Proceedings of the Edinburgh Mathematical Society (2010) **53**, 293–298 ©
DOI:10.1017/S0013091508000382 Printed in the United Kingdom

A SURFACE THAT CONTAINS MANY LORENTZIAN CIRCLES

GRACIELA S. BIRMAN AND GRACIELA M. DESIDERI

Núcleo Consolidado de Matemática Pura y Aplicada (NUCOMPA), Facultad de Ciencias Exacta, Universidad Nacional del Centro de la Provincia de Buenos Aires, Pinto 399, 7000 Tandil, Argentina
(gbirman@exa.unicen.edu.ar; gmdeside@exa.unicen.edu.ar)

(Received 30 March 2008)

Abstract The characterization of a surface by means of the circles contained in it has been studied by S. Izuyima, A. Takiyama, K. Ogiu, R. Takagi and N. Takeuchi, among others. The aim of this paper is to show some characterizations of a pseudosphere in Lorentz 3-space, assuming the existence of Lorentzian and Euclidean circles.

Keywords: pseudosphere; Lorentzian space; Lorentzian circle; Euclidean circle

2010 *Mathematics subject classification:* Primary 53C50
Secondary 53B30

1. Introduction

We aim to show some characterizations of the pseudosphere in Lorentzian 3-space, bearing in mind the causality condition of curves.

Some authors (see, for example, [**6**, **9**, **10**]) have studied the analogous problem in Euclidean 3-space; Izumiya and Takiyama [**4**, **5**] also generalized some results of [**6**, **9**] to Minkowski 3-space.

In [**6**], Ogiue and Takagi showed the following result, with several generalizations: let M be a surface in the Euclidean 3-space E^3, and suppose that, through each $p \in M$, there exist two circles of E^3 such that p is the unique point in common between them and such that they are contained in M near p; then M is locally a plane or a sphere.

Geometric characterizations of a sphere in E^3 are found in other papers. For example, in [**9**] Takeuchi showed the following: let M be a compact simply connected smooth surface in E^3, and suppose that, through each $p \in M$, there exist three circles of E^3 contained in M; then M is a sphere. In [**10**], Takeuchi studied a similar problem by replacing the condition 'three circles' by 'a circle contained in a normal plane'.

Many of the results obtained by Takeuchi [**9**] and Ogiue and Takagi [**6**] were generalized from E^3 to the Lorentz–Minkowski 3-space L^3 by Izuyima and Takiyama [**4**, **5**].

In [**5**], it is shown that if S is a time-like surface in L^3 and, for each point $p \in S$, the intersection of S and T_pS is two light-like lines, then S is a pseudosphere.

In [**4**], the following theorem is proved.

Theorem A. *Let S be a time-like surface in L^3. Suppose that, through each $p \in S$, there exist two pseudocircles γ_1, γ_2 of L^3 such that*

(i) *γ_1, γ_2 are contained in S in a neighbourhood of p,*

(ii) *$\gamma_1 \cup \gamma_2 = p$ (tangent to each other).*

Then S is locally a pseudosphere or a time-like plane.

In [**10**], Takeuchi stated that a sphere in E^3 is characterized as a compact smooth surface M which contains a circle through each point $p \in M$ contained in a normal plane of M at p.

One of the problems that arise when we want to generalize this result from the Euclidean 3-space to the Lorentzian 3-space is the fact that the pseudosphere is a non-compact surface. Thus, in our case, we characterize the pseudosphere via its normal sections using the shape operator [**7**].

The aim of this paper is to prove the following theorems.

Theorem 1.1. *Let M be a complete, simply connected, time-like surface in L^3. For each $p \in M$, suppose that there exist a Lorentzian circle and a Euclidean circle through p on M which have the same curvature; also suppose that they are the normal sections of the surface at p in the direction of a future-pointing time-like vector and a space-like vector, respectively. Then, M is a pseudosphere.*

The next two theorems are dual to each other.

Theorem 1.2. *Let M be a connected time-like surface in L^3. For each future-pointing time-like vector $\boldsymbol{u}$, suppose that $\boldsymbol{u}^\perp \cap M$ is a Euclidean circle on a normal section of the surface in the direction of $\boldsymbol{u}$. Then, M is locally a pseudosphere.*

Theorem 1.3. *Let M be a connected time-like surface in L^3. For each space-like vector $\boldsymbol{u}$, suppose that $\boldsymbol{u}^\perp \cap M$ is a Lorentzian circle on a normal section of the surface in the direction of $\boldsymbol{u}$. Then, M is locally a pseudosphere.*

2. Preliminaries

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ be two vectors in the three-dimensional vector space $\mathbb{R}^3$. The Lorentzian inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Thus, the square $\mathrm{d}s^2$ of an element of arc length is given by

$$\mathrm{d}s^2 = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2.$$

The space $\mathbb{R}^3$ furnished with this metric is called a Lorentz 3-space or Lorentz–Minkowski 3-space. We write L^3 or $\mathbb{R}^3_1$ instead of $(\mathbb{R}^3, \mathrm{d}s)$.

We say that a vector $\boldsymbol{x}$ in L^3 is time-like if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, space-like if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$ and light-like if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$.

Let $\boldsymbol{x} \in L^3$. Then $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$ is called the norm of vector $\boldsymbol{x}$.

A time-like vector $\boldsymbol{x}$ is called future-pointing if $\langle \boldsymbol{x}, \boldsymbol{e}_1 \rangle < 0$ and past-pointing if $\langle \boldsymbol{x}, \boldsymbol{e}_1 \rangle > 0$, where $\boldsymbol{e}_1 = (1, 0, 0)$.

For any $\boldsymbol{x}, \boldsymbol{y} \in L^3$, the Lorentzian vector product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined [**2**] by $\boldsymbol{x} \wedge \boldsymbol{y} = (x_3 y_2 - x_2 y_3, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$.

We say that $\boldsymbol{x}$ is orthogonal to $\boldsymbol{y}$ if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ and $\boldsymbol{x} \neq \boldsymbol{y} \neq \boldsymbol{0}$. Clearly, $\boldsymbol{x} \wedge \boldsymbol{y}$ is orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$.

We shall give a surface M in L^3 by expressing its coordinates x_i as functions of two parameters in a certain interval. We consider the functions x_i to be real functions of real variables.

The surface M is non-degenerate if, for each $p \in M$ and non-zero $\boldsymbol{x} \in T_p M$, there exists some $\boldsymbol{y} \in T_p M$ with $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \neq 0$. If, in addition, for each $p \in M$, the restriction $\langle \cdot , \cdot \rangle|_{T_p M \times T_p M}$ is positive definite, then we say that M is a space-like surface; if for each $p \in M$ the restriction $\langle \cdot , \cdot \rangle|_{T_p M \times T_p M}$ is a Lorentzian metric, then we say that M is a time-like surface [**1**]. If M is a space-like or time-like surface, then M is said to be a non-light-like surface.

The pseudosphere with radius $r > 0$ is the surface

$$S_1^2(r) = \{(x_1, x_2, x_3) \in L^3 : -x_1^2 + x_2^2 + x_3^2 = r^2\},$$

and the pseudohyperbolic space of radius $r > 0$ is the surface

$$H_0^2(r) = \{(x_1, x_2, x_3) \in L^3 : -x_1^2 + x_2^2 + x_3^2 = -r^2\}.$$

The pseudosphere $S_1^2(r)$ is a time-like surface.

A parametrized curve is called time-like, space-like or null if at every point, its tangent vector is time-like, space-like or null, respectively.

We say that the plane curve $\mathcal{C}(O, r) = \{X \in \Pi_{\mathrm{L}} : \langle \overrightarrow{OX}, \overrightarrow{OX} \rangle = r^2\}$ is a Lorentzian circle with centre O and radius r, where O is a point in a time-like plane $\Pi_{\mathrm{L}} \subset L^3$ and r is a positive real number. The Lorentzian circle $\mathcal{C}(O, r)$ is a time-like curve.

We say that the plane curve $\mathcal{E}(O, r) = \{X \in \Pi_{\mathrm{E}} : \langle \overrightarrow{OX}, \overrightarrow{OX} \rangle = r^2\}$ is a Euclidean circle with centre O and radius r, where O is a point in a space-like plane $\Pi_{\mathrm{E}} \subset L^3$ and r is a positive real number. The Euclidean circle $\mathcal{E}(O, r)$ is a space-like curve.

Let (x_1, x_2, x_3) be a coordinate system in L^3. In what follows, we consider the set

$$\left\{ \partial_1 = \frac{\partial}{\partial x_1},\ \partial_2 = \frac{\partial}{\partial x_2},\ \partial_3 = \frac{\partial}{\partial x_3} \right\}$$

to be an orthonormal basis or frame for L^3, that is, ∂_1, ∂_2 and ∂_3 are three mutually orthogonal vector fields such that $\langle \partial_1, \partial_1 \rangle = -1$ and $\langle \partial_2, \partial_2 \rangle = \langle \partial_3, \partial_3 \rangle = 1$.

The Levi-Cività connection $\bar{\nabla}$ of L^3 gives rise in a natural way to a function $\nabla : \Xi(M) \times \bar{\Xi}(M) \to \bar{\Xi}(M)$, called the induced connection on $M \subset L^3$. (See [**7**] for a definition of induced connection.)

Let $V \in \Xi(M)$ and $X \in \bar{\Xi}(M)$, and, for each $p \in M$, let $\bar{V}$ and $\bar{X}$ be smooth local extensions of V and X over a coordinate neighbourhood $\mathcal{U}$ of p in L^3. Then the induced connection $\nabla_V X$ is defined on each $\mathcal{U} \cap M$ to be a restriction of $\bar{\nabla}_{\bar{V}} \bar{X}$ to $\mathcal{U} \cap M$.

In [**7**], O'Neill defines the shape operator for hypersurfaces of semi-Riemannian manifolds. In particular, if N is the unit normal vector field on $M \subset L^3$, then the $(1,1)$ tensor field S on M is called the shape operator of $M \subset L^3$ derived from N such that, for all $V, W \in \Xi(M)$, $\langle S(V), W\rangle = \langle \mathit{\Pi}(V,W), N\rangle$.

At each point $p \in M$, S determines a linear operator $S : T_pM \to T_pM$ defined by

$$S(\partial_{t_i}) = -\bar{\nabla}_{\partial_{t_i}} N, \quad i = 1, 2,$$

where (t_1, t_2) is a coordinate system in M.

3. Main results

In [**8**] a definition of the normal section is given; we now generalize it from the Euclidean 3-space to the Lorentzian 3-space.

Definition 3.1. Let M be a non-light-like surface in L^3. A normal section at $p \in M$ is a curve which is the intersection of M with the plane at p through a tangent vector $\boldsymbol{u}_p$ and the normal vector $\boldsymbol{N}_p$ at p.

From now on we will assume that the curves are parametrized by the proper time parameter t, which is the Lorentzian counterpart of the arc-length parameter.

Some properties of the shape operator are stated in the following two lemmas, which will be used in the proof of Theorem 1.1.

Lemma 3.2. *Let σ be a non-light-like curve which is the intersection of a non-light-like surface M and a non-light-like plane $\mathit{\Pi}$. For each p on σ, $\boldsymbol{N}_p$ and $\boldsymbol{V}_p$ denote the normal vectors of M and $\mathit{\Pi}$, respectively. If $\langle \boldsymbol{N}_p, \boldsymbol{V}_p\rangle$ is constant along σ, then $S(\sigma') = \lambda\sigma'$, where $\lambda \in \mathbb{R}$.*

Proof. Without loss of generality, we consider $\boldsymbol{N}_p$ and $\boldsymbol{V}_p$ to be two unit vectors.
We have that

$$0 = \bar{\nabla}_{\sigma'}\langle \boldsymbol{N}_p, \boldsymbol{V}_p\rangle \;\Rightarrow\; \langle S(\sigma'), \boldsymbol{V}_p\rangle = 0,$$
$$0 = \bar{\nabla}_{\sigma'}\langle \boldsymbol{N}_p, \boldsymbol{N}_p\rangle \;\Rightarrow\; \langle S(\sigma'), \boldsymbol{N}_p\rangle = 0$$

and

$$\langle \sigma', \boldsymbol{V}_p\rangle = \langle \sigma', \boldsymbol{N}_p\rangle = 0.$$

Hence, $\boldsymbol{\sigma}'$ is a normal vector at p of the plane through $\boldsymbol{N}_p$ and $\boldsymbol{V}_p$.
Then

$$S(\boldsymbol{\sigma}') = \langle \boldsymbol{\sigma}', \boldsymbol{\sigma}'\rangle^{-1}\langle S(\boldsymbol{\sigma}'), \boldsymbol{\sigma}'\rangle\boldsymbol{\sigma}'.$$

□

Lemma 3.3. *Let σ be the normal section at p in the direction of $\boldsymbol{u}_p$. Then $k_n(\boldsymbol{u}_p) = \pm\kappa_\sigma(0)$, where $k_n(\boldsymbol{u}_p)$ denotes the normal curvature.*

Proof. We have that

$$k_n(\boldsymbol{u}_p) = \kappa_\sigma(0)\langle \boldsymbol{n}(0), \boldsymbol{N}_p\rangle = \pm\kappa_\sigma(0)$$

because $\boldsymbol{n}(0)|_{\boldsymbol{N}_p}$, where $\sigma(o) = p$, $\sigma'(0) = \boldsymbol{u}_p$ and $\ddot{\sigma}(0) = \kappa_\sigma(0)n(0)$. □

In [**4**] the causality condition of the curves is not mentioned in the definition of pseudocircles.

We say that a time-like (space-like) curve in L^3 is a Lorentzian (Euclidean) circle if there exists a time-like (space-like) plane π in L^3 such that this curve is a Lorentzian (Euclidean) circle of π in the classical way.

We now prove Theorem 1.1.

Proof of Theorem 1.1. Let α and β be a Lorentzian circle and a Euclidean circle through p on M that are the normal sections of the surface at p in the direction of a future-pointing vector and a space-like vector, respectively. Hence (cf. [**3, 7**]),

$$[S] = \begin{bmatrix} -\langle S(\alpha'), \alpha'\rangle & \langle S(\beta'), \alpha'\rangle \\ \langle S(\alpha'), \beta'\rangle & -\langle S(\beta'), \beta'\rangle \end{bmatrix}.$$

Since $\langle S(\alpha'), \beta'\rangle = \langle \lambda\alpha', N \wedge \alpha'\rangle = 0$, we have

$$[S] = \begin{bmatrix} k_n(\alpha') & 0 \\ 0 & k_n(\beta') \end{bmatrix}.$$

By Lemma 3.2, we have that

$$[S]_p = \begin{bmatrix} \kappa_\alpha(0) & 0 \\ 0 & \kappa_\beta(0) \end{bmatrix} \Rightarrow K_p = \det[S]_p \equiv 1/r^2 > 0 \quad \text{for all } p \in M.$$

On [**7**, p. 228] the Lorentz analogue of Hopf's Theorem states the following.

> A complete, simply connected, n-dimensional Lorentz manifold of constant curvature K is isometric to
>
> $$\begin{aligned} \text{the psuedosphere } & S_1^n(r) && \text{if } K = 1/r^2 \text{ and } n \geqslant 3, \\ \text{covering } & \widetilde{S_1^2}(r) && \text{if } K = 1/r^2 \text{ and } n = 2, \\ \text{Lorentz–Minkowski space } & \mathbb{R}_1^n && \text{if } K = 0, \\ \text{covering } & \widetilde{H_1^n}(r) && \text{if } K = -1/r^2. \end{aligned}$$

Hence, M is isometric to the covering $\widetilde{S_1^2}(r)$ of $S_1^2(r)$.

Since α is the normal section of M at p in the direction of a future-pointing vector, $M = S_1^2(r)$. □

Remark 3.4. If α is the normal section of M at p in the direction of a past-pointing vector, then $M = S_1^2(r)|_{-1}$. If the time-orientation is not considered, that is, if α is the normal section of M at p in the direction of a time-like vector, then $M = S_1^2(r)|_{\pm 1}$.

We now prove the Theorem 1.2.

Proof of Theorem 1.2. Let M be a connected time-like surface in L^3. For each future-pointing time-like vector $\boldsymbol{u}$, suppose that $\boldsymbol{u}^\perp \cap M$ is a Euclidean circle.

In particular, for each $p \in M$ there exist two future-pointing time-like vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that

(i) $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors with different directions,

(ii) $\boldsymbol{u}^\perp \cap M$ and $\boldsymbol{v}^\perp \cap M$ are two Euclidean circles such that p is the unique point in common between them.

By (ii) and Theorem A, M is locally a pseudosphere or a time-like plane. Thus, by (i) and Lemma 3.3, M is locally a pseudosphere. □

The proof of Theorem 1.3 is analogous to that of Theorem 1.2.

Acknowledgements. G.S.B. was partly supported by the Consejo Nacional de Investigaciones Científicas y Tecnológicas de la República Argentina.

References

1. J. K. Been, P. E. Ehrlich and K. L. Easley, *Global Lorentzian geometry*, 2nd edn (Marcel Dekker, New York, 1996).
2. G. S. Birman, On L^2 and L^3, *Elem. Math.* **43** (1988), 46–50.
3. G. S. Birman and G. M. Desideri, Laplacian on mean curvature vector fields for some non-light-like surfaces in the three-dimensional Lorentzian space, in *Actas del VII Congreso Dr. Antonio A. R. Monteiro*, pp. 27–33 (Instituto de Matemática, Universidad Nacional del Sur, 2003).
4. S. Izumiya and A. Takiyama, A time-like surface in Minkowski 3-space which contains pseudocircles, *Proc. Edinb. Math. Soc.* **40** (1997), 127–136.
5. S. Izumiya and A. Takiyama, A time-like surface in Minkowski 3-space which contains light-like lines, *J. Geom.* **64** (1999), 95–101.
6. K. Ogiue and R. Takagi, A submanifold which contains many extrinsic circles, *Tsukuba J. Math.* **8** (1984), 171–182.
7. B. O'Neill, *Semi-Riemannian geometry with applications to relativity* (Academic Press, New York, 1983).
8. D. J. Struik, *Lectures on classical differential geometry*, 2nd edn (Dover, New York, 1961).
9. N. Takeuchi, A sphere as a surface which contains many circles, *J. Geom.* **24** (1985), 123–130.
10. N. Takeuchi, A sphere as a surface which contains many circles, II, *J. Geom.* **34** (1989), 195–200.