HIGHLY SYMMETRIC HOMOGENEOUS SPACES

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We consider effective homogeneous spaces \( M = G/H \) where \( G \) is a compact connected Lie group, \( H \) is a closed subgroup and \( G \) acts effectively on \( M \) (i.e., \( H \) contains no non-trivial subgroup normal in \( G \)). It is known that \( \dim G \leq m^2/2 + m/2 \) where \( m = \dim M \) and that if \( \dim G = m^2/2 + m/2 \), then \( M \) is diffeomorphic to the standard sphere \( S^m \) or the standard real projective space \( RP^m \) [1]. In addition it has been shown that for fixed \( m \) there are gaps in the possible dimensions for \( G \) below the maximal bound [4; 5]. Using the Main Lemma of [3], we give a classification of all effective homogeneous spaces \( M = G/H \) with \( \dim G \geq m^2/4 + m/2 \), \( m \geq 19 \). In particular, if \( M \) is simply-connected we conclude that \( M = CP^n \), \( m = 2n \), or \( M = S^{m-k} \times V^k \), \( 0 \leq k \leq m/2 \), where \( V^k \) is a \( k \)-dimensional simply-connected homogeneous space.

**THEOREM.** Let \( M^m = G/H \) be an effective homogeneous space with \( \dim G \geq m^2/4 + m/2 \), \( m \geq 19 \). Then exactly one of the following holds:

1. \( M = CP^n \) (\( m = 2n \)) and \( G \) is locally isomorphic to \( SU(n+1) \).
2. \( M = CP^n \times S^1 \) (\( m = 2n + 1 \)) and \( G \) is locally isomorphic to \( U(n + 1) \).
3. \( M \) is a simple lens space finitely covered by \( S^{m+1} \) (\( m = 2n + 1 \)) and \( G \) is locally isomorphic to \( U(n + 1) \).

(In possibilities (4) through (6), \( V^k = G_2/H_2 \) denotes a \( k \)-dimensional homogeneous space, \( 0 \leq k \leq m/2 \), and \( G \) is locally isomorphic to \( \text{Spin}(m - k + 1) \times G_2 \).

**COROLLARY.** If \( M \) is simply-connected, then \( M = CP^n \) or \( M = S^{m-k} \times V^k \) where \( V^k \) is simply-connected.

**Proof of the Theorem.** Since \( G \) acts effectively on \( M \), we may apply the Main Lemma of [3]. Possibilities (1), (2) and (3) then correspond to cases (\( \alpha \)), (\( \beta \)) and (\( \gamma \)) of the Main Lemma. We are left with case (\( \delta \)) of the Main Lemma. Therefore \( G \) is locally isomorphic to \( \text{Spin}(m - k + 1) \times G_2 \), \( 0 \leq k \leq m/2 \),

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where \( G_1 = \text{Spin}(m - k + 1) \) acts \textit{almost effectively}, i.e. with finite kernel, on \( M \) with orbits which are some combination of fixed points, standard \((m - k)\)-spheres and standard real projective \((m - k)\)-spaces. It follows moreover from [4, Theorem 1] that \( \dim G_2 \leq k(k + 1)/2 \). Since the almost effective actions of \( G_1 \) and \( G_2 \) on \( M \) commute it is easily verified that if \( y = g_2 g_1 x \) for \( x, y \in M \) and \( g_1 \in G_1, g_2 \in G_2 \) then
\[
(G_1)_y = g_1 (G_1) g_1^{-1}
\]
where \((G_1)_y\) denotes the isotropy or stability subgroup of \( G_1 \) at \( y \). But \( G_1 \times G_2 \) is transitive on \( M \) and it follows from the observation above that all the orbits of the action of \( G_1 \) on \( M \) are of the same type. Let \( H_1 = (G_1)_x \) for some \( x \in M \). By Borel [2], \( M \) is a fibre bundle over \( M/G_1 \) with fibre \( G_1/H_1 \) (either \( S^{m-k} \) or \( \mathbb{R} P^{m-k} \)) and structural group \( N(H_1, G_1)/H_1 \). If \( G_1/H_1 = \mathbb{R} P^{m-k} \), then the structural group \( N(H_1, G_1)/H_1 \) is trivial and we have possibility (4). So we assume \( G_1/H_1 = S^{m-k} \). If the bundle is still trivial then of course we have possibility (5). So we are left with the case where \( M \) is the total space of a non-trivial \( S^{m-k} \) bundle over \( M/G_1 \). We can describe the bundle as follows [2]. Let \( F \) be the fixed point set of \( H_1 \) on \( M \) and let \( K = N(H_1, G_1)/H_1 \cong \mathbb{Z}_2 \). Now \( K \) acts freely on \( F \) and we have the principal \( K \)-bundle \( F \to M/G_1 \). The associated \( G_1/H_1 = S^{m-k} \) bundle is
\[
M = S^{m-k} \times_K F \to M/G_1.
\]
We show \( G_2 \) acts transitively on \( F \). The orbits of \( G_2 \) on \( M \) are all of the same type and \( G_2 \) leaves the subset \( F \) invariant. Since the actions of \( G_2 \) and \( K \) commute on \( F \) and \( G_2 \) acts transitively on \( M/G_1 = F/K \), \( K \) acts transitively on \( F/G_2 \). Therefore \( F/G_2 \) consists of either one or two points and, hence, there are either one or two orbits for the action of \( G_2 \) on \( F \). But if there are two orbits, \( K \) permutes these two orbits and
\[
M = S^{m-k} \times_K F = \mathbb{R} P^{m-k} \times F_0
\]
where \( F_0 \) is one of these two orbits. However, this would place us back in possibility (4). Therefore \( F = G_2/H_2 \).

Since \( K \) acts freely on \( F \), it is easily verified that \( K \) is isomorphic to a subgroup \( S \) of \( N(H_2, G_2)/H_2 \) and, in fact, the action of \( K \) on \( F \) is equivalent to the action of \( S \) on \( G_2/H_2 \) induced from the standard action of \( N(H_2, G_2)/H_2 \) on \( G_2/H_2 \). This completes the proof of the theorem.

For \( k \leq k_0 = \lfloor \frac{1}{2}((1 + 8m)^{1/2} - 3) \rfloor \),
\[
\frac{1}{2}(m - k)(m - k + 1) + \frac{1}{2}k(k + 1) < \frac{1}{2}(m - k + 1)(m - k + 2).
\]

Therefore it follows from the theorem that to list the effective homogeneous spaces \( G/H \) with \( \dim G \geq (m - k)(m - k + 1)/2, k \leq k_0 \), it is sufficient to list all homogeneous spaces of dimension less than or equal to \( k \). For small
values of $k$ this program is not difficult. As an example we list below all homogeneous spaces of dimensions one, two and three:

\[
\begin{align*}
  k = 1: & \quad S^1 \\
  k = 2: & \quad S^2, \text{RP}^2, T^2 \\
  k = 3: & \quad S^3, \text{RP}^3, T^3, \text{RP}^2 \times S^1, S^2 \times S^1, S^2 \times \mathbb{Z}_2, S^1, S^3/F \quad \text{(where } F \text{ is any finite subgroup of } S^3). \\
\end{align*}
\]

References


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