UNICITY THEOREMS FOR MEROMORPHIC OR ENTIRE FUNCTIONS III

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This paper studies the unique range set of meromorphic functions and shows that the set $S = \{w | w^{13} + w^{11} + 1 = 0\}$ is unique range set of meromorphic functions with 13 elements.

1. INTRODUCTION

By a meromorphic function we shall always mean a meromorphic function in the complex plane. We use the usual notations of Nevanlinna theory of meromorphic functions as explained in [4]. We use $E$ to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ ($r \to \infty$, $r \notin E$).

Let $f$ be a nonconstant meromorphic function and let $S$ be a subset of distinct elements in the complex plane. Define

$$E_f(S) = \bigcup_{a \in S} \{z | f(z) - a = 0\},$$

where each zero of $f(z) - a$ with multiplicity $m$ is repeated $m$ times in $E_f(S)$ (see [1]).

In 1976, Gross [2] proved that there exist three finite sets $S_j$ ($j = 1, 2, 3$) such that any two entire functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical, and asked the following question (see [2, Question 6]):

**QUESTION 1.** Can one find two (or possible even one) finite sets $S_j$ ($j = 1, 2$) such that any two nonconstant entire functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

Now it is natural to ask the following question:
QUESTION 2. Can one find two (or possible even one) finite sets $S_j$ ($j = 1, 2$) such that any two nonconstant meromorphic functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

Recently, the present author proved the following results which provide positive answers to Question 1.

**Theorem A.** (See [7, Theorem 3].) Let $S_1 = \{w \mid w^n - 1 = 0\}$, $S_2 = \{a, b\}$, where $n > 6$ is a positive integer, $a$ and $b$ are constants such that $ab \neq 0$, $a^n \neq b^n$, $a^{2n} \neq 1$, $b^{2n} \neq 1$ and $a^nb^n \neq 1$. Suppose that $f$ and $g$ are nonconstant entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$. Then $f \equiv g$.

**Theorem B.** (See [8, Theorem 1].) Let $S = \{w \mid w^n + aw^{n-m} + b = 0\}$, where $n$ and $m$ are two positive integers such that $n$ and $m$ have no common factors and $n \geq 2m + 5$, $a$ and $b$ are two nonzero constants such that the algebraic equation $w^n + aw^{n-m} + b = 0$ has no multiple roots. If $f$ and $g$ are nonconstant entire functions satisfying $E_f(S) = E_g(S)$, then $f \equiv g$.

Recently, the present author proved the following result which is a partial answer of Question 2.

**Theorem C.** (See [8, Theorem 2].) Let $S = \{w \mid w^n + aw^{n-m} + b = 0\}$, where $n$ and $m$ are two positive integers such that $m \geq 2$, $n \geq 2m + 7$ with $n$ and $m$ having no common factors, $a$ and $b$ are two nonzero constants such that the algebraic equation $w^n + aw^{n-m} + b = 0$ has no multiple roots. Suppose that $f$ and $g$ are nonconstant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$. Then $f \equiv g$.

The set $S$ such that for any two nonconstant meromorphic functions $f$ and $g$ the condition $E_f(S) = E_g(S)$ implies $f \equiv g$ is called a unique range set (URS in brief) of meromorphic functions. A similar definition for entire functions can be given. From Theorem B we immediately obtain the following result.

**Theorem B'.** Let $S$ be defined as in Theorem B. Then $S$ is a URS of entire functions.

As a special case of Theorem B', we deduce that the set $S = \{w \mid w^7 + w^6 + 1 = 0\}$ in a URS of entire functions with 7 elements. In this paper, we shall exhibit a URS of meromorphic functions with 13 elements. In fact, we prove more generally the following theorem, which provides a positive answer to Question 2.

**Theorem 1.** Let $S = \{w \mid w^n + aw^{n-m} + b = 0\}$, where $n$ and $m$ are two positive integers such that $n$ and $m$ have no common factors, $m \geq 2$ and $n > 2m + 8$, $a$ and $b$ are two nonzero constants such that the algebraic equation $w^n + aw^{n-m} + b = 0$ has no multiple roots. Then $S$ is a URS of meromorphic functions.
From Theorem 1 we immediately obtain that the set \( S = \{ w \mid w^{13} + w^{11} + 1 = 0 \} \) provides a URS of meromorphic functions with 13 elements, which provides a positive answer to Question 2.

2. SOME LEMMAS

LEMMA 1. (See [5].) Let \( f \) be a nonconstant meromorphic function, and let \( P(f) \) be a polynomial in \( f \) of the form

\[
P(f) = a_0 f^n + a_1 f^{n-1} + \ldots + a_{n-1} f + a_n,
\]

where \( a_0 \neq 0 \), \( a_1, \ldots, a_n \) are constants. Then

\[
T(r, P(f)) = nT(r, f) + S(r, f).
\]

In order to state the second lemma, we introduce the following notation.

Let \( F \) be a meromorphic function. We denote by \( n_1(r, 1/(F - a)) \) the number of simple \( a \)-points of \( F \) in \( |z| \leq r \). \( N_1(r, 1/(F - a)) \) is defined in terms of \( n_1(r, 1/(F - a)) \) in the usual way (see [6]).

Let \( F \) and \( G \) be two nonconstant meromorphic functions. If \( F \) and \( G \) have the same \( a \)-points with the same multiplicities, we say \( F \) and \( G \) share the value \( a \) CM (see [3]).

LEMMA 2. Let

\[
H = \left( \frac{F''}{F'} - \frac{2F''}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G''}{G - 1} \right),
\]

where \( F \) and \( G \) are two nonconstant meromorphic functions. If \( F \) and \( G \) share 1 CM, and \( H \neq 0 \), then

\[
N_1 \left( r, \frac{1}{F - 1} \right) \leq N \left( r, \frac{1}{H} \right).
\]

PROOF: Suppose that \( z_0 \) is a simple 1-point of \( F \). Let

\[
F(z) = 1 + a_1(z - z_0) + a_2(z - z_0)^2 + O((z - z_0)^3),
\]

\[
G(z) = 1 + b_1(z - z_0) + b_2(z - z_0)^2 + O((z - z_0)^3),
\]

where \( a_1 \neq 0 \) and \( b_1 \neq 0 \). Then an elementary calculation gives that

\[
H(z) = O(z - z_0),
\]

which proves that \( z_0 \) is a zero of \( H \). Thus,

\[
N_1 \left( r, \frac{1}{F - 1} \right) \leq N \left( r, \frac{1}{H} \right).
\]

\[\square\]
3. Proof of Theorem 1

Suppose that \( f \) and \( g \) are two nonconstant meromorphic functions satisfying \( E_f(S) = E_g(S) \). We proceed to prove \( f = g \).

Let

\[
F = -\frac{1}{b} f^{n-m} (f^m + a) \quad \text{and} \quad G = -\frac{1}{b} g^{n-m} (g^m + a).
\]

From Lemma 1, we have

\[
T(r, F) = n T(r, f) + S(r, f)
\]
and

\[
T(r, G) = n T(r, g) + S(r, g).
\]

Let

\[
T(r) = \max\{T(r, f), T(r, g)\}
\]
and

\[
S(r) = o(T(r)) \quad (r \to \infty, \ r \notin E).
\]

Noting \( S = \{w | w^n + aw^{n-m} + b = 0\} \), from \( E_f(S) = E_g(S) \) we get that \( F \) and \( G \) share the value 1 CM.

Let \( H \) be given by (1). If \( H \neq 0 \), from Lemma 2 we have

\[
N_1 \left( r, \frac{1}{F - 1} \right) \leq N \left( r, \frac{1}{H} \right) \leq T(r, H) + O(1).
\]

From (1) we obtain

\[
m(r, H) = S(r).
\]

From (2) we have

\[
F' = -\frac{1}{b} f^{n-m-1} (nf^m + a(n - m))f'.
\]
and

\[
G' = -\frac{1}{b} g^{n-m-1} (ng^m + a(n - m))g'.
\]
Since $F$ and $G$ share 1 CM, from (1), (7) and (8),

$$(9) \quad N(r, H) \leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{n fm + a(n - m)}\right) + N_0\left(r, \frac{1}{f'}\right)$$

$$+ \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{ngm + a(n - m)}\right) + N_0\left(r, \frac{1}{g'}\right)$$

$$\leq (m + 2)T(r, f) + (m + 2)T(r, g) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{g'}\right) + O(1),$$

where $N_0(r, 1/f')$ denotes the counting function corresponding to the zeros of $f'$ that are not zeros of $f$ and $F - 1$, $N_0(r, 1/g')$ denotes the counting function corresponding to the zeros of $g'$ that are not zeros of $g$ and $G - 1$. It follows from (5), (6) and (9) that

$$(10) \quad N_1\left(r, \frac{1}{F - 1}\right) \leq (m + 2)T(r, f) + (m + 2)T(r, g)$$

$$+ N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{g'}\right) + S(r).$$

Suppose that $w_1, w_2, \ldots, w_n$ are the distinct roots of the equation $w^n + aw^{n-m} + b = 0$. From (2) we have

$$(11) \quad F - 1 = -\frac{1}{b}(f - w_1)(f - w_2)\ldots(f - w_n)$$

and

$$(12) \quad G - 1 = -\frac{1}{b}(g - w_1)(g - w_2)\ldots(g - w_n).$$

By the second fundamental theorem, we deduce

$$(13) \quad nT(r, f) < \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^{n} \overline{N}\left(r, \frac{1}{f - w_j}\right) - N_0\left(r, \frac{1}{f'}\right) + S(r)$$

$$\leq 2T(r, f) + \overline{N}\left(r, \frac{1}{F - 1}\right) - N_0\left(r, \frac{1}{f'}\right) + S(r).$$

In the same manner as above, we have

$$(14) \quad nT(r, g) < 2T(r, g) + \overline{N}\left(r, \frac{1}{G - 1}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r).$$
It is obvious that

\begin{equation}
N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) = 2N \left( r, \frac{1}{F-1} \right)
\end{equation}

\begin{align*}
&\leq N_1 \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{F-1} \right) \\
&\leq N_1 \left( r, \frac{1}{F-1} \right) + T(r, F) + O(1) \\
&= N_1 \left( r, \frac{1}{F-1} \right) + nT(r, f) + S(r)
\end{align*}

and

\begin{equation}
N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) \leq N_1 \left( r, \frac{1}{F-1} \right) + nT(r, g) + S(r).
\end{equation}

From (10), (13), (14) and (15) we obtain

\[ nT(r, g) \leq (m + 4)T(r, f) + (m + 4)T(r, g) + S(r). \]

From (10), (13), (14) and (16) we obtain

\[ nT(r, f) \leq (m + 4)T(r, f) + (m + 4)T(r, g) + S(r). \]

Thus,

\begin{equation}
T(r) \leq (m + 4)T(r, f) + (m + 4)T(r, g) + S(r) \\
\leq (2m + 8)T(r) + S(r).
\end{equation}

Since \( n > 2m + 8 \), (17) is a contradiction. From this we derive \( H = 0 \). By integration we have from (1),

\[ \frac{1}{G-1} = \frac{A}{F-1} + B, \]

where \( A \neq 0 \) and \( B \) are constants. Thus,

\begin{equation}
G = \frac{(B + 1)F + (A - B - 1)}{BF + (A - B)}.
\end{equation}

From (18),

\[ T(r, G) = T(r, F) + O(1) \]

and

\begin{equation}
T(r) = T(r, f) + S(r, f).
\end{equation}
From (2) we have
\begin{align}
(20) \quad &N(r, F) = N(r, f) \leq T(r), \\
(21) \quad &N(r, G) = N(r, g) \leq T(r), \\
(22) \quad &N \left( r, \frac{1}{F} \right) = N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f^{m+1} + a} \right) \leq (m + 1)T(r) + O(1), \\
(23) \quad &N \left( r, \frac{1}{G} \right) = N \left( r, \frac{1}{g} \right) + N \left( r, \frac{1}{g^{m+1} + a} \right) \leq (m + 1)T(r) + O(1).
\end{align}

We discuss the following three cases.

**CASE I.** Suppose that \( B \neq 0, -1 \).

If \( A - B - 1 \neq 0 \), from (18) we have
\[
N \left( r, \frac{1}{F + \frac{A-B-1}{B+1}} \right) = N \left( r, \frac{1}{G} \right).
\]

From this and the second fundamental theorem, we have
\[
T(r, F) < N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F + \frac{A-B-1}{B+1}} \right) + S(r, F)
\]
\[
= N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + S(r, F).
\]

Combining this with (3), (19), (20), (22) and (23), we obtain
\[
nT(r) < (2m + 3)T(r) + S(r),
\]
which contradicts the assumption \( n > 2m + 8 \). Thus \( A - B - 1 = 0 \). From (18),
\[
G = \frac{(B + 1)F}{BF + 1}.
\]

From this we have
\[
N \left( r, \frac{1}{F + \frac{1}{B}} \right) = N(r, G).
\]

Again from the second fundamental theorem, we obtain
\[
T(r, F) < N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F + \frac{1}{B}} \right) + S(r, F)
\]
\[
= N(r, F) + N \left( r, \frac{1}{F} \right) + N(r, G) + S(r, F).
\]

Combining this with (3), (19), (20), (21) and (22), we obtain
\[
nT(r) < (m + 3)T(r) + S(r),
\]
which is impossible.
CASE II. Suppose that $B = -1$.

From (18) we have

\[
G = \frac{A}{-F + (A + 1)}.
\]

If $A + 1 \neq 0$, from (24) we obtain

\[
\bar{N}\left(r, \frac{1}{F - (A + 1)}\right) = \bar{N}(r, G).
\]

Thus, in the same manner as above, we have a contradiction. From this we obtain $A + 1 = 0$. Again from (24) we derive $F \cdot G \equiv 1$. This and (2) yield

\[
f^{n-m}(f - a_1)(f - a_2)\cdots(f - a_m)g^{n-m}(g^m + a) \equiv b^2,
\]

where $a_1, a_2, \ldots, a_m$ are the distinct roots of the equation $\omega^m + a = 0$.

Suppose that $z_0$ is a zero of $f$ of order $p$. From (25) we know that $z_0$ is a pole of $g$. Suppose that $z_0$ is a pole of $g$ of order $q$. From (25) we obtain

\[
(n - m)p = nq.
\]

Noting that $n$ and $m$ have no common factors, from (26) we get $n \leq p$. Thus,

\[
\bar{N}\left(r, \frac{1}{f}\right) \leq \frac{1}{n}N\left(r, \frac{1}{f}\right) \leq \frac{1}{n}T(r, f) + O(1).
\]

Suppose that $z_j (j = 1, 2, \ldots, m)$ is a zero of $f - a_j$ of order $p_j$. From (25) we know that $z_j$ is a pole of $g$. Suppose that $z_j$ is a pole of $g$ of order $q_j$. From (25) we obtain

\[
p_j = nq_j.
\]

Thus $n \leq p_j$ and hence

\[
\bar{N}\left(r, \frac{1}{f - a_j}\right) \leq \frac{1}{n}N\left(r, \frac{1}{f - a_j}\right) \leq \frac{1}{n}T(r, f) + O(1).
\]

By the second fundamental theorem, from (27) and (28) we have

\[
(m - 1)T(r, f) < \bar{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^{m} \bar{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f)
\]

\[
\leq \frac{m + 1}{n}T(r, f) + S(r, f),
\]

which is impossible.
CASE III. Suppose that $B = 0$.

From (18) we have

$$G = \frac{F + (A - 1)}{A}.$$  

If $A - 1 \neq 0$, from (29) we obtain

$$\overline{N}(r, \frac{1}{F + (A - 1)}) = \overline{N}(r, \frac{1}{G}).$$

Thus, in the same manner as above, we have a contradiction. From this we obtain $A - 1 = 0$. Again from (29) we derive $F \equiv G$. This and (2) yield

$$f^n - g^n = -a(f^{n-m} - g^{n-m}).$$

If $f^n \neq g^n$, from (30) we obtain

$$g^m = -a(h - v)(h - v^2)\ldots(h - v^{n-m-1}) \overline{N}(r, \frac{1}{h - u^j}) \leq \frac{1}{m}N\left(r, \frac{1}{h - u^j}\right) \leq \frac{1}{2}T(r, h) + O(1).$$

By the second fundamental theorem, from (32) we obtain

$$(n - 3)T(r, h) < \sum_{j=1}^{n-1} \overline{N}\left(r, \frac{1}{h - u^j}\right) + S(r, h) \leq \frac{n-1}{2}T(r, h) + S(r, h),$$

which is impossible. Thus $f^n \equiv g^n$ and $f^{n-m} \equiv g^{n-m}$. However, since $n$ and $m$ have no common factors, we get $f \equiv g$.

This completes the proof of Theorem 1.
4. SUPPLEMENT OF THEOREM 1

It is reasonable to ask: What can be said if $m = 1$ in Theorem 1? In this section, we prove the following theorem, which is a supplement of Theorem 1.

**Theorem 2.** Let $S = \{w \mid w^n + aw^{n-1} + b = 0\}$, where $n > 10$ is a positive integer, $a$ and $b$ are two nonzero constants such that the algebraic equation $w^n + aw^{n-1} + b = 0$ has no multiple roots. If $f$ and $g$ are two distinct nonconstant meromorphic functions satisfying $E_f(S) = E_g(S)$, then

$$f = -\frac{ah(h^{n-1} - 1)}{h^n - 1} \quad \text{and} \quad g = -\frac{a(h^{n-1} - 1)}{h^n - 1},$$

where $h$ is a nonconstant meromorphic function.

**Proof:** Let

$$F = -\frac{1}{b} f^{n-1}(f + a) \quad \text{and} \quad G = -\frac{1}{b} g^{n-1}(g + a).$$

Proceeding as in the proof of Theorem 1, we have $F \cdot G \equiv 1$ or $F \equiv G$. We distinguish the following two cases.

**Case I.** Assume $F \cdot G \equiv 1$.

From (33) we have

$$f^{n-1}(f + a)g^{n-1}(g + a) \equiv b^2.$$ 

Suppose that $z_0$ is a zero of $f$ of order $p$. From (34) we know that $z_0$ is a pole of $g$. Suppose that $z_0$ is a pole of $g$ of order $q$. From (34) we obtain $(n - 1)p = nq$. From this we get $n \leq p$. Thus

$$\frac{1}{n} T(r, f) + O(1) \leq \frac{1}{n} T(r, f) + N(r, \frac{1}{f}).$$

Suppose that $z_1$ is a zero of $f + a$ of order $p_1$. From (34) we know that $z_1$ is a pole of $g$. Suppose that $z_1$ is a pole of $g$ of order $q_1$. From (34) we obtain $p_1 = nq_1$. Thus $n \leq p_1$ and hence

$$\frac{1}{n} T(r, f + a) \leq \frac{1}{n} T(r, f + a) + O(1).$$

In the same manner as above, we have

$$\frac{1}{n} T(r, g) + O(1),$$

and

$$\frac{1}{n} T(r, g + a) + O(1).$$

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From (34) one sees easily that the poles of $f$ can only be from the zeros of $g$ and $g + a$. Consequently,
\[
\overline{N}(r, f) \leq \overline{N}\left( r, \frac{1}{g} \right) + \overline{N}\left( r, \frac{1}{g + a} \right).
\]
From this, (37) and (38) we obtain
\[
(39) \quad \overline{N}(r, f) \leq \frac{2}{n} T(r, g) + O(1).
\]
By the first fundamental theorem and Lemma 1, from (34) we have
\[
T(r, g) = T(r, f) + S(r, f).
\]
From this and (39) we obtain
\[
(40) \quad \overline{N}(r, f) \leq \frac{2}{n} T(r, f) + S(r, f).
\]
By the second fundamental theorem, from (35), (36) and (40) we get
\[
T(r, f) < \overline{N}\left( r, \frac{1}{f} \right) + \overline{N}\left( r, \frac{1}{f + a} \right) + \overline{N}(r, f) + S(r, f)
\leq \frac{4}{n} T(r, f) + S(r, f),
\]
which is impossible.

**CASE II.** Assume $F \equiv G$.

From (33) we have
\[
(41) \quad f^n - g^n \equiv -a(f^{n-1} - g^{n-1}).
\]
Noting $f \neq g$, from (41) we obtain
\[
(42) \quad g = -\frac{a(h^{n-1} - 1)}{h^n - 1},
\]
where $h = f/g$. From (42) we know that $h$ is a nonconstant meromorphic function. Thus, from (42) we have
\[
f = -\frac{ah(h^{n-1} - 1)}{h^n - 1}.
\]

This completes the proof of Theorem 2. \qed
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