A NOTE ON THE POLIGNAC NUMBERS

HAO PAN

(Received 8 August 2013; accepted 13 September 2013; first published online 13 March 2014)

Abstract
Suppose that \( k_0 \geq 3.5 \times 10^6 \) and \( \mathcal{H} = \{h_1, \ldots, h_{k_0}\} \) is admissible. Then, for any \( m \geq 1 \), the set \( \{m(h_j - h_i) : h_i < h_j\} \) contains at least one Polignac number.

Keywords and phrases: Polignac number, admissible set.

1. Introduction
A recent huge breakthrough in prime number theory is Zhang’s brilliant work (see [4]), which asserts that
\[
\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 7 \times 10^7,
\]
where \( p_n \) denotes the \( n \)th prime. For a set \( \mathcal{H} = \{h_1, h_2, \ldots, h_{k_0}\} \) of positive integers, we say that \( \mathcal{H} \) is admissible if \( \nu_p(\mathcal{H}) < p \) for every prime \( p \), where \( \nu_p(\mathcal{H}) \) denotes the number of distinct residue classes occupied by those \( h_i \) modulo \( p \). Zhang proved that if \( k_0 \geq 3.5 \times 10^6 \) and \( \mathcal{H} = \{h_1, \ldots, h_{k_0}\} \) is admissible, then, for sufficiently large \( x \), there exists \( n \in [x, 2x] \) such that \( \{n + h_1, n + h_2, \ldots, n + h_{k_0}\} \) contains at least two primes. He also constructed an admissible \( \mathcal{H} = \{h_1, \ldots, h_{k_0}\} \) such that \( \max_{i,j} |h_j - h_i| \leq 7 \times 10^7 \).

In fact, one may have the following ‘cheap’ extension of Zhang’s theorem.

**Theorem 1.1.** Let \( k_0 \geq 3.5 \times 10^6 \) and \( A > 0 \). Suppose that \( x \) is sufficiently large and \( 1 \leq q \leq (\log x)^A \). If \( \mathcal{H} = \{h_1, \ldots, h_{k_0}\} \) is admissible and \( (q, h_1h_2 \cdots h_{k_0}) = 1 \), there exists \( n \in [x, 2x] \) such that \( \{qn + h_1, qn + h_2, \ldots, qn + h_{k_0}\} \) contains at least two primes.

The proof of Theorem 1.1 is just a copy of Zhang’s original one. The only modification is to set
\[
P(n) = \prod_{i=1}^{k_0} (qn + h_i) \quad \text{and} \quad \Omega = \prod_{\text{prime } p | q} \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right) \cdot \prod_{\text{prime } p} \left(1 - \frac{1}{p}\right)^{-k_0}.
\]

The author is supported by the National Natural Science Foundation of China (grant no. 11271185).
© 2014 Australian Mathematical Publishing Association Inc. 0004-9727/2014 $16.00

500
As an immediate consequence of Theorem 1.1, for $0 \leq b < q$ with $(b, q) = 1$,

$$\liminf_{n \to \infty} \frac{p_{n+1}^{(b, q)} - p_n^{(b, q)}}{q} \leq 7 \times 10^7,$$

(1.1)

where $p_n^{(b, q)}$ denotes the $n$th prime of the form $qm + b$. In fact, suppose that prime $p \not| q$ and $\{h_1, \ldots, h_{k_0}\}$ does not cover the residue class $c$ modulo $p$. Then $\{b + qh_1, \ldots, b + qh_{k_0}\}$ does not cover $(b + cq)$ modulo $p$, as well. And if $p \mid q$, then evidently $\{b + qh_1, \ldots, b + qh_{k_0}\}$ does not cover 0 modulo $p$. That is, the admissibility of $\{h_1, \ldots, h_{k_0}\}$ always implies that of $\{b + qh_1, \ldots, b + qh_{k_0}\}$. Thus (1.1) easily follows from Theorem 1.1.

However, the main purpose of this short note is to give another application of Theorem 1.1, concerning Polignac numbers [3]. A positive even number $d$ is called a Polignac number if there exist infinitely many $n$ such that $p_{n+1} - p_n = d$. Of course, it is believed that every positive even number is a Polignac number. And Zhang’s theorem shows that the smallest Polignac number is not greater than $7 \times 10^7$.

Recently, combining Zhang’s techniques with some lemmas from [1], Pintz [2] proved that the set of all Polignac numbers has a positive lower density. We shall now show that this lower density is at least $2 \times 10^{-21}$. In fact, we have the following theorem.

**Theorem 1.2.** Suppose that $k_0 \geq 3.5 \times 10^6$ and $\mathcal{H} = \{h_1, \ldots, h_{k_0}\}$ is admissible. Let $\sigma(\mathcal{H}) = \{h_j - h_i : h_i < h_j\}$. Then, for any $m \geq 1$, the set $m \cdot \sigma(\mathcal{H}) = \{md : d \in \sigma(\mathcal{H})\}$ contains at least one Polignac number.

Evidently, by taking $k_0 = 3.5 \times 10^6$ and $\max_{i,j} |h_i - h_j| \leq 7 \times 10^7$ in Theorem 1.2, we can get that the lower density of all Polignac numbers is at least

$$\frac{2}{k_0^2 \cdot \max_{i,j} |h_j - h_i|} \geq 2 \times 10^{-21}. $$

**2. Proof of Theorem 1.2**

Without loss of generality, assume that $h_1 < h_2 < \cdots < h_{k_0}$. Let

$$X = \{a \in [mh_1, mh_{k_0}] : a \equiv mh_1 \pmod{2}, a \not\in \{mh_1, \ldots, mh_{k_0}\}\}.$$

Assume that $X = \{a_1, a_2, \ldots, a_l\}$. Arbitrarily choose distinct primes $p_1, p_2, \ldots, p_l > mh_{k_0}$. Let $b > 0$ be an integer such that $b \equiv 1 \pmod{m}$ and $b \equiv -a_j \pmod{p_j}$ for $1 \leq j \leq l$. Let $q = mp_1 p_2 \cdots p_l$. Since $(b, m) = 1$, $\{b + mh_1, \ldots, b + mh_{k_0}\}$ is admissible. And for each $j$, noting that $p_j \mid b + a_j$ and $p_j > mh_{k_0}$, we must have

$$\prod_{i=1}^{k_0} (b + mh_i) \equiv 0 \pmod{p_j}.$$
That is, \( q \) is coprime to \((b + mh_1)(b + mh_2) \cdots (b + mh_{k_0})\). By Theorem 1.1, there exist infinitely many \( n \) such that \( \{qn + b + mh_1, qn + b + mh_2, \ldots, qn + b + mh_{k_0}\} \) contains at least two primes.

Let \( n_1, n_2, n_3, \ldots \) be all such positive integers \( n \). For each \( s \geq 1 \), noting that \( \{qn_s + b + mh_1, \ldots, qn_s + b + mh_{k_0}\} \) contains at least two primes, we may choose a pair \((i_s, j_s)\) with \( i_s < j_s \) such that both \( qn_s + b + mh_{i_s} \) and \( qn_s + b + mh_{j_s} \) are prime, but \( qn_s + b + mh_k \) is composite for all \( i_s < k < j_s \). Since \( 1 \leq i_s < j_s \leq k_0 \), clearly there exists a pair \((i_s, j_s)\) such that the set \( \{s : (i_s, j_s) = (i_s, j_s)\} \) is infinite. That is, \( qn + b + mh_{i_s} \) and \( qn + b + mh_{j_s} \) are prime for infinitely many \( n \). But according to the definition of \( q \), for any \( a_j \in (mh_{i_s}, mh_{j_s}) \), \( qn + b + a_j \equiv 0 \) (mod \( p_j \)), that is, \( qn + b + a_j \) cannot be prime. So \( qn + b + mh_{i_s} \) and \( qn + b + mh_{j_s} \) must be two consecutive primes, that is, \( m(h_{j_s} - h_{i_s}) \) is a Polignac number. We are done.

Acknowledgements

I am grateful to the anonymous referee for his/her useful suggestions on this paper. I also thank Professor Zhi-Wei Sun for his helpful discussions on Zhang’s theorem.

References


HAO PAN, Department of Mathematics, Nanjing University, Nanjing 210093, PR China
e-mail: haopan1979@gmail.com