Invariant Subspaces on $\mathbb{T}^N$ and $\mathbb{R}^N$

Michio Seto

Abstract. Let $N$ be an integer which is larger than one. In this paper we study invariant subspaces of $L^2(\mathbb{T}^N)$ under the double commuting condition. A main result is an $N$-dimensional version of the theorem proved by Mandrekar and Nakazi. As an application of this result, we have an $N$-dimensional version of Lax’s theorem.

1 Invariant Subspaces of $L^2(\mathbb{T}^N)$

Let $N$ be an integer which is larger than one, and $\mathbb{T}^N$ denote the torus, the Cartesian product of $N$ unit circles in $\mathbb{C}$, that is,

$$\mathbb{T}^N = \{ z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N : |z_i| = 1 (i = 1, 2, \ldots, N) \}.$$

$L^2(\mathbb{T}^N)$ will denote the usual Lebesgue space with respect to the normalized Lebesgue measure $\mu$ of $\mathbb{T}^N$, and let $\alpha$ denote a multi-index that is an ordered $N$-tuple $\alpha = (\alpha_1, \ldots, \alpha_N)$ of integers $\alpha_j$. $H^2(\mathbb{T}^N) = \bigotimes_1^N H^2(\mathbb{T})$ will denote the Hardy space over $\mathbb{T}^N$, that is, $H^2(\mathbb{T}^N)$ is the space of all $f$ in $L^2(\mathbb{T}^N)$ whose Fourier coefficients

$$\hat{f}(\alpha) = \int_{\mathbb{T}^N} f(z) z^\alpha \, d\mu$$

are 0 whenever at least one component of $\alpha$ is negative. For integers $1 \leq i_1, \ldots, i_k \leq N$, we define a closed subspace $H^2_{z_{i_1} \cdots z_{i_k}}(\mathbb{T}^N)$ of $L^2(\mathbb{T}^N)$ as follows:

$$H^2_{z_{i_1} \cdots z_{i_k}}(\mathbb{T}^N) = \bigvee_{m_1, \ldots, m_k < 0} z_{i_1}^{m_1} \cdots z_{i_k}^{m_k} H^2(\mathbb{T}^N),$$

where $\bigvee$ denotes closed vector span.

Definition 1 A closed subspace $\mathcal{M}$ of $L^2(\mathbb{T}^N)$ is said to be an invariant subspace of $L^2(\mathbb{T}^N)$ if $z_i \mathcal{M} \subseteq \mathcal{M}$ for any $i = 1, \ldots, N$. $V_i$ denotes the restriction on $\mathcal{M}$ of the multiplication operator $L_{z_i}$ on $L^2(\mathbb{T}^N)$ by $z_i$.

Mandrekar [5] and Nakazi [9] characterized the invariant subspaces of $L^2(\mathbb{T}^2)$ under the condition that $V_1$ commutes with $V_2^*$.

Theorem 1 ([9]) Let $\mathcal{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$. If $\mathcal{M}$ satisfies the condition that $V_1 V_2^* = V_2^* V_1$, then one and only one of the following occurs.

Received by the editors January 23, 2002; revised May 30, 2002.

AMS subject classification: Primary: 47A15; secondary: 47B47.

Keywords: invariant subspaces.

Invariant Subspaces on $\mathbb{T}^N$ and $\mathbb{R}^N$

(i) $M = \chi_E L^2(T^2) \oplus \chi_{E_i} \phi_i H^2_{2,2,z}(T^2)$,
(ii) $M = \chi_E L^2(T^2) \oplus \chi_{E_i} \phi_i H^2_{2,2,z}(T^2)$,
(iii) $M = \phi H^2(T^2),$

where $\phi_i$ and $\phi$ are unimodular functions, $\chi_E$ denotes the characteristic function of $E$, $\chi_{E_i}$ is the characteristic function of $E_i$ which depends only on a variable $z_i$.

We consider the condition which is analogous to the case $N = 2$, that is, each $V_i$ commutes with $V_i^*$ for any $i \neq i$. Since $V_i$ is an isometry on $M$ for each $i = 1, 2, \ldots, N$, we have the following Wold decomposition,

$$V_i = V_i^{(u)} + V_i^{(o)} \quad \text{on } M = M^{(u)} \oplus M^{(o)},$$

where $V_i^{(u)} = V_i | M^{(u)}$ is unitary, $V_i^{(o)} = V_i | M^{(o)}$ is a unilateral shift, and

$$M^{(u)} = \left\{ f \in M : \|V_i^{(k)} f\| = \|f\| \quad (k \geq 1) \right\} = \bigcap_{k=1}^{\infty} \left\{ f \in M : V_i^{(k)} V_i^{(k)} f = f \right\} = \bigcap_{k=1}^{\infty} V_i^{(k)} M$$

is the maximal reducing subspace on which its restriction is unitary. Let $P_i$ denote the projection from $M$ onto $M^{(u)}$. Then $P_i$ is in the center of $\mathbb{R}(V_i)$, where $\mathbb{R}(V_i)$ is the von Neumann algebra generated by $V_i$ (cf. [12]).

The following lemma is well known.

**Lemma 1** Let $M$ and $N$ be invariant subspaces of $L^2(\mathbb{T}^N)$. If $M$ is orthogonal to $N$, then $fg = 0$ for any $f \in M$ and any $g \in N$.

**Proof** For any $f \in M$ and any $g \in N$, $\int_{\mathbb{T}^N} f \overline{g} \sigma \, d\sigma = 0$.

In the following argument, we deal with the case where $N = 3$, because it is difficult to describe invariant subspaces under the double commuting condition in general. (But we will be aware that our proof in the case where $N = 3$ can be applied to the general case.) This is a complicated problem and, in the later remark, we shall reduce this complication.

**Theorem 2** Let $M$ be an invariant subspace of $L^2(\mathbb{T}^3)$. If $M$ satisfies the condition that $V_i V_j^* = V_j^* V_i$ for $i \neq j$, then one and only one of the following occurs.

(i) $M = \chi_E L^2(T^3) \oplus \chi_{E_i}, \phi_i H^2_{2,2,3}(T^3) \oplus \chi_{E_j}, \phi_j H^2_{2,2,3}(T^3) \oplus \chi_{E_k}, \phi_k H^2_{2,2,3}(T^3)$,
(ii) $M = \chi_E L^2(T^3) \oplus \chi_{E_i}, \phi_i H^2_{2,2,3}(T^3) \oplus \chi_{E_j}, \phi_j H^2_{2,2,3}(T^3) \oplus \chi_{E_k}, \phi_k H^2_{2,2,3}(T^3)$,
(iii) $M = \chi_E L^2(T^3) \oplus \chi_{E_i}, \phi_i H^2_{2,2,3}(T^3) \oplus \chi_{E_j}, \phi_j H^2_{2,2,3}(T^3) \oplus \chi_{E_k}, \phi_k H^2_{2,2,3}(T^3)$,
(iv) $M = \chi_E L^2(T^3) \oplus \chi_{E_i}, \phi_i H^2_{2,2,3}(T^3) \oplus \chi_{E_j}, \phi_j H^2_{2,2,3}(T^3) \oplus \chi_{E_k}, \phi_k H^2_{2,2,3}(T^3)$,
(v) $M = \phi H^2(T^3),$

where $\phi_i$ and $\phi$ are unimodular functions, $\chi_E$ denotes the characteristic function of $E$, $\chi_{E_i}$ is the characteristic function of $E_i$ which depends only on two variables $z_i$ and $z_j$, $\chi_E$ is the characteristic function of $E$ which depends only on a variable $z_i$.  

https://doi.org/10.4153/CMB-2004-011-5 Published online by Cambridge University Press
Proof The following argument is a slight modification of the proof given by Mandrekar [5] and Nakazi [9].
Suppose that $V_i V_j^* = V_j^* V_i$ $(i \neq j)$. By the property of Wold decomposition, we have the following:

$$M = \sum_{a,b,c \in \{0,1\}} p^{(a)}_1 p^{(b)}_2 p^{(c)}_3 M,$$

where $p^{(0)}_i = P_i$ and $p^{(1)}_i = P_i^\perp = I - P_i$. We note that $P_i$ commutes with $P_j$ because $V_i \in \sigma(V_j)'$ and $V_j \in \sigma(V_i)'$ by the assumption. Hence it suffices to describe $P_1 P_2 P_3 M$, $P_1 P_2 P_3^\perp M$, $P_1 P_2^\perp P_3 M$, and $P_1^\perp P_2 P_3 M$. Let $M_1$, $M_2$, $M_3$ and $M_4$ denote the above four subspaces, respectively.

(1) By the Wiener-Tauberian theorem, $M_1 = \chi_L L^2(T^3)$.

(2) For $M_2 = P_1 P_2 P_3^\perp M$, we have the following decomposition:

$$M_2 = \sum_{k=0}^{\infty} V_k^j (M_2 \ominus V_3 M_2),$$

where we note that the restriction of $V_1$ and $V_2$ on $M_2 \ominus V_3 M_2$ are unitary operators. For $f \in M_2 \ominus V_3 M_2$, we have $\int |f|^2 z_1^j z_2^k d\mu = 0$ for $k \neq 0$. Hence $|f|$ is independent of $z_3$. Then we have

$$\bigvee_{i,j \in \mathbb{Z}} \phi_{z_i^j f} = \phi_f \chi_{E(f)} L^2(T^2),$$

where, for any measurable function $g$, a measurable set $E(g)$ and a unimodular function $\phi_g$ are defined as follows:

$$E(g) = \{ z \in T^N : g(z) \neq 0 \},$$

$$\phi_g = \begin{cases} g/|g| & (g \neq 0) \\ 1 & (g = 0). \end{cases}$$

Since there exists a function $F$ in $M_2 \ominus V_3 M_2$ which has the maximal support in $M_2 \ominus V_3 M_2$, that is, $E(f) \subseteq E(F)$, for any $f \in M_2 \ominus V_3 M_2$, we have $M_2 \ominus V_3 M_2 = \phi_F \chi_{E(F)} L^2(T^2)$. Hence

$$M_2 = \sum_{k=0}^{\infty} V_k^j \phi_F \chi_{E(F)} L^2(T^2) = \chi_{E(F)} \phi_F H^2_{z_1 z_2}(T^3).$$

(3) For $M_3 = P_1 P_2^\perp P_3^\perp M$, we have the following decomposition:

$$M_3 = \sum_{j,k \geq 0} V_2^j V_3^k \{(M_3 \ominus V_2 M_3) \cap (M_3 \ominus V_3 M_3)\}.$$
Invariant Subspaces on $T^N$ and $R^N$

Let $(M_3 \otimes V_2 M_3) \cap (M_3 \otimes V_3 M_3)$ be denoted by $N$. The restriction of $V_1$ on $N$ is unitary. For $f \in N$, we have $\int |f|^2 z_1^j z_2^k d\mu = 0$ for all $(j, k) \neq (0, 0)$. Hence $|f|$ is independent of $z_2$ and $z_3$. Then we have

$$\int_{i \in \mathbb{Z}} z_i^j \phi_f|f| = \phi_f \chi_{E(f)} L^2(T),$$

where $\phi_f$ and $\chi_{E(f)}$ are the same defined in the case (2). Since there exists a function $F$ in $N$ which has the maximal support in $N$, we have $N = \phi_F \chi_{E(F)} L^2(T)$. Hence

$$M_3 = \sum_{j, k \geq 0} V_2^j V_3^k \phi_F \chi_{E(F)} L^2(T) = \chi_{E(F)} \phi_F H^2_{z_1}(T^3).$$

(4) For $M_4 = P_1^+ P_2^+ P_3^+ M$, we have the following decomposition:

$$M_4 = \sum_{i, j, k \geq 0} V_1^i V_2^j V_3^k \{(M_4 \otimes V_1 M_4) \cap (M_4 \otimes V_2 M_4) \cap (M_4 \otimes V_3 M_4)\}.$$

Let $(M_4 \otimes V_1 M_4) \cap (M_4 \otimes V_2 M_4) \cap (M_4 \otimes V_3 M_4)$ be denoted by $N'$. For $q \in N'$ such that $\|q\|_{L^2(T^3)} = 1$, we have $\int |q|^2 z_1^i z_2^j z_3^k d\mu = 0$ for all $(i, j, k) \neq (0, 0, 0)$. Hence $|q| = 1$ and dim $N' = 1$. We have

$$M_4 = \sum_{i, j, k \geq 0} V_1^i V_2^j V_3^k q = q H^2(T^3).$$

Combining those results that we got in (1), (2), (3) and (4), and using Lemma 1, we have the conclusion.

**Remark 1** In this remark we shall simplify complication in the general case. First, we shall consider the following figures, and call their arms $z_1$, $z_2$ and $z_3$, respectively as follows:

Terminology

Next we shall identify function spaces with figures by the correspondence of the indices of the function spaces to the arms of the figures. Where we note that $L^2(T^3) = H^2_{z_1, z_2, z_3}(T^3)$. Since the function spaces which appeared in Theorem 2 can be identified

https://doi.org/10.4153/CMB-2004-011-5 Published online by Cambridge University Press
with some figures by this correspondence, we have the following simple description of Theorem 2:

(i)

(ii)

(iii)

(iv)

(v)

By the same method, applying this identification to Theorem 1, we have the following:

(i) \[ z_2 \oplus z_1 \]

(ii) \[ \oplus \]

(iii) \[ \circ \]

Similarly, for the case where \( N = 4 \), we have 13 invariant subspaces. Since there is a rule induced by Lemma 1, under the double commuting condition, the research of invariant subspaces can be reduced to a combinatorial problem.

2 Invariant Subspaces of \( L^2(\mathbb{R}^N) \)

Let \( N \) be an integer which is larger than one. \( L^2(\mathbb{R}^N) \) will denote the usual Lebesgue space with respect to the Lebesgue measure \( dx = dx_1 \cdot dx_2 \cdots dx_N \) on the usual \( N \)-dimensional Euclidean space \( \mathbb{R}^N \), and let \( \alpha \) denote a multi-index that is an ordered \( N \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_N) \) of real numbers \( \alpha_j \). Let \( H^2(\mathbb{R}^N) = \bigotimes_1^N H^2(\mathbb{R}) \) be the space of all \( f \) in \( L^2(\mathbb{R}^N) \) whose Fourier transform

\[
\hat{f}(\alpha) = \hat{f}(\alpha) = \int_{\mathbb{R}^N} f(x) e^{-i(\alpha, x)} \, dx
\]

is 0 whenever at least one component of \( \alpha \) is negative, where \( x = (x_1, \ldots, x_N) \) is in \( \mathbb{R}^N \) and \( \langle \ , \ \rangle \) denotes the usual inner product in \( \mathbb{R}^N \). Note that our \( H^2(\mathbb{R}^N) \) is different from the usual Hardy space on \( \mathbb{R}^N \).
Invariant Subspaces on $\mathbb{T}^N$ and $\mathbb{R}^N$

We define a closed subspace $H^2_{\mathbf{x}_1, \ldots, \mathbf{x}_k}(\mathbb{R}^N)$ as follows:

$$H^2_{\mathbf{x}_1, \ldots, \mathbf{x}_k}(\mathbb{R}^N) = \bigotimes_{1 \leq j \leq k} L^2(\mathbb{R}, dx_j) \bigotimes_{1 \leq i, j \neq 1, \ldots, k} H^2(\mathbb{R}, dx_i).$$

**Definition 2** A closed subspace $\mathcal{M}$ of $L^2(\mathbb{R}^N)$ is said to be an invariant subspace of $L^2(\mathbb{R}^N)$ if $e^{is\mathbf{x}}\mathcal{M} \subseteq \mathcal{M}$ for any $j = 1, \ldots, N$ and any $s \geq 0$. For $s \geq 0$, $S_j(s)$ denotes the restriction on $\mathcal{M}$ of the multiplication operator $L_{e^{is\mathbf{x}_j}}$ on $L^2(\mathbb{R}^N)$ by $e^{is\mathbf{x}_j}$.

In this section, we shall show an $N$-dimensional version of Lax’s theorem (cf. Lax [4]). The essential idea of our proof was given by Hoffman in [2], he proved Lax’s theorem as a corollary of famous Beurling’s theorem by using a linear fractional transformation from $\mathbb{R}$ to $\mathbb{T}$. In Section 1, we considered the double commuting condition, and in this section we shall consider a similar condition for $S_j(s)$ and $S_k(t)$, that is,

$$S_j(s)S_k(t) = S_k(t)S_j(s)$$

for any $j \neq k$ and $s, t \geq 0$. We first consider the case $N = 2$, let $S_i = S_i(t)$ and $T_i = T_i(t)$, for short.

**Theorem 3** Let $\mathcal{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$. If $\mathcal{M}$ satisfies the condition that $S_iT_i^* = T_i^*S_i$ for any $s, t \geq 0$, then one and only one of the following occurs.

(i) $\mathcal{M} = \chi_E L^2(\mathbb{T}^2) \oplus \chi_{E_j} \phi_1 H^2(\mathbb{T}^2)$,
(ii) $\mathcal{M} = \chi_E L^2(\mathbb{T}^2) \oplus \chi_{E'} \phi_2 H^2(\mathbb{T}^2)$,
(iii) $\mathcal{M} = q H^2(\mathbb{T}^2)$,

where $\phi_1$ and $q$ are unimodular functions, $\chi_E$ denotes the characteristic function of $E$, $\chi_{E_j}$ is the characteristic function of $E_j$ which depends only on the variable $x_j$.

**Proof** Let $H_S$ be the generator of $S_i$. $H_S$ is a densely defined closed symmetric operator on the domain $\mathcal{D}(H_S)$, and $H_S$ is the multiplication operator by $x$ on $\mathcal{D}(H_S)$. $V_{x_i}$ denotes the Cayley transform of $H_S$, that is,

$$V_{x_i} = c(H_S) = (H_S - iI)(H_S + iI)^{-1}.$$  

Then $V_{x_i}$ is the multiplication by $(x_i - i)/(x_i + i)$ on $\mathcal{M}$, that is, for all $f \in \mathcal{M}$

$$V_{x_i} f = \frac{x_i - i}{x_i + i} f$$

and $V_{x_i}$ is an isometry on $\mathcal{M}$. Similarly, we have an isometry $V_{x_i}$ on $\mathcal{M}$ as follows:

$$V_{x_i} f = \frac{x_i - i}{x_i + i} f \quad (f \in \mathcal{M}).$$

Since $\{S_i\}_{t \geq 0}$ and $\{T_i\}_{t \geq 0}$ are semi-groups of isometries on $\mathcal{M}$, we have the following integral representations of $V_{x_i}$ and $V_{x_i}$, respectively:

$$I - V_{x_i} = 2 \int_0^\infty e^{-t} S_i \, dt \quad \text{and} \quad I - V_{x_i} = 2 \int_0^\infty e^{-t} T_i \, dt.$$
Hence $V_s \in \mathcal{R}\{\{S_i\}_{i \geq 0}\}$ and $V_x \in \mathcal{R}\{\{T_i\}_{i \geq 0}\}$, where $\mathcal{R}\{\{S_i\}_{i \geq 0}\}$ (resp. $\mathcal{R}\{\{T_i\}_{i \geq 0}\}$) is the von Neumann algebra generated by $\{S_i\}_{i \geq 0}$ (resp. $\{T_i\}_{i \geq 0}$). Since $S_s T_s^* = T_s^* S_s$ for any $s, t \geq 0$, we have $V_s V_s^* = V_s^* V_s$. Here we construct an isometric operator $U$ from $L^2(\mathbb{R}^2)$ onto $L^2(T^2)$ as follows:

$$U: \frac{1}{\pi^2} (x_1 - i)^k (x_2 - i)^l \mapsto z_1^i z_2^j$$

where $z_1$ and $z_2$ are the coordinate functions on $T^2$ (cf. [2]). Especially, $U \left( H^2(\mathbb{R}^2) \right) = H^2(T^2)$, $U \left( H_x(\mathbb{R}^2) \right) = H^2_x(T^2)$ and $U \left( H_z(\mathbb{R}^2) \right) = H^2_z(T^2)$. Since $V_s = U^* V_x U$ and $V_x = U^* V_s U$, we have $V_s \mapsto V_x \mapsto U^* V_s U$, which depends only on two variables $x_1$ and $x_2$, respectively. Then, we have that $V_s V_s^* = V_x^* V_x$ if and only if $V_s V_x^* = V_x^* V_s$. Since $U(\mathcal{M})$ is an invariant subspace of $L^2(T^2)$, and by Theorem 1, we have the conclusion.

By the same way as in the proof of Theorem 3, we have the following:

**Theorem 4** Let $\mathcal{M}$ be an invariant subspace of $L^2(\mathbb{R}^3)$. If $\mathcal{M}$ satisfies the condition that $S_j(s) S_j(t)^* = S_k(s) S_k(t)$ for any $j \neq k$ and $s, t \geq 0$, then one and only one of the following occurs.

\[ (i) \quad \mathcal{M} = \chi_x L^2(\mathbb{R}^3) \oplus \chi_{E_{x,j}} \phi_1 H^2_{x_1,x_2}(\mathbb{R}^3) \oplus \chi_{E_{y,j}} \phi_2 H^2_{y_1,y_2}(\mathbb{R}^3) \oplus \chi_{E_{z,j}} \phi_3 H^2_{z_1,z_2}(\mathbb{R}^3), \]

\[ (ii) \quad \mathcal{M} = \chi_x L^2(\mathbb{R}^3) \oplus \chi_{E_{x,j}} \phi_1 H^2_{x_1,x_2}(\mathbb{R}^3) \oplus \chi_{E_{y,j}} \phi_2 H^2_{y_1,y_2}(\mathbb{R}^3) \oplus \chi_{E_{z,j}} \phi_3 H^2_{z_1,z_2}(\mathbb{R}^3), \]

\[ (iii) \quad \mathcal{M} = \chi_x L^2(\mathbb{R}^3) \oplus \chi_{E_{x,j}} \phi_1 H^2_{x_1,x_2}(\mathbb{R}^3) \oplus \chi_{E_{y,j}} \phi_2 H^2_{y_1,y_2}(\mathbb{R}^3) \oplus \chi_{E_{z,j}} \phi_3 H^2_{z_1,z_2}(\mathbb{R}^3), \]

\[ (iv) \quad \mathcal{M} = \chi_x L^2(\mathbb{R}^3) \oplus \chi_{E_{x,j}} \phi_1 H^2_{x_1,x_2}(\mathbb{R}^3) \oplus \chi_{E_{y,j}} \phi_2 H^2_{y_1,y_2}(\mathbb{R}^3) \oplus \chi_{E_{z,j}} \phi_3 H^2_{z_1,z_2}(\mathbb{R}^3), \]

\[ (v) \quad \mathcal{M} = q H^2(\mathbb{R}^3), \]

where $\phi_1$ and $q$ are unimodular functions, $\chi_E$ denotes the characteristic function of $E$, $\chi_{E_{x,j}}$ is the characteristic function of $E_{x,j}$ which depends only on two variables $x_1$ and $x_2$, $\chi_{E_{y,j}}$ is the characteristic function of $E_{y,j}$ which depends only on two variables $x_1$ and $x_2$.

**Acknowledgements** The author thanks Professor Takashi Yoshino for his helpful and valuable suggestions. Also the author is grateful to the referee for his comments.

**References**


Invariant Subspaces on $\mathbb{T}^N$ and $\mathbb{R}^N$


Mathematical Institute
Tohoku University
Sendai 980-8578
Japan
email: s98m21@math.tohoku.ac.jp