax).*

PROOF. We proceed by induction on s. If $s = 0$, then $\mathrm{H}^0_{\mathfrak{a}}(M, N) \cong \Gamma_{\mathfrak{a}}(\operatorname{Hom}(M, N))$ and $\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M, N)))$ is equal to the finite (respectively minimax) R-module

$$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Hom}_R(M, N)) \cong \operatorname{Hom}_R(M/\mathfrak{a}M, N).$$

Suppose that $s > 0$ and that the case $s - 1$ is settled. We have that $\operatorname{Ext}^j_R(M/\mathfrak{a}M, \Gamma_{\mathfrak{a}}(N))$ is finite (respectively minimax) for all $j \geq 0$. Using the exact sequence $0 \to \Gamma_{\mathfrak{a}}(N) \to N \to N/\Gamma_{\mathfrak{a}}(N) \to 0$, we get that $\operatorname{Ext}^s_R(M/\mathfrak{a}M, N/\Gamma_{\mathfrak{a}}(N))$ is finite (respectively minimax). On the other hand, $\mathrm{H}^0_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)) = 0$ and for all $i > 0$ and all $j \geq 0$,

$$\operatorname{Ext}^j_R(M/\mathfrak{a}M, \mathrm{H}^i_{\mathfrak{a}}(N/\Gamma_{\mathfrak{a}}(N))) \cong \operatorname{Ext}^j_R(M/\mathfrak{a}M, H^i_{\mathfrak{a}}(N)).$$

Thus we may assume that $\Gamma_{\mathfrak{a}}(N) = 0$. Let E be an injective hull of N and put $T = E/N$. Then $\Gamma_{\mathfrak{a}}(E) = 0$ and $\operatorname{Hom}_R(M/\mathfrak{a}M, E) = 0$. Consequently, $\operatorname{Ext}^i_R(M/\mathfrak{a}M, T) \cong \operatorname{Ext}^{i+1}_R(M/\mathfrak{a}M, N)$ and $\mathrm{H}^i_{\mathfrak{a}}(M, T) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M, N)$ for all $i \geq 0$. Now the induction hypothesis yields that $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{s-1}_{\mathfrak{a}}(M, T))$ is finite (respectively minimax) and hence $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is finite (respectively minimax). □

COROLLARY 3.2. *Let s be a nonnegative integer. Let $\mathfrak{a}$ be an ideal of R and let M and N be two R-modules. Let L be a submodule of $\mathrm{H}^s_{\mathfrak{a}}(M, N)$ such that $\operatorname{Ext}^1_R(R/\mathfrak{a}, L)$ is a finite (respectively minimax) R-module. If $\operatorname{Ext}^s_R(M/\mathfrak{a}M, N)$ is a finite (respectively minimax) R-module and $\mathrm{H}^i_{\mathfrak{a}}(N)$ is $(\mathfrak{a}, M)$-cofinite for all $i < s$, then the module $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N)/L)$ is finite (respectively minimax). In particular, $\mathrm{H}^s_{\mathfrak{a}}(M, N)/L$ has finitely many associated primes.*

PROOF. Let L be a submodule of $\mathrm{H}^s_{\mathfrak{a}}(M, N)$ such that $\operatorname{Ext}^1_R(R/\mathfrak{a}, L)$ is a finite (respectively minimax) R-module. The short exact sequence $0 \to L \to \mathrm{H}^s_{\mathfrak{a}}(M, N) \to \mathrm{H}^s_{\mathfrak{a}}(M, N)/L \to 0$ induces the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N)) \to \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N)/L) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, L).$$

Since by Theorem 3.1 the left-hand term and by hypothesis the right-hand term are finite (respectively minimax), we have that $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N)/L)$ is finite (respectively minimax). For the last statement note that

$$\operatorname{Supp}(\mathrm{H}^s_{\mathfrak{a}}(M, N)/L) \subseteq \operatorname{Supp}(\mathrm{H}^s_{\mathfrak{a}}(M, N)) \subseteq \mathrm{V}(\mathfrak{a}).$$

Therefore $\operatorname{Ass}(\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N)/L)) = \operatorname{Ass}(\mathrm{H}^s_{\mathfrak{a}}(M, N)/L)$ is a finite set. □

The following corollary is the main result of Brodmann and Lashgari [6].

COROLLARY 3.3. *Let M be a finite R-module. Let s be a nonnegative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is finite for each $i < s$. Then for any finite submodule N of $\mathrm{H}^s_{\mathfrak{a}}(M)$, the set $\operatorname{Ass}(\mathrm{H}^s_{\mathfrak{a}}(M)/N)$ has finitely many elements.*

One can define the term $(\mathfrak{a}, M)$-coartinian by replacing 'Artinian' with 'finite' in our definition of $(\mathfrak{a}, M)$-cofinite. Then a similar proof as for Theorem 3.1 implies the following.

THEOREM 3.4. *Let s be a nonnegative integer. Let M and N be two R-modules such that $\mathrm{H}^i_{\mathfrak{a}}(N)$ is $(\mathfrak{a}, M)$-coartinian for all $i < s$. If $\operatorname{Ext}^s_R(M/\mathfrak{a}M, N)$ is an Artinian R-module, then $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is Artinian.*

Using [1, Theorem 2.9], we are able to express this result in several equivalent situations. Note that by [1, Example 2.4(b)] the class of Artinian R-modules is closed under taking submodules, quotients, extensions and injective hulls. Hence it satisfies condition C_a in the notation of [1, Theorem 2.9].

COROLLARY 3.5. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Let s be a nonnegative integer. Let M and N be two R-modules such that $M/\mathfrak{a}M$ is finite and $\operatorname{Ext}^s_R(M/\mathfrak{a}M, N)$ is Artinian. Then in any of the following cases, $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is Artinian:*

(a) *$\mathrm{H}^i_{\mathfrak{a}}(N)$ is Artinian for all $i < s$;*
(b) *$\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ is Artinian for all $i < s$;*
(c) *$\operatorname{Ext}^i_R(T, N)$ is Artinian for all $i < s$ and each finite module T such that $\operatorname{Supp}_R(T) \subseteq \mathrm{V}(\mathfrak{a})$;*
(d) *there is a finite R-module T with $\operatorname{Supp}_R(T) = \mathrm{V}(\mathfrak{a})$ such that $\operatorname{Ext}^i_R(T, N)$ is Artinian for all $i < s$;*
(e) *$\mathrm{H}^i(x_1, \ldots, x_r, N)$ is Artinian for all $i < s$ where $x_1, \ldots, x_r$ generate $\mathfrak{a}$;*
(f) *$\mathrm{H}^i_{\mathfrak{a}}(T, N)$ is Artinian for each finite R-module T and for all $i < s$.*

The following theorem is the second main result of this paper.

THEOREM 3.6. *Let M and N be finite R-modules. If $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is a minimax module for all $i < s$, then $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is finite.*

PROOF. First we prove the theorem under the additional assumption that $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite module for all $i < s$. If $s = 0$, then $\mathrm{H}^0_{\mathfrak{a}}(M, N) \cong \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M, N))$ is a finite R-module. Now suppose that $s > 0$. The short exact sequence $0 \to \Gamma_{\mathfrak{a}}(N) \to N \to N/\Gamma_{\mathfrak{a}}(N) \to 0$ induces the exact sequence

$$\mathrm{H}^t_{\mathfrak{a}}(M, N) \to \mathrm{H}^t_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)) \to \mathrm{H}^{t+1}_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)).$$

For $t < s$ the R-module $\mathrm{H}^t_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite and minimax and by [13, Lemma 1.1] the R-module $\mathrm{H}^{t+1}_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N))$ is finite. Thus by [15, Corollary 4.4] we have that $\mathrm{H}^t_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N))$ is an $\mathfrak{a}$-cofinite and minimax R-module. Thus without loss of generality we can assume that $\Gamma_{\mathfrak{a}}(N) = 0$. Choosing an arbitrary N-regular element x in $\mathfrak{a}$ and using an argument similar to the proof of [4, Lemma 2.2], we obtain that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N))$ is finite.

Next suppose that $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is minimax module for all $i < s$. In view of the first part of the proof, it is enough to show that $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all $i < s$. We proceed by induction on i. The case $i = 0$ is obvious as $\mathrm{H}^0_{\mathfrak{a}}(M, N)$ is finite. Thus let $i > 0$, and assume the result has been proved for smaller values of i. By the inductive hypothesis, $\mathrm{H}^j_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for $j < i$. Thus by the first part, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M, N))$ is finite. Therefore by [15, Proposition 4.3], $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite. Hence $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite minimax for all $i < s$. Now the assertion follows from the first part. □

By an argument similar to the proof of Corollary 3.2 we have the following corollary.

COROLLARY 3.7. *Let s be a nonnegative integer. Let $\mathfrak{a}$ be an ideal of R and let M and N be two finite R-modules. Let L be a submodule of $\mathrm{H}^s_{\mathfrak{a}}(M, N)$ such that $\mathrm{Ext}^1_R(R/\mathfrak{a}, L)$ is a finite R-module. If $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is minimax for all $i < s$, then the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M, N)/L)$ is finite. In particular, $\mathrm{H}^s_{\mathfrak{a}}(M, N)/L$ has finitely many associated primes.*

Applying Theorem 3.6, we have the following result; see [5, Theorem 2.2].

COROLLARY 3.8. *Let $\mathfrak{a}$ be an ideal of R and let s be a nonnegative integer. Let N be an R-module such that $\mathrm{Ext}^s_R(R/\mathfrak{a}, N)$ is a finite R-module. If $\mathrm{H}^i_{\mathfrak{a}}(N)$ is minimax for all $i < s$ then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(N))$ is a minimax module. Furthermore, if L is a finite R-module such that $\mathrm{Supp}(L) \subseteq \mathrm{V}(\mathfrak{a})$, then $\mathrm{Hom}_R(L, \mathrm{H}^s_{\mathfrak{a}}(N))$ is a minimax module.*

In the same way we can apply Theorem 3.6 to deduce the following result; see [4, Theorem 2.3].

COROLLARY 3.9. *Let R be a Noetherian ring, M a nonzero finite R-module and $\mathfrak{a}$ an ideal of R. Let s be a nonnegative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is minimax for all $i < s$. Then the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is finite. In particular, $\mathrm{Ass}_R(\mathrm{H}^s_{\mathfrak{a}}(M))$ is finite.*

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