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Invariant Subspaces on $\mathbb{T}^N$ and $\mathbb{R}^N$

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Abstract. Let N be an integer which is larger than one. In this paper we study invariant subspaces of $L^2(\mathbb{T}^N)$ under the double commuting condition. A main result is an N-dimensional version of the theorem proved by Mandrekar and Nakazi. As an application of this result, we have an N-dimensional version of Lax's theorem.

1 Invariant Subspaces of $L^2(\mathbb{T}^N)$

Let N be an integer which is larger than one, and $\mathbb{T}^N$ denote the torus, the Cartesian product of N unit circles in $\mathbb{C}$, that is,

$$\mathbb{T}^N = \{z = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N : |z_i| = 1 (i = 1, 2, \dots, N)\}.$$

$L^2(\mathbb{T}^N)$ will denote the usual Lebesgue space with respect to the normalized Lebesgue measure μ of $\mathbb{T}^N$, and let α denote a multi-index that is an ordered N-tuple $\alpha = (\alpha_1, \dots, \alpha_N)$ of integers α_j. $H^2(\mathbb{T}^N) = \bigotimes^N H^2(\mathbb{T})$ will denote the Hardy space over $\mathbb{T}^N$, that is, $H^2(\mathbb{T}^N)$ is the space of all f in $L^2(\mathbb{T}^N)$ whose Fourier coefficients

$$\hat{f}(\alpha) = \int_{\mathbb{T}^N} f(z)\bar{z}^\alpha \, d\mu$$

are 0 whenever at least one component of α is negative. For integers $1 \le i_1, \dots, i_k \le N$, we define a closed subspace $H^2_{z_{i_1},\dots,z_{i_k}}(\mathbb{T}^N)$ of $L^2(\mathbb{T}^N)$ as follows:

$$H^2_{z_{i_1},\dots,z_{i_k}}(\mathbb{T}^N) = \bigvee_{m_1,\dots,m_k<0} z_{i_1}^{m_1} \cdots z_{i_k}^{m_k} H^2(\mathbb{T}^N),$$

where $\bigvee$ denotes closed vector span.

Definition 1 A closed subspace $\mathcal{M}$ of $L^2(\mathbb{T}^N)$ is said to be an invariant subspace of $L^2(\mathbb{T}^N)$ if $z_i\mathcal{M} \subseteq \mathcal{M}$ for any $i = 1, \dots, N$. V_i denotes the restriction on $\mathcal{M}$ of the multiplication operator L_{z_i} on $L^2(\mathbb{T}^N)$ by z_i.

Mandrekar [5] and Nakazi [9] characterized the invariant subspaces of $L^2(\mathbb{T}^2)$ under the condition that V_1 commutes with V_2^*.

Theorem 1 **([9])** *Let $\mathcal{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$. If $\mathcal{M}$ satisfies the condition that $V_1V_2^* = V_2^*V_1$, then one and only one of the following occurs.*

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(i) $\mathcal{M} = \chi_E L^2(\mathbb{T}^2) \oplus \chi_{E_1}\phi_1 H^2_{z_1}(\mathbb{T}^2)$,
(ii) $\mathcal{M} = \chi_E L^2(\mathbb{T}^2) \oplus \chi_{E_2}\phi_2 H^2_{z_2}(\mathbb{T}^2)$,
(iii) $\mathcal{M} = qH^2(\mathbb{T}^2)$,

where ϕ_i and q are unimodular functions, χ_E denotes the characteristic function of E, χ_{E_j} is the characteristic function of E_j which depends only on a variable z_j.

We consider the condition which is analogous to the case $N = 2$, that is, each V_i commutes with V_j^* for any $j \neq i$. Since V_i is an isometry on $\mathcal{M}$ for each $i = 1, 2, \ldots, N$, we have the following Wold decomposition,

$$V_i = V_i^{(u)} \oplus V_i^{(o)} \quad \text{on} \quad \mathcal{M} = \mathcal{M}^{(u)} \oplus \mathcal{M}^{(o)},$$

where $V_i^{(u)} = V_i|\mathcal{M}^{(u)}$ is unitary, $V_i^{(o)} = V_i|\mathcal{M}^{(o)}$ is a unilateral shift, and

$$\begin{aligned}\mathcal{M}^{(u)} &= \{f \in \mathcal{M} : \|V_i^{*k} f\| = \|f\| \ (k \geq 1)\} \\ &= \bigcap_{k=1}^{\infty} \{f \in \mathcal{M} : V_i^k V_i^{*k} f = f\} = \bigcap_{k=1}^{\infty} V_i^k \mathcal{M}\end{aligned}$$

is the maximal reducing subspace on which its restriction is unitary. Let P_i denote the projection from $\mathcal{M}$ onto $\mathcal{M}^{(u)}$. Then P_i is in the center of $\mathcal{R}(V_i)$, where $\mathcal{R}(V_i)$ is the von Neumann algebra generated by V_i (*cf.* [12]).

The following lemma is well known.

Lemma 1 *Let $\mathcal{M}$ and $\mathcal{N}$ be invariant subspaces of $L^2(\mathbb{T}^N)$. If $\mathcal{M}$ is orthogonal to $\mathcal{N}$, then $fg = 0$ for any $f \in \mathcal{M}$ and any $g \in \mathcal{N}$.*

Proof For any $f \in \mathcal{M}$ and any $g \in \mathcal{N}$, $\int_{\mathbb{T}^N} f\bar{g}\bar{z}^{\alpha}\, d\sigma = 0$. ■

In the following argument, we deal with the case where $N = 3$, because it is difficult to describe invariant subspaces under the double commuting condition in general. (But we will be aware that our proof in the case where $N = 3$ can be applied to the general case.) This is a complicated problem and, in the later remark, we shall reduce this complication.

Theorem 2 *Let $\mathcal{M}$ be an invariant subspace of $L^2(\mathbb{T}^3)$. If $\mathcal{M}$ satisfies the condition that $V_i V_j^* = V_j^* V_i$ for $i \neq j$, then one and only one of the following occurs.*

(i) $\mathcal{M} = \chi_E L^2(\mathbb{T}^3) \oplus \chi_{E_{1,2}}\phi_1 H^2_{z_1,z_2}(\mathbb{T}^3) \oplus \chi_{E_{3,1}}\phi_2 H^2_{z_3,z_1}(\mathbb{T}^3) \oplus \chi_{E_{2,3}}\phi_3 H^2_{z_2,z_3}(\mathbb{T}^3)$,
(ii) $\mathcal{M} = \chi_E L^2(\mathbb{T}^3) \oplus \chi_{E_{1,2}}\phi_1 H^2_{z_1,z_2}(\mathbb{T}^3) \oplus \chi_{E_{3,1}}\phi_2 H^2_{z_3,z_1}(\mathbb{T}^3) \oplus \chi_{E_1}\phi_3 H^2_{z_1}(\mathbb{T}^3)$,
(iii) $\mathcal{M} = \chi_E L^2(\mathbb{T}^3) \oplus \chi_{E_{1,2}}\phi_1 H^2_{z_1,z_2}(\mathbb{T}^3) \oplus \chi_{E_{2,3}}\phi_2 H^2_{z_2,z_3}(\mathbb{T}^3) \oplus \chi_{E_2}\phi_3 H^2_{z_2}(\mathbb{T}^3)$,
(iv) $\mathcal{M} = \chi_E L^2(\mathbb{T}^3) \oplus \chi_{E_{2,3}}\phi_1 H^2_{z_2,z_3}(\mathbb{T}^3) \oplus \chi_{E_{3,1}}\phi_2 H^2_{z_3,z_1}(\mathbb{T}^3) \oplus \chi_{E_3}\phi_3 H^2_{z_3}(\mathbb{T}^3)$,
(v) $\mathcal{M} = qH^2(\mathbb{T}^3)$,

where ϕ_i and q are unimodular functions, χ_E denotes the characteristic function of E, $\chi_{E_{i,j}}$ is the characteristic function of $E_{i,j}$ which depends only on two variables z_i and z_j, χ_{E_i} is the characteristic function of E_i which depends only on a variable z_i.

Proof The following argument is a slight modification of the proof given by Mandrekar [5] and Nakazi [9].

Suppose that $V_iV_j^* = V_j^*V_i$ $(i \neq j)$. By the property of Wold decomposition, we have the following:

$$\mathcal{M} = \sum_{a,b,c\in\{0,1\}} P_1^{(a)}P_2^{(b)}P_3^{(c)}\mathcal{M},$$

where $P_i^{(0)} = P_i$ and $P_i^{(1)} = P_i^{\perp} = I - P_i$. We note that P_i commutes with P_j because $V_i \in \mathcal{R}(V_j)'$ and $V_j \in \mathcal{R}(V_i)'$ by the assumption. Hence it suffices to describe $P_1P_2P_3\mathcal{M}$, $P_1P_2P_3^{\perp}\mathcal{M}$, $P_1P_2^{\perp}P_3^{\perp}\mathcal{M}$ and $P_1^{\perp}P_2^{\perp}P_3^{\perp}\mathcal{M}$. Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$ denote the above four subspaces, respectively.

(1) By the Wiener-Tauberian theorem, $\mathcal{M}_1 = \chi_E L^2(\mathbb{T}^3)$.

(2) For $\mathcal{M}_2 = P_1P_2P_3^{\perp}\mathcal{M}$, we have the following decomposition:

$$\mathcal{M}_2 = \sum_{k=0}^{\infty} V_3^k(\mathcal{M}_2 \ominus V_3\mathcal{M}_2),$$

where we note that the restriction of V_1 and V_2 on $\mathcal{M}_2 \ominus V_3\mathcal{M}_2$ are unitary operators. For $f \in \mathcal{M}_2 \ominus V_3\mathcal{M}_2$, we have $\int |f|^2 z_1^i z_2^j z_3^k \, d\mu = 0$ for $k \neq 0$. Hence $|f|$ is independent of z_3. Then we have

$$\bigvee_{i,j\in\mathbb{Z}} z_1^i z_2^j f = \phi_f \chi_{E(f)} L^2(\mathbb{T}^2),$$

where, for any measurable function g, a measurable set $E(g)$ and a unimodular function ϕ_g are defined as follows:

$$E(g) = \{z \in \mathbb{T}^N : g(z) \neq 0\},$$

$$\phi_g = \begin{cases} g/|g| & (g \neq 0) \\ 1 & (g = 0). \end{cases}$$

Since there exists a function F in $\mathcal{M}_2 \ominus V_3\mathcal{M}_2$ which has the maximal support in $\mathcal{M}_2 \ominus V_3\mathcal{M}_2$, that is, $E(f) \subseteq E(F)$, for any $f \in \mathcal{M}_2 \ominus V_3\mathcal{M}_2$, we have $\mathcal{M}_2 \ominus V_3\mathcal{M}_2 = \phi_F \chi_{E(F)} L^2(\mathbb{T}^2)$. Hence

$$\mathcal{M}_2 = \sum_{k=0}^{\infty} V_3^k \phi_F \chi_{E(F)} L^2(\mathbb{T}^2) = \chi_{E(F)} \phi_F H^2_{z_1,z_2}(\mathbb{T}^3).$$

(3) For $\mathcal{M}_3 = P_1P_2^{\perp}P_3^{\perp}\mathcal{M}$, we have the following decomposition:

$$\mathcal{M}_3 = \sum_{j,k\geq 0} V_2^j V_3^k \{(\mathcal{M}_3 \ominus V_2\mathcal{M}_3) \cap (\mathcal{M}_3 \ominus V_3\mathcal{M}_3)\}.$$

Let $(\mathcal{M}_3 \ominus V_2\mathcal{M}_3) \cap (\mathcal{M}_3 \ominus V_3\mathcal{M}_3)$ be denoted by $\mathcal{N}$. The restriction of V_1 on $\mathcal{N}$ is unitary. For $f \in \mathcal{N}$, we have $\int |f|^2 z_1^i z_2^j z_3^k \, d\mu = 0$ for all $(j,k) \neq (0,0)$. Hence $|f|$ is independent of z_2 and z_3. Then we have

$$\bigvee_{i\in\mathbb{Z}} z_1^i \phi_f |f| = \phi_f \chi_{E(f)} L^2(\mathbb{T}),$$

where ϕ_f and $\chi_{E(f)}$ are the same defined in the case (2). Since there exists a function F in $\mathcal{N}$ which has the maximal support in $\mathcal{N}$, we have $\mathcal{N} = \phi_F \chi_{E(F)} L^2(\mathbb{T})$. Hence

$$\mathcal{M}_3 = \sum_{j,k\geq 0} V_2^j V_3^k \phi_F \chi_{E(F)} L^2(\mathbb{T}) = \chi_{E(F)} \phi_F H^2_{z_1}(\mathbb{T}^3).$$

(4) For $\mathcal{M}_4 = P_1^\perp P_2^\perp P_3^\perp \mathcal{M}$, we have the following decomposition:

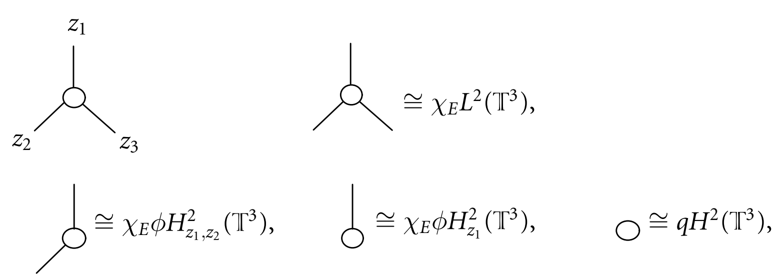
$$\mathcal{M}_4 = \sum_{i,j,k\geq 0} V_1^i V_2^j V_3^k \{(\mathcal{M}_4 \ominus V_1\mathcal{M}_4) \cap (\mathcal{M}_4 \ominus V_2\mathcal{M}_4) \cap (\mathcal{M}_4 \ominus V_3\mathcal{M}_4)\}.$$

Let $(\mathcal{M}_4 \ominus V_1\mathcal{M}_4) \cap (\mathcal{M}_4 \ominus V_2\mathcal{M}_4) \cap (\mathcal{M}_4 \ominus V_3\mathcal{M}_4)$ be denoted by $\mathcal{N}'$. For $q \in \mathcal{N}'$ such that $\|q\|_{L^2(\mathbb{T}^3)} = 1$, we have $\int |q|^2 z_1^i z_2^j z_3^k \, d\mu = 0$ for all $(i,j,k) \neq (0,0,0)$. Hence $|q| = 1$ and $\dim \mathcal{N}' = 1$. We have

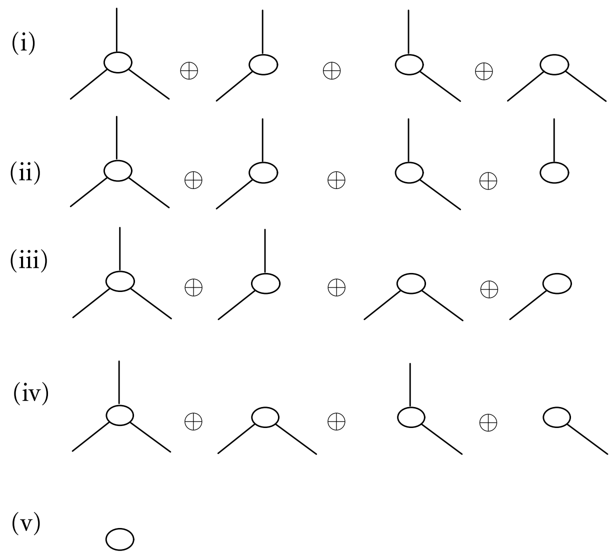
$$\mathcal{M}_4 = \sum_{i,j,k\geq 0} V_1^i V_2^j V_3^k q = qH^2(\mathbb{T}^3).$$

Combining those results that we got in (1), (2), (3) and (4), and using Lemma 1, we have the conclusion. ∎

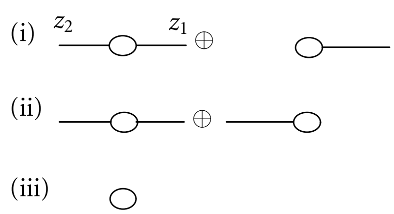
Remark 1 In this remark we shall simplify complication in the general case. First, we shall consider the following figures, and call their arms z_1, z_2 and z_3, respectively as follows:



Next we shall identify function spaces with figures by the correspondence of the indices of the function spaces to the arms of the figures. Where we note that $L^2(\mathbb{T}^3) = H^2_{z_1,z_2,z_3}(\mathbb{T}^3)$. Since the function spaces which appeared in Theorem 2 can be identified with some figures by this correspondence, we have the following simple description of Theorem 2:



By the same method, applying this identification to Theorem 1, we have the following:



Similarly, for the case where $N = 4$, we have 13 invariant subspaces. Since there is a rule induced by Lemma 1, under the double commuting condition, the research of invariant subspaces can be reduced to a combinatorial problem.

2 Invariant Subspaces of $L^2(\mathbb{R}^N)$

Let N be an integer which is larger than one. $L^2(\mathbb{R}^N)$ will denote the usual Lebesgue space with respect to the Lebesgue measure $dx = dx_1\, dx_2 \cdots dx_N$ on the usual N-dimensional Euclidean space $\mathbb{R}^N$, and let α denote a multi-index that is an ordered N-tuple $\alpha = (\alpha_1, \dots, \alpha_N)$ of real numbers α_j. Let $H^2(\mathbb{R}^N) = \bigotimes^N H^2(\mathbb{R})$ be the space of all f in $L^2(\mathbb{R}^N)$ whose Fourier transform

$$\mathfrak{F}(f)(\alpha) = \hat{f}(\alpha) = \int_{\mathbb{R}^N} f(x) e^{-i\langle \alpha, x\rangle}\, dx$$

is 0 whenever at least one component of α is negative, where $x = (x_1, \dots, x_N)$ is in $\mathbb{R}^N$ and $\langle\ ,\ \rangle$ denotes the usual inner product in $\mathbb{R}^N$. Note that our $H^2(\mathbb{R}^N)$ is different from the usual Hardy space on $\mathbb{R}^N$.

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We define a closed subspace $H^2_{x_{i_1},\dots,x_{i_k}}(\mathbb{R}^N)$ as follows:

$$H^2_{x_{i_1},\dots,x_{i_k}}(\mathbb{R}^N) = \bigotimes_{1\le j\le k} L^2(\mathbb{R}, dx_{i_j}) \bigotimes_{1\le i\le N, i\ne i_1,\dots,i_k} H^2(\mathbb{R}, dx_i).$$

Definition 2 A closed subspace $\mathcal{M}$ of $L^2(\mathbb{R}^N)$ is said to be an invariant subspace of $L^2(\mathbb{R}^N)$ if $e^{isx_j}\mathcal{M} \subseteq \mathcal{M}$ for any $j = 1, \dots, N$ and any $s \ge 0$. For $s \ge 0$, $S_j(s)$ denotes the restriction on $\mathcal{M}$ of the multiplication operator $L_{e^{isx_j}}$ on $L^2(\mathbb{R}^N)$ by e^{isx_j}.

In this section, we shall show an N-dimensional version of Lax's theorem (*cf.* Lax [4]). The essential idea of our proof was given by Hoffman in [2], he proved Lax's theorem as a corollary of famous Beurling's theorem by using a linear fractional transformation from $\mathbb{R}$ to $\mathbb{T}$. In Section 1, we considered the double commuting condition, and in this section we shall consider a similar condition for $S_j(s)$ and $S_k(t)$, that is,

$$S_j(s)S_k(t)^* = S_k(t)^*S_j(s)$$

for any $j \ne k$ and $s, t \ge 0$. We first consider the case $N = 2$, let $S_s = S_1(s)$ and $T_t = S_2(t)$, for short.

Theorem 3 *Let $\mathcal{M}$ be an invariant subspace of $L^2(\mathbb{R}^2)$. If $\mathcal{M}$ satisfies the condition that $S_sT_t^* = T_t^*S_s$ for any $s, t \ge 0$, then one and only one of the following occurs.*

(i) $\mathcal{M} = \chi_E L^2(\mathbb{R}^2) \oplus \chi_{E_1}\phi_1 H^2_{x_1}(\mathbb{R}^2)$,
(ii) $\mathcal{M} = \chi_E L^2(\mathbb{R}^2) \oplus \chi_{E_2}\phi_2 H^2_{x_2}(\mathbb{R}^2)$,
(iii) $\mathcal{M} = qH^2(\mathbb{R}^2)$,

where ϕ_i and q are unimodular functions, χ_E denotes the characteristic function of E, χ_{E_j} is the characteristic function of E_j which depends only on the variable x_j.

Proof Let H_S be the generator of S_s. H_S is a densely defined closed symmetric operator on the domain $\mathcal{D}(H_S)$, and H_S is the multiplication operator by x on $\mathcal{D}(H_S)$. V_{x_1} denotes the Cayley transform of H_S, that is,

$$V_{x_1} = c(H_S) = (H_S - iI)(H_S + iI)^{-1}.$$

Then V_{x_1} is the multiplication by $(x_1 - i)/(x_1 + i)$ on $\mathcal{M}$, that is, for all $f \in \mathcal{M}$

$$V_{x_1} f = \frac{x_1 - i}{x_1 + i} f$$

and V_{x_1} is an isometry on $\mathcal{M}$. Similarly, we have an isometry V_{x_2} on $\mathcal{M}$ as follows:

$$V_{x_2} f = \frac{x_2 - i}{x_2 + i} f \quad (f \in \mathcal{M}).$$

Since $\{S_s\}_{s\ge 0}$ and $\{T_t\}_{t\ge 0}$ are semi-groups of isometries on $\mathcal{M}$, we have the following integral representations of V_{x_1} and V_{x_2}, respectively:

$$I - V_{x_1} = 2\int_0^\infty e^{-s} S_s\, ds \quad \text{and} \quad I - V_{x_2} = 2\int_0^\infty e^{-t} T_t\, dt.$$

Hence $V_{x_1} \in \mathcal{R}(\{S_s\}_{s\geq 0})$ and $V_{x_2} \in \mathcal{R}(\{T_t\}_{t\geq 0})$, where $\mathcal{R}(\{S_s\}_{s\geq 0})$ (resp. $\mathcal{R}(\{T_t\}_{t\geq 0})$) is the von Neumann algebra generated by $\{S_s\}_{s\geq 0}$ (resp. $\{T_t\}_{t\geq 0}$). Since $S_sT_t^* = T_t^*S_s$ for any $s, t \geq 0$, we have $V_{x_1}V_{x_2}^* = V_{x_2}^*V_{x_1}$. Here we construct an isometric operator U from $L^2(\mathbb{R}^2)$ onto $L^2(\mathbb{T}^2)$ as follows:

$$U\colon \frac{1}{\pi^2}\frac{(x_1-i)^k}{(x_1+i)^{k+1}}\frac{(x_2-i)^l}{(x_2+i)^{l+1}} \longmapsto z_1^k z_2^l$$

where z_1 and z_2 are the coordinate functions on $\mathbb{T}^2$ (*cf.* [2]). Especially, $U\big(H^2(\mathbb{R}^2)\big) = H^2(\mathbb{T}^2)$, $U\big(H_{x_1}(\mathbb{R}^2)\big) = H^2_{z_1}(\mathbb{T}^2)$ and $U\big(H_{x_2}(\mathbb{R}^2)\big) = H^2_{z_2}(\mathbb{T}^2)$. Since $V_{x_1} = U^*V_{z_1}U$ and $V_{x_2} = U^*V_{z_2}U$, where V_{z_1} and V_{z_2} denote the multiplication operators on $U(\mathcal{M})$ by the coordinate functions z_1 and z_2, respectively. Then, we have that $V_{x_1}V_{x_2}^* = V_{x_2}^*V_{x_1}$ if and only if $V_{z_1}V_{z_2}^* = V_{z_2}^*V_{z_1}$. Since $U(\mathcal{M})$ is an invariant subspace of $L^2(\mathbb{T}^2)$, and by Theorem 1, we have the conclusion. ■

By the same way as in the proof of Theorem 3, we have the following:

Theorem 4 *Let $\mathcal{M}$ be an invariant subspace of $L^2(\mathbb{R}^3)$. If $\mathcal{M}$ satisfies the condition that $S_j(s)S_k(t)^* = S_k(t)^*S_j(s)$ for any $j \neq k$ and $s, t \geq 0$, then one and only one of the following occurs.*

(i) $\mathcal{M} = \chi_E L^2(\mathbb{R}^3) \oplus \chi_{E_{1,2}}\phi_1 H^2_{x_1,x_2}(\mathbb{R}^3) \oplus \chi_{E_{3,1}}\phi_2 H^2_{x_3,x_1}(\mathbb{R}^3) \oplus \chi_{E_{2,3}}\phi_3 H^2_{x_2,x_3}(\mathbb{R}^3)$,
(ii) $\mathcal{M} = \chi_E L^2(\mathbb{R}^3) \oplus \chi_{E_{1,2}}\phi_1 H^2_{x_1,x_2}(\mathbb{R}^3) \oplus \chi_{E_{3,1}}\phi_2 H^2_{x_3,x_1}(\mathbb{R}^3) \oplus \chi_{E_1}\phi_3 H^2_{x_1}(\mathbb{R}^3)$,
(iii) $\mathcal{M} = \chi_E L^2(\mathbb{R}^3) \oplus \chi_{E_{1,2}}\phi_1 H^2_{x_1,x_2}(\mathbb{R}^3) \oplus \chi_{E_{2,3}}\phi_2 H^2_{x_2,x_3}(\mathbb{R}^3) \oplus \chi_{E_2}\phi_3 H^2_{x_2}(\mathbb{R}^3)$,
(iv) $\mathcal{M} = \chi_E L^2(\mathbb{R}^3) \oplus \chi_{E_{2,3}}\phi_1 H^2_{x_2,x_3}(\mathbb{R}^3) \oplus \chi_{E_{3,1}}\phi_2 H^2_{x_3,x_1}(\mathbb{R}^3) \oplus \chi_{E_3}\phi_3 H^2_{x_3}(\mathbb{R}^3)$,
(v) $\mathcal{M} = qH^2(\mathbb{R}^3)$,

where ϕ_i and q are unimodular functions, χ_E denotes the characteristic function of E, $\chi_{E_{i,j}}$ is the characteristic function of $E_{i,j}$ which depends only on two variables x_i and x_j, χ_{E_i} is the characteristic function of E_i which depends only on a variable x_i.

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