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LOGARITHMIC COEFFICIENTS PROBLEMS IN FAMILIES RELATED TO STARLIKE AND CONVEX FUNCTIONS

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Abstract

Let $\mathcal{S}$ be the family of analytic and univalent functions f in the unit disk $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$, and let $\gamma_n(f) = \gamma_n$ denote the logarithmic coefficients of $f \in \mathcal{S}$. In this paper we study bounds for the logarithmic coefficients for certain subfamilies of univalent functions. Also, we consider the families $\mathcal{F}(c)$ and $\mathcal{G}(c)$ of functions $f \in \mathcal{S}$ defined by

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 1 - \frac{c}{2} \quad \text{and} \quad \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{c}{2}, \quad z \in \mathbb{D},$$

for some $c \in (0, 3]$ and $c \in (0, 1]$, respectively. We obtain the sharp upper bound for $|\gamma_n|$ when $n = 1, 2, 3$ and f belongs to the classes $\mathcal{F}(c)$ and $\mathcal{G}(c)$, respectively. The paper concludes with the following two conjectures:

- If $f \in \mathcal{F}(-1/2)$, then $|\gamma_n| \le 1/n(1 - (1/2^{n+1}))$ for $n \ge 1$, and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{\pi^2}{6} + \frac{1}{4}\,\mathrm{Li}_2\left(\frac{1}{4}\right) - \mathrm{Li}_2\left(\frac{1}{2}\right),$$

  where $\mathrm{Li}_2(x)$ denotes the dilogarithm function.
- If $f \in \mathcal{G}(c)$, then $|\gamma_n| \le c/2n(n+1)$ for $n \ge 1$.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and let $\mathcal{S}$ denote the subclass of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. The logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined by the formula

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad \text{for } z \in \mathbb{D}. \tag{1.1}$$

We use $\gamma_n(f) = \gamma_n$ when there is no confusion. These coefficients play an important role for various estimates in the theory of univalent functions, and some authors use γ_n in place of $2\gamma_n$. Louis de Branges [2] showed (see also [1] and [6]) that for each $n \geq 1$,

$$\sum_{k=1}^{n} k(n-k+1)|\gamma_n|^2 \leq \sum_{k=1}^{n} \frac{n-k+1}{k},$$

where equality holds if and only if f has the form $z/(1-e^{i\theta}z)^2$ for some $\theta \in \mathbb{R}$. It is known that this proves the famous Bieberbach–Robertson–Milin conjectures about Taylor coefficients of $f \in \mathcal{S}$ in its most general form. See [1]. For another (shorter) version of de Branges' proof, we refer to [7]. Note that for $f(z) = z/(1-e^{i\theta}z)^2$ we have $\gamma_n = e^{ni\theta}/n$ for $n = 1, 2, \ldots$. The idea of studying the logarithmic coefficients helped Kayumov [11] to solve Brennans conjecture for conformal mappings. In this note, we will discuss the logarithmic coefficients problem for certain subfamilies of univalent functions and derive some similar inequalities. We begin with some preliminaries.

A function $f \in \mathcal{S}$ is called starlike if $f(\mathbb{D})$ is a domain that is starlike with respect to the origin. Every starlike function is characterized by the condition $\mathrm{Re}\,(zf'(z)/f(z)) > 0$ for $z \in \mathbb{D}$. A function $f \in \mathcal{S}$ is convex when the function $g = zf'$ is starlike. A function $f \in \mathcal{A}$ is said to be close to convex if there exist a real number θ and a function $g \in \mathcal{S}^*$ such that $\mathrm{Re}\,(e^{i\theta} zf'(z)/g(z)) > 0$ in $\mathbb{D}$. We denote by $\mathcal{S}^*, \mathcal{C}$ and $\mathcal{K}$, the class of starlike functions, convex functions and close-to-convex functions, respectively. Functions in the class $\mathcal{K}$ of all close-to-convex functions are known to be univalent in $\mathbb{D}$. The role of $\mathcal{S}$, together with its subfamilies, and their importance concerning strongly starlike functions in geometric function theory are well documented. See, for example, [3, 4, 9, 22].

If $f \in \mathcal{S}$, then it is known that $|\gamma_1| \leq 1$ and

$$|\gamma_2| = \tfrac{1}{2}|a_3 - \tfrac{1}{2}a_2^2| \leq \tfrac{1}{2}(1 + 2e^{-2}) = 0.635\ldots$$

by using the Fekete–Szegö inequality (see [4, Theorem 3.8]). For $n \geq 3$, the logarithmic coefficients problem seems much harder. The inequality $|\gamma_n| \leq 1/n$ holds for $f \in \mathcal{S}^*$ but is not true for the full class $\mathcal{S}$, even in order of magnitude (see [4, Theorem 8.4]). Indeed, there exists a bounded function $f \in \mathcal{S}$ with $\gamma_n \neq O(n^{-0.83})$. On the other hand, Roth [20] established the following sharp inequality for $f \in \mathcal{S}$:

$$\sum_{n=1}^{\infty} p_n|\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{p_n}{n^2}, \quad p_n = \left(\frac{n}{n+1}\right)^2. \tag{1.2}$$

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This inequality is a source of many new inequalities for the logarithmic coefficients of $f \in \mathcal{S}$, such as

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{\pi^2}{6}.$$

Elhosh [5] proved that if $f \in \mathcal{K}$, then $|\gamma_n| \le 1/n$. However, Girela [8] pointed out that this bound is false for the class $\mathcal{K}$ when $n \ge 2$. He proved that there exists a function $f \in \mathcal{K}$ such that $|\gamma_n| > 1/n$ for $n \ge 2$. It remains an open problem to find the correct order of growth of $|\gamma_n|$ for $f \in \mathcal{S}$ even for $f \in \mathcal{K}$. In a recent paper [15], the authors considered the inequality of type (1.2) and the order of growth of $|\gamma_n|$ for the class $\mathcal{G}(c)$, which is defined below, and also for some other subclasses of $\mathcal{S}$.

Let $\mathcal{B}$ denote the class of all analytic functions ϕ in $\mathbb{D}$ which satisfy the condition $w(0) = 0$ and $|\phi(z)| < 1$ for $z \in \mathbb{D}$. Functions in $\mathcal{B}$ are called Schwarz functions. Let f and g be two analytic functions in $\mathbb{D}$. We say that f is *subordinate* to g, written as $f \prec g$, if there exists a function $\phi \in \mathcal{B}$ such that $f(z) = g(\phi(z))$ for $z \in \mathbb{D}$. In particular, if g is univalent in $\mathbb{D}$, then $f \prec g$ is equivalent to $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$. See [4, 16].

(1) The class $\mathcal{S}^*(A, B)$ is defined by

$$\mathcal{S}^*(A, B) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \text{ for } z \in \mathbb{D} \right\},$$

where $A \in \mathbb{C}, -1 \le B \le 0$ and $A \ne B$. The class $\mathcal{S}^*(A, B)$ with the restriction $-1 \le B < A \le 1$ has been studied by Janowski [10]. In particular, for $B = -1$ and $A = e^{i\alpha}(e^{i\alpha} - 2\beta \cos \alpha)$, $0 \le \beta < 1$, the class $\mathcal{S}^*(A, B)$ reduces to the class of spirallike functions of order β, denoted by $\mathcal{S}_\alpha(\beta)$, so that

$$\mathcal{S}_\alpha(\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{-i\alpha} \frac{zf'(z)}{f(z)} \right) > \beta \cos \alpha, \text{ for } z \in \mathbb{D} \right\},$$

where $\beta \in [0, 1)$ and $\alpha \in (-\pi/2, \pi/2)$. Each function in $\mathcal{S}_\alpha(\beta)$ is univalent in $\mathbb{D}$ (see [12]). Clearly, $\mathcal{S}_\alpha(\beta) \subset \mathcal{S}_\alpha(0) \subset \mathcal{S}$ whenever $0 \le \beta < 1$. Functions in $\mathcal{S}_\alpha(0)$ are called *α-spirallike*, but they do not necessarily belong to the starlike family $\mathcal{S}^*$. The class $\mathcal{S}_\alpha(0)$ was introduced by Špaček [21] (see also [4]).

(2) The class $\mathcal{SS}^*_\alpha$ of strongly starlike functions is defined by [3, 22]

$$\mathcal{SS}^*_\alpha := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \left(\frac{1 + z}{1 - z} \right)^\alpha, \text{ for } z \in \mathbb{D} \right\},$$

for $0 < \alpha \le 1$. For $\alpha = 1$, the class $\mathcal{SS}^*_\alpha$ reduces to the class of starlike functions.

(3) The class $\mathcal{F}(c)$ is defined by

$$\begin{aligned} \mathcal{F}(c) &:= \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > 1 - \frac{c}{2}, z \in \mathbb{D} \right\} \\ &= \{ f \in \mathcal{A} : zf' \in \mathcal{S}^*(c - 1, -1) \}, \end{aligned}$$

for some $c \in (0, 3]$. If we set $\alpha = 1 - c/2 \in [0, 1)$, in this choice the family as $\mathcal{F}(c)$ is well known and is referred to as the family of convex functions of order α.

Clearly, the family $\mathcal{F}(2)$ is the usual class of normalized convex functions. In particular, for $c = 3$, we have the class $\mathcal{F}(3)$ which has attracted the attention of many in recent years (see [17] and the references therein). Also it is important to point out that functions in $\mathcal{F}(3)$ are known to be convex in one direction (and hence, univalent and close to convex) but are not necessarily starlike in $\mathbb{D}$ [23].

(4) The class $\mathcal{G}(c)$ is defined by

$$\begin{aligned}\mathcal{G}(c) &:= \left\{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{c}{2}, z \in \mathbb{D}\right\} \\ &= \{f \in \mathcal{A} : zf' \in \mathcal{S}^*(-(1+c), -1)\}\end{aligned}$$

for some $c \in (0, 1]$. Set $\mathcal{G}(1) =: \mathcal{G}$. It is known that $\mathcal{G} \subset \mathcal{S}^*$ and thus, functions in $\mathcal{G}(c)$ are starlike. This class has been studied extensively in the recent past (see, for instance, [14, 15] and the references therein).

This paper is organized as follows. In Section 2 we state our main results along with two conjectures concerning the logarithmic coefficients bound for $\mathcal{F}(c)$ and $\mathcal{G}(c)$, respectively. These conjectures have been verified for the first three logarithmic coefficients. In Section 2 we have proved the logarithmic coefficients bound completely for the families $\mathcal{S}^*(A, B)$ and $\mathcal{SS}^*_\alpha$ along with inequalities of the type (1.2) for these families. Our final results in Section 2 concern the families $\mathcal{F}(c)$ and $\mathcal{G}(c)$, where we obtain sharp estimates for the initial three coefficients, and nonsharp estimates for the fourth and fifth coefficients. In Section 3 we recall a few important lemmas which are useful in the sequel. The proofs of our main results are presented in Section 4.

2. Main results

Theorem 2.1. *For $-1 \le B < A \le 1$ and $B \ne 0$, the logarithmic coefficients of $f \in \mathcal{S}^*(A, B)$ satisfy the inequalities*

$$|\gamma_n| \le \frac{A - B}{2n}, \quad \text{for } n \ge 1, \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \le \left(\frac{A - B}{2B}\right)^2 \mathrm{Li}_2(B^2), \tag{2.2}$$

where $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$ denotes the dilogarithm function. Inequality (2.1) is sharp for the function $k_{A,B;n}(z) = z(1 + Bz^n)^{(A-B)/nB}$ and inequality (2.2) is sharp for the function $k_{A,B;1}(z)$.

Corollary 2.2. *If $f \in \mathcal{S}^*(A, -A)$ for $0 < A \le 1$, then we have*

$$|\gamma_n| \le \frac{A}{n}, \quad \text{for } n \ge 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |\gamma_n|^2 \le \mathrm{Li}_2(A^2).$$

The first and second inequalities are sharp for the functions $k_{A;n}(z) = z/(1 - Az^n)^{2/n}$ and $k_{A;1}(z)$, respectively.

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Theorem 2.1 for the case $B = 0$ takes the following form.

COROLLARY 2.3. *Let $0 < A \le 1$ and $f \in \mathcal{A}$ satisfy the inequality $|(zf'(z)/f(z)) - 1| < A$, $z \in \mathbb{D}$, that is, $f \in \mathcal{S}^*(A, 0)$. Then the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_n| \le \frac{A}{2n}, \quad \text{for } n \ge 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{A^2}{4}.$$

Both inequalities are sharp for the functions $k_{A;n}(z) = ze^{Az^n/n}$ and $k_{A;1}(z)$, respectively.

Our next result, which uses the method of proof of [1, Theorem 2.5] and [4, Theorem 6.3], establishes an inequality of the type (1.2) for the class $\mathcal{S}^*(A, B)$.

THEOREM 2.4. *Let $f \in \mathcal{S}^*(A, B)$ for $-1 \le B < A \le 1$, and let $t \le 2$. Then we have*

$$\sum_{n=1}^{\infty} (n+1)^t |\gamma_n|^2 \le \left(\frac{A - B}{2B}\right)^2 \sum_{n=1}^{\infty} \frac{(n+1)^t}{n^2} |B|^{2n}.$$

THEOREM 2.5. *For $|\alpha| < \pi/2$ and $\beta \in [0, 1)$, the logarithmic coefficients of $f \in \mathcal{S}_\alpha(\beta)$ satisfy the inequalities*

$$|\gamma_n| \le \frac{(1-\beta)}{n} \cos\alpha, \quad \text{for } n \ge 1, \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{\pi^2}{6} (1-\beta)^2 \cos^2\alpha. \tag{2.4}$$

Both inequalities are sharp for the function $f_{\alpha,\beta}(z) = z/(1-z)^{2(1-\beta)\cos\alpha}$.

For the case of strongly starlike functions, we have the following theorem.

THEOREM 2.6. *Let $0 < \alpha \le 1$ and*

$$A_n(\alpha) = \sum_{k=1}^{n} \binom{n-1}{k-1} \binom{\alpha}{k} 2^k.$$

Then the logarithmic coefficients γ_n of $f \in \mathcal{SS}^*_\alpha$ satisfy the inequalities

$$|\gamma_n| \le \frac{\alpha}{n}, \quad n \ge 1, \tag{2.5}$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{1}{4} \sum_{n=1}^{\infty} \frac{|A_n(\alpha)|^2}{n^2}. \tag{2.6}$$

Both inequalities are sharp for the function f given by $zf'(z)/f(z) = ((1+z^n)/(1-z^n))^\alpha$ and $zf'(z)/f(z) = ((1+z)/(1-z))^\alpha$, respectively.

Our next two theorems concern the two important classes $\mathcal{F}(c)$ and $\mathcal{G}(c)$.

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THEOREM 2.7. *Let $f \in \mathcal{F}(c)$ for $c \in (0,3]$. Then the logarithmic coefficients γ_n of f, for $n = 1, 2, \dots, 5$, satisfy the inequalities*

$$\begin{cases} |\gamma_1| \le \dfrac{c}{4}, \\ |\gamma_2| \le \dfrac{1}{48}(4c + c^2), \\ |\gamma_3| \le \dfrac{1}{48}(2c + c^2), \end{cases}$$

$$|\gamma_4| \le \begin{cases} \dfrac{1}{40}\Big[c + \dfrac{c^2}{18}\Big(13 + \dfrac{c}{2} - \dfrac{c^2}{8}\Big)\Big] & \text{for } c \in (0, 2.61818), \\ \dfrac{1}{40}\Big[c + \dfrac{2c^2}{9} + \dfrac{c^2}{2}|I_2|\Big] & \text{for } c \in (2.71569, 3], \end{cases}$$

and

$$|\gamma_5| \le \begin{cases} \dfrac{1}{60}\Big[c + +\dfrac{c^2}{12}\Big(11 + c - \dfrac{c^2}{4}\Big)\Big] & \text{for } c \in (0, 1.31148), \\ \dfrac{1}{60}\Big[c + \dfrac{c^2}{2} + \dfrac{c^3}{24} + \dfrac{5c^2}{12}|I_3|\Big] & \text{for } c \in (1.35541, 3], \end{cases}$$

where

$$|I_2| \le \frac{54 + c}{27}\Big(\frac{4(54 + c)}{3(288 + 8c - c^2)}\Big)^{1/2}$$

and

$$|I_3| \le \frac{30 + c}{15}\Big(\frac{2(30 + c)}{3(80 + 4c - c^2)}\Big)^{1/2}.$$

The first three inequalities are sharp for the functions

$$f_c(z) = \begin{cases} \dfrac{(1 - z)^{1-c} - 1}{c - 1}, & \text{for } c \ne 1, \\ -\log(1 - z), & \text{for } c = 1. \end{cases}$$

Moreover, the bounds for $|\gamma_4|$ are sharp for $c \le 144/55$, whereas the bounds for $|\gamma_5|$ are sharp for $c \le 80/61$.

If we take $c = 3$ in Theorem 2.7, then we obtain the logarithmic coefficients bound for the class $\mathcal{F}(3)$.

COROLLARY 2.8. *Let $\mathcal{F}(3)$, that is, $\mathrm{Re}(1 + (zf''(z)/f'(z))) > -1/2$ in $\mathbb{D}$. Then we have*

$$|\gamma_1| \le \tfrac{3}{4}, \quad |\gamma_2| \le \tfrac{7}{16}, \quad |\gamma_3| \le \tfrac{5}{16},$$
$$|\gamma_4| \le \tfrac{1}{40}\Big(5 + \tfrac{19}{2}\sqrt{\tfrac{76}{303}}\Big) \approx 0.243945$$

and

$$|\gamma_5| \le \tfrac{1}{60}\Big(\tfrac{69}{8} + \tfrac{33}{4}\sqrt{\tfrac{22}{83}}\Big) \approx 0.2145050.$$

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The first three inequalities are sharp for the function

$$f_3(z) = \frac{z - (z^2/2)}{(1-z)^2}$$

which is extremal for the class $\mathcal{F}(3)$. *For this function, we have*

$$\gamma_k(f_3) = \Big(1 - \frac{1}{2^{k+1}}\Big)\frac{1}{k}, \quad \text{for } k \in \mathbb{N}. \tag{2.7}$$

The logarithmic coefficients of f_3 given by (2.7) give that

$$\begin{aligned} 4\sum_{n=1}^{\infty}|\gamma_n(f_3)|^2 &= \sum_{n=1}^{\infty}\Big(2 - \frac{1}{2^n}\Big)^2\frac{1}{n^2} \\ &= 4\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{(1/4)^n}{n^2} - 4\sum_{n=1}^{\infty}\frac{(1/2)^n}{n^2} \\ &= \frac{4\pi^2}{6} + \mathrm{Li}_2\Big(\frac{1}{4}\Big) - 4\,\mathrm{Li}_2\Big(\frac{1}{2}\Big), \end{aligned}$$

where $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$ denotes the dilogarithm function. These observations lead us to the following conjecture.

CONJECTURE 2.9. *The logarithmic coefficients* γ_n *of* $f \in \mathcal{F}(3)$ *satisfy the inequalities*

$$|\gamma_n| \le \frac{1}{n}\Big(1 - \frac{1}{2^{n+1}}\Big), \quad \text{for } n \ge 1,$$

and

$$\sum_{n=1}^{\infty}|\gamma_n|^2 \le \frac{\pi^2}{6} + \frac{1}{4}\,\mathrm{Li}_2\Big(\frac{1}{4}\Big) - \mathrm{Li}_2\Big(\frac{1}{2}\Big).$$

Equalities in these inequalities are attained for the function $f_3(z)$ *given as above.*

In [15], the authors considered the logarithmic coefficients of the functions f in the class $\mathcal{G}(c)$ for some $c \in (0, 1]$ and obtained the estimate

$$|\gamma_n| \le \frac{c}{2(c+1)n}, \quad n \in \mathbb{N}.$$

Among other things, they conjectured that for $c = 1$ the inequalities

$$|\gamma_n| \le \frac{1}{2(n+1)n}, \quad n \in \mathbb{N},$$

are valid, where equality is attained for $f'(z) = (1 - z^n)^{1/n}$. In [15], this conjecture was proved only in the case $n = 1$. In the sequel, we shall consider the cases $n = 1, 2, 3$ in the families $\mathcal{G}(c)$ using a method similar to the proof of Theorem 2.7.

THEOREM 2.10. *Let $f \in \mathcal{G}(c)$ for $c \in (0, 1]$. Then the logarithmic coefficients γ_n of f satisfy the inequalities*

$$|\gamma_1| \le \frac{c}{4}, \quad |\gamma_2| \le \frac{c}{12}, \quad \text{and} \quad |\gamma_3| \le \frac{c}{24}.$$

The equalities are attained for $f'(z) = (1 - z^n)^{c/n}$, $n = 1, 2, 3$.

These results led us to a generalization of the conjecture mentioned above.

CONJECTURE 2.11. *The logarithmic coefficients γ_n of the functions in $\mathcal{G}(c)$, $c \in (0, 1]$, satisfy the inequalities*

$$|\gamma_n| \le \frac{c}{2n(n+1)}, \quad n \in \mathbb{N}.$$

Equality is attained for $f'(z) = (1 - z^n)^{c/n}$.

3. Lemmas

LEMMA 3.1 [13, Corollary 3.1d.1, page 76]. *Let h be starlike in $\mathbb{D}$, with $h(0) = 0$ and $a \neq 0$. If an analytic function $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ satisfies the subordination relation*

$$\frac{zp'(z)}{p(z)} \prec h(z),$$

then

$$p(z) \prec q(z) = a \exp\left[n^{-1} \int_0^z \frac{h(t)}{t}\, dt\right].$$

LEMMA 3.2 [4, Theorem 6.3, page 192] (see also [19, Rogosinski's Theorem II (i)]). *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be analytic in $\mathbb{D}$, and suppose that $f \prec g$, where g is univalent in $\mathbb{D}$. Then*

$$\sum_{k=1}^{n} |a_k|^2 \le \sum_{k=1}^{n} |b_k|^2, \quad n = 1, 2, \ldots.$$

LEMMA 3.3 [4, Theorem 6.4 (i), page 195] (see also [19, Rogosinski's Theorem X]). *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be analytic in $\mathbb{D}$. Suppose that $f \prec g$, where g is univalent and convex in $\mathbb{D}$. Then*

$$|a_n| \le |g'(0)| = |b_1|, \quad n = 1, 2, \ldots.$$

Our next lemma, due to Prokhorov and Szynal [18], is crucial in the investigation of the fourth and fifth logarithmic coefficients bound for $\mathcal{F}(c)$ and $\mathcal{G}(c)$.

LEMMA 3.4 [18, Lemma 2]. *Let $\phi(z) = \sum_{k=1}^{\infty} c_k z^k \in \mathcal{B}$ be a Schwarz function and*

$$\Psi(\phi) = |c_3 + \mu c_1 c_2 + \upsilon c_1^3|.$$

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Then for any real number μ and υ, we have the following sharp estimate:

$$\Psi(\phi) \le \Phi(\mu,\upsilon) = \begin{cases} 1, & \text{if } (\mu,\upsilon) \in D_2, \\ |\upsilon|, & \text{if } (\mu,\upsilon) \in D_6, \\ \dfrac{2}{3}(|\mu|+1)\Big(\dfrac{|\mu|+1}{3(|\mu|+1+\upsilon)}\Big)^{1/2}, & \text{if } (\mu,\upsilon) \in D_9, \end{cases}$$

where

$$D_2 = \{(\mu,\upsilon) \in \mathbb{R}^2 : \tfrac{1}{2} \le |\mu| \le 2,\ \tfrac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \le \upsilon \le 1\},$$
$$D_6 = \{(\mu,\upsilon) \in \mathbb{R}^2 : 2 \le |\mu| \le 4,\ \upsilon \ge \tfrac{1}{12}(\mu^2+8)\},$$
$$D_9 = \Big\{(\mu,\upsilon) \in \mathbb{R}^2 : |\mu| \ge 2, -\frac{2}{3}(|\mu|+1) \le \upsilon \le \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}\Big\}.$$

4. Proofs of the main results

4.1. Proof of Theorem 2.1. Suppose $f \in \mathcal{S}^*(A,B)$ for $-1 \le B < A \le 1$. Then by the definition of $\mathcal{S}^*(A,B)$, we get

$$z\Big(\log\Big(\frac{f(z)}{z}\Big)\Big)' = \frac{zf'(z)}{f(z)} - 1 \prec \frac{(A-B)z}{1+Bz}, \quad z \in \mathbb{D}, \tag{4.1}$$

which, in terms of the logarithmic coefficients γ_n of f defined by (1.1), is equivalent to

$$\sum_{n=1}^{\infty}(2n\gamma_n)z^n \prec \begin{cases} \dfrac{(A-B)}{B}\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}B^n z^n, & \text{if } -1 \le B < A \le 1, B \ne 0, \\ Az, & \text{if } B = 0, \end{cases}$$
$$=: G(z). \tag{4.2}$$

Since G is convex in $\mathbb{D}$ with $G'(0) = A - B$, it follows from Lemma 3.3 that

$$2n|\gamma_n| \le |G'(0)| = |A-B|, \quad \text{for } n \ge 1,$$

which implies the desired inequality (2.1). The equality holds for the function $k_{A,B;n}(z) = z(1+Bz^n)^{(A-B)/nB}$. We have

$$\log\Big(\frac{k_{A,B;n}(z)}{z}\Big) = \frac{A-B}{nB}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}B^k}{k}(z^n)^k.$$

Let $g(z) := z/f(z)$ which is a nonvanishing analytic function in $\mathbb{D}$ with $g(0) = 1$, and it has the series representation

$$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

It is clear from (4.1) that g satisfies the relation

$$\frac{zg'(z)}{g(z)} = 1 - \frac{zf'(z)}{f(z)} \prec \frac{-(A-B)z}{1+Bz} =: \phi(z), \quad z \in \mathbb{D}.$$

Note that ϕ is convex in $\mathbb{D}$ and $\phi(0) = 0$. By using the subordination result of Lemma 3.1, we get

$$g(z) := \frac{z}{f(z)} \prec q_{A,B}(z) = \exp\biggl(\int_0^z \frac{\phi(t)}{t} dt\biggr). \tag{4.3}$$

It is a simple exercise to compute that

$$q_{A,B}(z) = \begin{cases} e^{-Az}, & \text{for } B = 0, \\ (1 + Bz)^{1-(A/B)}, & \text{for } B \neq 0. \end{cases}$$

We can rewrite relation (4.3) as

$$\frac{f(z)}{z} \prec \frac{1}{q_{A,B}(z)},$$

which, in terms of the logarithmic coefficients γ_n of f defined by (1.1), is equivalent to (compare with (4.2))

$$2\sum_{n=1}^{\infty} \gamma_n z^n \prec \begin{cases} \biggl(\dfrac{A-B}{B}\biggr) \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{B^n}{n} z^n = \biggl(\frac{A-B}{B}\biggr) \log(1 + Bz), & \text{for } B \neq 0, \\ Az, & \text{for } B = 0. \end{cases}$$

Using Lemma 3.2, we obtain that

$$\begin{aligned} 4\sum_{n=1}^{k} |\gamma_n|^2 &\leq \biggl(\frac{A-B}{B}\biggr)^2 \sum_{n=1}^{k} \frac{B^{2n}}{n^2}, \qquad \text{for } B \neq 0 \\ &\leq (A-B)^2 \frac{\mathrm{Li}_2(B^2)}{B^2}. \end{aligned}$$

Letting $k \to \infty$, we get

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{(A-B)^2}{4} \frac{\mathrm{Li}_2(B^2)}{B^2},$$

where $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} (x^n/n^2)$. For $x = 0$, we let $\mathrm{Li}_2(x)/x$ as the limit value 1. This proves the desired assertion (2.2). The equality holds for the function $k_{A,B}$ defined by

$$k_{A,B}(z) := \begin{cases} ze^{Az}, & \text{for } B = 0, \\ z(1 + Bz)^{(A/B)-1}, & \text{for } B \neq 0. \end{cases}$$

Indeed, for the function $k_{A,B}$, we have

$$\begin{aligned} \log\biggl(\frac{k_{A,B}(z)}{z}\biggr) &= \begin{cases} \biggl(\dfrac{A-B}{B}\biggr) \log(1 + Bz), & \text{for } B \neq 0, \\ Az, & \text{for } B = 0, \end{cases} \\ &= 2\sum_{n=1}^{\infty} \gamma_n(k_{A,B}) z^n, \end{aligned}$$

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where

$$\gamma_n(k_{A,B}) = \begin{cases} (-1)^{n-1}\left(\dfrac{A-B}{2B}\right)\dfrac{B^n}{n}, & \text{for } B \neq 0, \\ \dfrac{A}{2}, & \text{for } B = 0. \end{cases}$$

This completes the proof of Theorem 2.1. □

We remark that when $A = 1 - 2\beta$ and $B = -1$ in Theorem 2.1, we obtain [15, Remark 1].

Remark 4.1. From relation (4.2) and using Rogosinski's theorem (see [4, Theorem 6.3]), we obtain

$$4\sum_{n=1}^{\infty} n^2|\gamma_n|^2 \leq \left|\frac{A-B}{B}\right|^2 \sum_{n=1}^{\infty} |B|^{2n}, \quad \text{for } B \neq -1,$$

and so

$$\sum_{n=1}^{\infty} n^2|\gamma_n|^2 \leq \frac{(A-B)^2}{4(1-B^2)}. \tag{4.4}$$

4.2. Proof of Theorem 2.4. We recall from formula (4.2) and Rogosinski's result (see also [16, Theorem 2.2] and [4, Theorem 6.3]), that for $k \in \mathbb{N}$ the inequalities

$$4\sum_{n=1}^{k} n^2|\gamma_n|^2 \leq \left(\frac{A-B}{B}\right)^2 \sum_{n=1}^{k} |B|^{2n} \tag{4.5}$$

are valid. This implies inequality (4.4) as well, if $B \neq -1$. We consider (4.5) for $k = 1, 2, \ldots, N$, and multiply the kth inequality by the factor

$$\frac{(k+1)^t}{k^2} - \frac{(k+2)^t}{(k+1)^2} > 0, \quad \text{if } k = 1, 2, \ldots, N-1,$$

and by $(N+1)^t/N^2$ for $k = N$. Then summation of the inequalities multiplied yields

$$\sum_{k=1}^{N} (k+1)^t|\gamma_k|^2 \leq \left(\frac{A-B}{2B}\right)^2 \sum_{k=1}^{N} \frac{(k+1)^t}{k^2} B^{2k},$$

for $N = 1, 2, \ldots$. Allowing $N \to \infty$, we see that the proof of the theorem is complete. □

Remark 4.2. Here is an alternate approach to prove the inequality (2.2) (compare, for example, [1, Theorem 2.5]). If we take $t = 0$ in Theorem 2.4, then we obtain

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \left(\frac{A-B}{2B}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} B^{2n} = \left(\frac{A-B}{2B}\right)^2 \mathrm{Li}_2(B^2).$$

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4.3. Proof of Theorem 2.5. Suppose $f \in \mathcal{S}_\alpha(\beta)$. Then by the definition of $\mathcal{S}_\alpha(\beta)$, we obtain that

$$\frac{zf'(z)}{f(z)} - 1 \prec \frac{1 + [e^{2i\alpha} - 2\beta e^{i\alpha}\cos\alpha]z}{1-z} - 1$$
$$= 2(1-\beta)e^{i\alpha}\cos\alpha\Big(\frac{z}{1-z}\Big) =: G_1(z), \tag{4.6}$$

which, in terms of the logarithmic coefficients γ_n of f defined by (1.1), is equivalent to

$$2\sum_{n=1}^{\infty} n\gamma_n z^n \prec G_1(z).$$

Since G_1 is convex in $\mathbb{D}$ and $G_1'(0) = 2(1-\beta)e^{i\alpha}\cos\alpha$, Rogosinski's result (see Lemma 3.3) gives inequality (2.3).

From (4.6), g defined by $g(z) = z/f(z)$ satisfies the relation

$$\frac{zg'(z)}{g(z)} \prec -G_1(z), \quad z \in \mathbb{D}.$$

Note that G_1 is convex in $\mathbb{D}$ with $G_1(0) = 0$. By Lemma 3.1, we get

$$g(z) = \frac{z}{f(z)} \prec q_{\alpha,\beta}(z) := \exp\Big(-\int_0^z \frac{G_1(t)}{t}\,dt\Big),$$

or equivalently,

$$\frac{f(z)}{z} \prec \frac{1}{q_{\alpha,\beta}(z)} = (1-z)^{-\gamma} =: \frac{f_{\alpha,\beta}(z)}{z},$$

where $\gamma = 2(1-\beta)\cos\alpha$. It is easy to see that $f_{\alpha,\beta} \in \mathcal{S}_\alpha(\beta)$. From (1.1), we have

$$2\sum_{n=1}^{\infty}\gamma_n z^n \prec \gamma\sum_{n=1}^{\infty}\frac{z^n}{n}.$$

Then by Lemma 3.2, we obtain

$$\sum_{n=1}^{k}|\gamma_n|^2 \le \frac{|\gamma|^2}{4}\sum_{n=1}^{k}\frac{1}{n^2} \le \frac{|\gamma|^2}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

If we allow $k \to \infty$, we get inequality (2.4). The equality holds in inequalities (2.3) and (2.4) for the function $f_{\alpha,\beta}(z) = z/(1-z)^\gamma$, $\gamma = 2(1-\beta)\cos\alpha$. □

4.4. Proof of Theorem 2.6. Suppose $f \in \mathcal{SS}_\alpha^*$ and $g(z) = z/f(z)$. Then g is a nonvanishing analytic function in $\mathbb{D}$ and, by the definition of $\mathcal{SS}_\alpha^*$, we get

$$\frac{zg'(z)}{g(z)} = 1 - \frac{zf'(z)}{f(z)} \prec 1 - \Big(\frac{1+z}{1-z}\Big)^\alpha =: \phi_1(z), \quad z \in \mathbb{D}. \tag{4.7}$$

Using Lemma 3.1, we obtain

$$g(z) = \frac{z}{f(z)} \prec q_\alpha(z) := \exp\Big(\int_0^z \frac{\phi_1(t)}{t}\Big)\,dt,$$

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which is equivalent to

$$\log\Big(\frac{f(z)}{z}\Big) \prec \log\Big(\frac{1}{q_\alpha(z)}\Big) = -\int_0^z \frac{\phi_1(t)}{t}\,dt.$$

We consider the function

$$\Big(\frac{1+z}{1-z}\Big)^\alpha = 1 + \sum_{n=1}^\infty A_n(\alpha) z^n. \tag{4.8}$$

Using this, we have

$$\int_0^z \frac{\phi_1(t)}{t}\,dt = -\sum_{n=1}^\infty A_n(\alpha)\frac{z^n}{n},$$

where

$$A_n(\alpha) = \sum_{k=1}^n \binom{n-1}{k-1}\binom{\alpha}{k} 2^k, \quad \text{for } n \ge 1.$$

As in the previous case, by Rogosinski's theorem (see Lemma 3.2), we obtain

$$\sum_{n=1}^\infty |\gamma_n|^2 \le \frac{1}{4}\sum_{n=1}^\infty \frac{|A_n(\alpha)|^2}{n^2}.$$

The proof of (2.6) is complete.

We now prove inequality (2.5). From (4.7) and (4.8), we also get

$$\frac{zf'(z)}{f(z)} - 1 \prec -\phi_1(z) = \sum_{n=1}^\infty A_n(\alpha) z^n,$$

which, in terms of the logarithmic coefficients γ_n of f defined by (1.1), is equivalent to

$$2\sum_{n=1}^\infty n\gamma_n z^n \prec \sum_{n=1}^\infty A_n(\alpha) z^n.$$

Using Lemma 3.3, we find that

$$2n|\gamma_n| \le |-\phi_1'(0)| = |A_1(\alpha)| = 2\alpha,$$

and the desired inequality (2.5) follows. Equality occurs in inequalities (2.5) and (2.6) if $f \in \mathcal{A}$ is given by

$$\frac{zf'(z)}{f(z)} = \Big(\frac{1+z^n}{1-z^n}\Big)^\alpha \quad \text{and} \quad \frac{zf'(z)}{f(z)} = \Big(\frac{1+z}{1-z}\Big)^\alpha,$$

respectively. □

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4.5. Proof of Theorem 2.7. Let $f \in \mathcal{F}(c)$ for $c \in (0, 3]$. Consider the identity

$$\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right) = \left(\frac{zf'(z)}{f(z)}\right)^2 + z\left(\frac{zf'(z)}{f(z)}\right)'. \tag{4.9}$$

We may now set

$$1 + \frac{zf''(z)}{f'(z)} = \sum_{n=0}^{\infty} \beta_n z^n, \tag{4.10}$$

where $\beta_0 = 1$. By relation (1.1), we have

$$\frac{zf'(z)}{f(z)} = 1 + 2\sum_{n=1}^{\infty} n\gamma_n z^n = \delta_0 + \sum_{n=1}^{\infty} \delta_n z^n, \tag{4.11}$$

where $\delta_0 = 1$ and $\delta_n = 2n\gamma_n$. Using (4.10) and (4.11), we can write (4.9) in series form as

$$\left(\sum_{n=0}^{\infty} \beta_n z^n\right)\left(\sum_{n=0}^{\infty} \delta_n z^n\right) = \left(\sum_{n=0}^{\infty} \delta_n z^n\right)^2 + \sum_{n=0}^{\infty} n\delta_n z^n.$$

Using the Cauchy product of power series, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \delta_n\beta_{n-k}\right)z^n &= \delta_0\beta_0 + (\delta_0\beta_1 + \delta_1\beta_0)z + (\delta_0\beta_2 + \delta_1\beta_1 + \delta_2\beta_0)z^2 + \cdots \\ &= (\delta_0 + \delta_1 z + \delta_2 z^2 + \delta_3 z^3 + \cdots)^2 + \delta_1 z + 2\delta_2 z^2 + 3\delta_3 z^3 + \cdots .\end{aligned}$$

As $\beta_0 = 1 = \delta_0$, this is equivalent to

$$\begin{aligned}&1 + (\delta_1 + \beta_1)z + (\delta_2 + \delta_1\beta_1 + \beta_2)z^2 + (\beta_3 + \delta_1\beta_2 + \delta_2\beta_1 + \delta_3)z^3 + \cdots \\ &\quad = 1 + (2\delta_1 + \delta_1)z + (2\delta_2 + \delta_1^2 + 2\delta_2)z^2 + (2\delta_3 + 2\delta_1\delta_2 + 3\delta_3)z^3 + \cdots .\end{aligned} \tag{4.12}$$

First, we compare the coefficients of z^n for $n = 1, 2, 3$ and get (by using $\gamma_n = \delta_n/(2n)$) that

$$\gamma_1 = \frac{\delta_1}{2} = \frac{\beta_1}{4}, \tag{4.13}$$

$$\gamma_2 = \frac{\delta_2}{4} = \frac{1}{12}\left(\beta_2 + \frac{\beta_1^2}{4}\right), \tag{4.14}$$

and

$$\gamma_3 = \frac{\delta_3}{6} = \frac{1}{24}\left(\beta_3 + \frac{\beta_1\beta_2}{2}\right). \tag{4.15}$$

By the definition of $f \in \mathcal{F}(c)$ and (4.10), we get

$$\sum_{k=0}^{\infty} \beta_k z^k = 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{cz}{1-z}, \quad z \in \mathbb{D}. \tag{4.16}$$

But then, by Rogosinski's theorem (see, for instance, [4, page 195, Theorem 6.4]), we obtain that

$$|\beta_k| \le c, \quad k \in \mathbb{N},$$

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which also follows from the estimate on the coefficients of functions with real part bigger than $\alpha = 1 - c/2 < 1$:

$$|\beta_k| \le 2(1-\alpha) = c, \quad k \in \mathbb{N}.$$

The estimate here is sharp. Using this inequality, the above identities (4.13)–(4.15) imply

$$|\gamma_1| \le \frac{c}{4}, \quad |\gamma_2| \le \frac{1}{48}(4c + c^2) \quad \text{and} \quad |\gamma_3| \le \frac{1}{48}(2c + c^2).$$

These inequalities are sharp and the equality follows from (4.13)–(4.15) by substituting $\beta_k = c$ for $k = 1, 2, 3$. The extremal function can be derived by integration of the equation

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{cz}{1-z},$$

which results in

$$f(z) = \frac{(1-z)^{1-c} - 1}{c-1} =: f_c(z), \quad \text{for } c \ne 1,$$

and $f_1(z) = -\log(1-z)$ for $c = 1$.

To get estimates for the fourth and fifth logarithmic coefficients, by (4.16), we use the fact that

$$\sum_{k=1}^{\infty} \beta_k z^k = \frac{c\phi(z)}{1 - \phi(z)}, \tag{4.17}$$

where $\phi \in \mathcal{B}$ is a Schwarz function. Note that equality is attained in the estimate $|\beta_k| \le c$, if $\phi(z) = e^{i\theta} z$. If we write $\phi(z) = \sum_{k=1}^{\infty} c_k z^k$, then identity (4.17) implies

$$\beta_1 = cc_1, \quad \beta_2 = c(c_2 + c_1^2) \quad \text{and} \quad \beta_3 = c(c_3 + 2c_1c_2 + c_1^3).$$

In view of these, expressions of the form

$$I_1 = \beta_3 + a\beta_1\beta_2 + b\beta_1^3$$

can be written in the form

$$I_1 = c[c_3 + (2 + ac)c_1c_2 + c_1^3(1 + ac + bc^2)]. \tag{4.18}$$

We now consider the functional

$$\Psi(\phi) = |c_3 + \mu c_1 c_2 + \upsilon c_1^3|, \quad \mu \text{ and } \upsilon \text{ are real},$$

within the class $\mathcal{B}$. In [18], Prokhorov and Szynal found the precise bound for the functional $\Psi(\phi)$ which we recalled in Lemma 3.4 and we use this to find the fourth and the fifth logarithmic coefficients bound for the class $\mathcal{F}(c)$.

We now calculate the fourth logarithmic coefficient bound for the class $\mathcal{F}(c)$. From (4.12), we compare the coefficients of z^4 and obtain

$$\beta_4 + \beta_3\delta_1 + \beta_2\delta_2 + \beta_1\delta_3 + \delta_4 = 6\delta_4 + 2\delta_1\delta_3 + \delta_2^2.$$

Using (4.13) and (4.14) and then simplifying, we get

$$5\delta_4 = \beta_4 + \beta_3\delta_1 + \beta_2\delta_2 - \delta_2^2 = \beta_4 + \beta_3\delta_1 + \delta_2(\beta_2 - \delta_2)$$
$$= \beta_4 + \frac{\beta_3\beta_1}{2} + \frac{1}{3}\Big(\beta_2 + \frac{\beta_1^2}{4}\Big)\Big[\beta_2 - \frac{1}{3}\Big(\beta_2 + \frac{\beta_1^2}{4}\Big)\Big].$$

Since $\delta_4 = 8\gamma_4$, we have

$$40|\gamma_4| = \Big|\beta_4 + \frac{\beta_3\beta_1}{2} + \frac{1}{9}\Big(2\beta_2^2 + \frac{\beta_1^2\beta_2}{4} - \frac{\beta_1^4}{16}\Big)\Big|$$
$$\leq |\beta_4| + \frac{2}{9}|\beta_2|^2 + \frac{|\beta_1|}{2}\Big|\beta_3 + \frac{\beta_1\beta_2}{18} - \frac{\beta_1^3}{72}\Big|$$
$$\leq c + \frac{2c^2}{9} + \frac{c^2}{2}\Big|c_3 + \Big(2 + \frac{c}{18}\Big)c_1c_2 + \Big(1 + \frac{c}{18} - \frac{c^2}{72}\Big)c_1^3\Big|$$

(using (4.18))

$$= c + \frac{2c^2}{9} + \frac{c^2}{2}|I_2|, \tag{4.19}$$

where

$$I_2 := c_3 + \mu c_1c_2 + \upsilon c_1^3, \quad \mu = 2 + \frac{c}{18} \quad \text{and} \quad \upsilon = 1 + \frac{c}{18} - \frac{c^2}{72}. \tag{4.20}$$

Since our aim is to get an upper bound that corresponds to $|c_1| = 1$, we seek values of c such that

$$|I_2| \leq |\upsilon|. \tag{4.21}$$

We shall consider cases where

$$\upsilon = 1 + \frac{c}{18} - \frac{c^2}{72} \geq 0.$$

From [18] (see D_6 in Lemma 3.4) we deduce that (4.21) is satisfied if

$$1 + \frac{c}{18} - \frac{c^2}{72} \geq \frac{1}{12}\Big[\Big(2 + \frac{c}{18}\Big)^2 + 8\Big].$$

This condition is equivalent to $0 < c \leq 144/55 = 2.61818$. Unfortunately $c = 3$ does not lie within this range. Nevertheless one may use Lemma 3.4. From (4.19) and (4.21), we get

$$40|\gamma_4| \leq c + \frac{2c^2}{9} + \frac{c^2}{2}|\upsilon|, \qquad \text{for } c \in (0, 2.61818).$$

Since υ is positive, we therefore obtain

$$|\gamma_4| \leq \frac{1}{40}\Big[c + \frac{c^2}{18}\Big(13 + \frac{c}{2} - \frac{c^2}{8}\Big)\Big], \qquad \text{for } c \in (0, 2.61818). \tag{4.22}$$

Further discussion in Lemma 3.4 reveals that among the cases therein we have to choose the case D_9. For the case D_9, the inequality

$$-\frac{2}{3}(|\mu|+1) \leq \upsilon \leq \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}$$

is true for $c \in (2.71569, 3]$. Clearly the left-hand side of the inequality is true for $c \in (0, 3]$, whereas the right-hand side of the inequality holds only for $c \in (2.71569, 3]$. In view of this reasoning, from (4.20), we find that

$$|I_2| \leq \frac{2}{3}\Big(3+\frac{c}{18}\Big)\bigg[\frac{3+\frac{c}{18}}{3(4+\frac{2c}{18}-\frac{c^2}{72})}\bigg]^{1/2} = \frac{54+c}{27}\bigg[\frac{4(54+c)}{3(288+8c-c^2)}\bigg]^{1/2}, \tag{4.23}$$

for $c \in (2.71569, 3]$. From (4.19), we obtain

$$|\gamma_4| \leq \frac{1}{40}\Big(c+\frac{2c^2}{9}+\frac{c^2}{2}|I_2|\Big), \quad \text{for } c \in (2.71569, 3]. \tag{4.24}$$

Hence, from (4.22) and (4.24), we obtain the desired inequality.

For example, if we let $c = 3$ in (4.24) and (4.23), then (4.24) gives

$$|\gamma_4| \leq \tfrac{1}{40}\Big(3+2+\tfrac{19}{2}\sqrt{\tfrac{76}{303}}\Big) = \tfrac{1}{40}\Big(5+\tfrac{19}{2}\sqrt{\tfrac{76}{303}}\Big) \approx 0.243945.$$

On the other hand, for the function f_3, we have $\gamma_4(f_3) = \frac{31}{128} \approx 0.2421875$ (see (2.7)).

We now calculate the fifth logarithmic coefficient bound for the class $\mathcal{F}(c)$. From (4.12), we compare the coefficients of z^5 and obtain

$$\beta_5+\beta_4\delta_1+\beta_3\delta_2+\beta_2\delta_3+\beta_1\delta_4+\delta_5 = 2\delta_5+2\delta_1\delta_4+2\delta_2\delta_3+5\delta_5 = 7\delta_5+2\delta_1\delta_4+2\delta_2\delta_3.$$

Using relations (4.13)–(4.15) and then simplifying, we obtain

$$\begin{aligned}6\delta_5 &= \beta_5+\beta_4\delta_1+\delta_2(\beta_3-\delta_3)+\delta_3(\beta_2-\delta_2)\\ &= \beta_5+\tfrac{1}{2}\beta_1\beta_4+\tfrac{1}{24}\beta_1^2\beta_3+\tfrac{5}{12}\beta_2(\beta_3+\tfrac{1}{10}\beta_1\beta_2-\tfrac{1}{20}\beta_1^3).\end{aligned}$$

If we take similar steps to those above and use (4.18), we arrive at the estimate

$$\begin{aligned}|\gamma_5| &\leq \frac{1}{60}\bigg[c+\frac{c^2}{2}+\frac{c^3}{24}+\frac{5c^2}{12}\bigg|c_3+\Big(2+\frac{c}{10}\Big)c_1c_2+\Big(1+\frac{c}{10}-\frac{c^2}{20}\Big)c_1^3\bigg|\bigg]\\ &= \frac{1}{60}\bigg[c+\frac{c^2}{2}+\frac{c^3}{24}+\frac{5c^2}{12}|I_3|\bigg],\end{aligned} \tag{4.25}$$

where

$$I_3 := c_3+\mu c_1c_2+\upsilon c_1^3, \quad \mu = 2+\frac{c}{10} \quad \text{and} \quad \upsilon = 1+\frac{c}{10}-\frac{c^2}{20}. \tag{4.26}$$

Our aim is to find an upper bound for $|I_3|$. Since

$$\upsilon \geq \frac{1}{12}\bigg[\Big(2+\frac{c}{10}\Big)^2+8\bigg]$$

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for $0 < c \leq 80/61 = 1.31148\ldots$, we have proved the sharp estimate for these values of the parameter c. Lemma 3.4 gives $|I_3| \leq |\upsilon|$, $c \in (0, 1.31148)$, for the case D_6. Since υ is positive, we therefore have

$$|\gamma_5| \leq \frac{1}{60}\left[c + \frac{c^2}{2} + \frac{c^3}{24} + \frac{5c^2}{12}\upsilon\right]$$
$$= \frac{1}{60}\left[c + \frac{c^2}{12}\left(11 + c - \frac{c^2}{4}\right)\right], \quad \text{for } c \in (1.31148, 3]. \tag{4.27}$$

We see that this corresponds again to the case D_9 in Lemma 3.4. The case D_9 holds for $c \in (1.35541, 3]$. In view of this, from (4.26), we get

$$|I_3| \leq \frac{2}{3}\left(3 + \frac{c}{10}\right)\left[\frac{3 + \frac{c}{10}}{3(4 + \frac{2c}{10} - \frac{c^2}{20})}\right]^{1/2} = \frac{30 + c}{15}\left[\frac{2(30 + c)}{3(80 + 4c - c^2)}\right]^{1/2} \tag{4.28}$$

for $c \in (1.35541, 3]$. From (4.25), we find that

$$|\gamma_5| \leq \frac{1}{60}\left[c + \frac{c^2}{2} + \frac{c^3}{24} + \frac{5c^2}{12}|I_3|\right], \quad \text{for } c \in (1.35541, 3]. \tag{4.29}$$

Hence, from (4.27) and (4.29), we obtain the desired inequality. For instance, if we take $c = 3$, then from (4.28) we get

$$|I_3| \leq \tfrac{11}{5}\sqrt{\tfrac{22}{83}},$$

which, by (4.25), gives

$$|\gamma_5| \leq \tfrac{1}{60}\left(3 + \tfrac{9}{2} + \tfrac{27}{24} + \tfrac{33}{4}\sqrt{\tfrac{22}{83}}\right) = \tfrac{1}{60}\left(\tfrac{69}{8} + \tfrac{33}{4}\sqrt{\tfrac{22}{83}}\right) \approx 0.2145050.$$

On the other hand, in this case, $\gamma_5(f_3) = \frac{63}{320} \approx 0.196875$ (see (2.7)). □

REMARK 4.3. The above estimates contain sharp estimates for the logarithmic coefficients of convex functions of order α. In particular, we have the sharp estimates for $|\gamma_n|$, $n = 1, 2, 3, 4$, for convex functions, that is, for the case $\alpha = 0$, and sharp estimates for $|\gamma_n|$, $n = 1, 2, 3, 4, 5$, for convex functions of order $\alpha = 1/2$.

4.6. Proof of Theorem 2.10. Choosing similar abbreviations to those in Theorem 2.7, $f \in \mathcal{G}(c)$ if and only if

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 - \frac{cz}{1 - z}.$$

Hence, we get

$$\beta_1 = -cc_1, \quad \beta_2 = -c(c_2 + c_1^2) \quad \text{and} \quad \beta_3 = -c(c_3 + 2c_1c_2 + c_1^3).$$

This results in the relation

$$\gamma_1 = \frac{\beta_1}{4} = \frac{-cc_1}{4}$$

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so that $|\gamma_1| \le c/4$, where equality is attained for $f'(z) = (1-z)^c$. Further,

$$\gamma_2 = \frac{-c}{12}\left[c_2 + \left(1 - \frac{c}{4}\right)c_1^2\right].$$

To get an estimate for $|\gamma_2|$, we use the triangle inequality together with the well-known inequality $|c_2| \le 1 - |c_1|^2$ for $\phi(z) = \sum_{k=1}^{\infty} c_k z^k$ in $\mathcal{B}$. This implies

$$|\gamma_2| \le \frac{c}{12}\left(1 - \frac{c}{4}|c_1|^2\right) \le \frac{c}{12}.$$

Equality is attained here for the function $f'(z) = (1-z^2)^{c/2}$. In the case $n = 3$, we consider

$$\gamma_3 = \frac{-c}{24}\left[c_3 + \left(2 - \frac{c}{2}\right)c_2c_1 + \left(1 - \frac{c}{2}\right)c_1^3\right].$$

We see that this corresponds to the case D_2 in Lemma 3.4, and from this we obtain $|\gamma_3| \le c/24$. Here, equality is attained for $f'(z) = (1-z^3)^{c/3}$.

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