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# A solvability condition for finite groups with nilpotent maximal subgroups

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Let $G$ be a finite group with a nilpotent maximal subgroup $S$ and let $P$ denote the 2-Sylow subgroup of $S$. It is shown that if $P \cap Q$ is a normal subgroup of $P$ for any 2-Sylow subgroup $Q$ of $G$, then $G$ is solvable.

Thompson [6] has shown that if a finite group $G$ has a nilpotent maximal subgroup $S$ of odd order, then $G$ is solvable. Janko [3] has extended this result by proving that if the 2-Sylow subgroup of $S$ has class $\leq 2$, then $G$ is solvable. We note here another condition under which $G$ is solvable.

THEOREM. *Let $G$ be a finite group with a nilpotent maximal subgroup $S$ and let $P$ denote the 2-Sylow subgroup of $S$. Suppose that given any 2-Sylow subgroup $Q$ of $G$, $P \cap Q$ is a normal subgroup of $P$. Then $G$ is solvable.*

Proof. Assume the theorem is false and let $G$ be a minimal counterexample. Let $\text{rad}(G)$ denote the radical of $G$ (the largest solvable normal subgroup of $G$) and distinguish two cases.

Case 1. $\text{rad}(G) = 1$. Then $G$ has a non-abelian minimal normal subgroup $M$.

LEMMA [4]. *If a finite group $G$ contains a nilpotent maximal subgroup $S$ and a non-abelian minimal normal subgroup $M$ and if $P$ is*

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*the* 2-*Sylow subgroup of* $S$ *, then* $S = P\mathrm{rad}(G)$ *,* $G = MP\mathrm{rad}(G)$ *and* $P$ *is a* 2-*Sylow subgroup of* $G$ *.*

By the lemma, $G = MP$ , $P$ is a 2-Sylow subgroup of $G$ and $P$ is a maximal subgroup of $G$ . Let $R$ be a 2-Sylow subgroup of $G$ such that $R \neq P$ . If $R \cap P \neq 1$ , then $P \subseteq N_G(R \cap P)$ by hypothesis. Write $R = gPg^{-1}$ for some $g$ in $G$ . $P \cap gPg^{-1}$ is normal in $P$ so $g(P \cap g^{-1}Pg)g^{-1} = R \cap P$ is normal in $R$ . Hence $\langle R, P \rangle \subseteq N_G(R \cap P)$ . But $P$ is maximal in $G$ so $R \cap P$ is normal in $G$ , contradicting $\mathrm{rad}(G) = 1$ . Thus $R \cap P = 1$ , and from Theorem 1.4, p. 302 of [2], we conclude that $G$ has exactly one conjugate class of involutions.

By [6], $P$ has at least one involution. Suppose that $P$ has more than one involution. Let $a$ be an involution in the center of $P$ and pick an involution $b$ in $P$ such that $a \neq b$ . Let $A = \{a\}$ and $B = \{b\}$ . Since $G$ has only one conjugate class of involutions, $A$ and $B$ are subsets of $P$ conjugate in $G$ , and Corollary 2.7, p. 245 of [2], implies that there exist 2-Sylow subgroups $Q_i$ of $G$ and subsets $A = A_0, A_1, \ldots, A_m = B$ of $P$ such that $A_i \subseteq P \cap Q_i$ and $A_i = A_{i-1}^{y_i}$ for some $y_i$ in $N_G(P \cap Q_i)$ , $1 \leq i \leq m$ . Thus $P \cap Q_i \neq 1$ and since $P$ is disjoint from its distinct conjugates, $P = Q_i$ and $N_G(P \cap Q_i) = N_G(P)$ . However $P$ is maximal in $G$ and $\mathrm{rad}(G) = 1$ , so $N_G(P) = P$ , $y_i$ is in $P$ for each $i$ , and $b$ is conjugate to $a$ in $P$ , a contradiction. Therefore $P$ has exactly one involution.

Since $M$ is a non-abelian minimal normal subgroup of $G$ , we may write $M = M_1 \times \ldots \times M_t$ , $t \geq 1$ , where the $M_i$ are mutually isomorphic non-abelian simple groups, by Theorem 4.4.4 of [5]. Since $P$ is a 2-Sylow subgroup of $G$ and $M_i$ is a subnormal subgroup of $G$ , $M_i \cap P$ is a 2-Sylow subgroup of $M_i$ . $M_i$ cannot be solvable, so $M_i \cap P \neq 1$ by the Feit-Thompson theorem. Thus $M_i \cap P$ has exactly one involution and is therefore cyclic or generalized quaternion, by Theorem 9.7.3 of [5]. If $M_i \cap P$ is cyclic then $M_i$ has a normal 2-complement by

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Theorem 6.2.11 of [5], a contradiction. If $M_i \cap P$ is generalized quaternion, [1] implies that $M_i$ is not simple, a contradiction. Case 1 is now eliminated.

Case 2. $\text{rad}(G) \neq 1$. We assert that given any 2-Sylow subgroup $Q$ of $G$, $P \cap Q\text{rad}(G)$ is normal in $P$. For if $P$ is normal in $G$, $P$ is contained in $Q$ and $P \cap Q\text{rad}(G) = P$. If $P$ is not normal in $G$ then $N_G(P) = S$ and $P$ is a 2-Sylow subgroup of $G$. Now $Q = gPg^{-1}$ for some $g$ in $G$ and $Q \subseteq gSg^{-1}$. Since $gSg^{-1}$ is a nilpotent maximal subgroup of $G$ and $G$ is not solvable, $\text{rad}(G) \subseteq gSg^{-1}$. So $Q$ is the 2-Sylow subgroup of the nilpotent group $Q\text{rad}(G)$. It follows that $P \cap Q\text{rad}(G) \subseteq P \cap Q$. Thus $P \cap Q\text{rad}(G) = P \cap Q$, a normal subgroup of $P$ by hypothesis.

Let $\bar{G} = G/\text{rad}(G)$, $\bar{S} = S/\text{rad}(G)$ and $\bar{P} = P\text{rad}(G)/\text{rad}(G)$. $\bar{S}$ is a nilpotent maximal subgroup of $\bar{G}$ and $\bar{P}$ is the 2-Sylow subgroup of $\bar{S}$. Let $\bar{Q}$ be any 2-Sylow subgroup of $\bar{G}$. Write $\bar{Q} = Q\text{rad}(G)/\text{rad}(G)$ for some 2-Sylow subgroup $Q$ of $G$. Since $P \cap Q\text{rad}(G)$ is normal in $P$, $\bar{P} \cap \bar{Q}$ is normal in $\bar{P}$. Since $G$ was a minimal counterexample to the theorem and $|\bar{G}| < |G|$, $\bar{G}$ is solvable. Hence $G$ is solvable, a contradiction. This completes the proof of the theorem.

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