THE FRATTINI SUBGROUP OF A p-GROUP

P.L. Manley

We prove that the Frattini subgroups are trivial for finite groups whose orders are not divisible by squares of a prime.

If G is a group, we define its Frattini subgroup $\phi(G)$ as the intersection of all the maximal subgroups of G. An element $x \in G$ is a nongenerator of G if whenever $G = \langle X, x \rangle$, where $X \subseteq G$, then $G = \langle X \rangle$.

PROPOSITION 1. $\phi(G)$ *is the set of nongenerators of* G, *and is a characteristic subgroup of* G.

COROLLARY 2. *If* $\phi(G)$ *is a finite subgroup of* G, *then every set that, in conjunction with* $\phi(G)$, *generates* G *is itself a generating set for* G.

If Π is a set of prime numbers, we say that a group is a Π-group if $|G|$ is divisible only by primes in Π. A subgroup H of a group G is a Hall subgroup of G provided that H is a Π-group and $|G : H|$ is divisible by no primes in Π.

PROPOSITION 3. (Schur-Zassenhaus Theorem). *If* H *is a normal Hall* Π-*subgroup of a group* G *then* G *has a Hall* Π'-*subgroup* K *which is a complement to* H *in* G.

We state our theorem.

THEOREM 4. *If* G *is a finite group of order* $|G| = p_1 \ldots p_n$ *where* p_i *are prime numbers such that* $p_i \neq p_j$ $(i, j = 1, \ldots, n)$ *then* $\phi(G)$ *is the identity subgroup*.

Proof. Since G is a finite group it follows that $\phi(G)$ is a proper subgroup. We proceed by contradiction. Suppose that $\phi(G)$ is not the identity subgroup. We may assume without loss of generality that $|\phi(G)| = p_1 \ldots p_m$ $(1 \leq m < n)$. The exponent of $\phi(G)$ in G is $p_{m+1} \ldots p_n$ and is relatively prime with its order. Since $\phi(G)$ is a characteristic subgroup of G then $\phi(G)$ is a normal subgroup of G. Now by the Schur-Zassenhaus Theorem there exists a subgroup H of G with $|H| = p_{m+1} \ldots p_n$ such that $G = \phi(G)\,H$. But by Corollary 2 we have that $G = \phi(G)\,H$ implies that $G = H$ which is impossible so that $\phi(G)$ is the identity subgroup.

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University of Windsor

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