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Yuri Berest and Peter Samuelson

Compositio Math. **152** (2016), 1333–1384.

[doi:10.1112/S0010437X16007314](https://doi.org/10.1112/S0010437X16007314)



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# Double affine Hecke algebras and generalized Jones polynomials

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## ABSTRACT

In this paper we propose and discuss implications of a general conjecture that there is a natural action of a rank 1 double affine Hecke algebra on the Kauffman bracket skein module of the complement of a knot  $K \subset S^3$ . We prove this in a number of nontrivial cases, including all  $(2, 2p + 1)$  torus knots, the figure eight knot, and all 2-bridge knots (when  $q = \pm 1$ ). As the main application of the conjecture, we construct three-variable polynomial knot invariants that specialize to the classical colored Jones polynomials introduced by Reshetikhin and Turaev. We also deduce some new properties of the classical Jones polynomials and prove that these hold for all knots (independently of the conjecture). We furthermore conjecture that the skein module of the unknot is a submodule of the skein module of an arbitrary knot. We confirm this for the same example knots, and we show that this implies that the colored Jones polynomials of  $K$  satisfy an inhomogeneous recursion relation.

## 1. Introduction

In this paper we introduce new connections between the representation theory of double affine Hecke algebras and the colored Jones polynomials of a knot in  $S^3$ . These connections can be motivated by the following general considerations.

If  $K$  is a knot in  $S^3$ , the most natural algebraic invariant of  $K$  is the fundamental group  $\pi_1(S^3 \setminus K)$  of the complement. This group is not a complete invariant, since complements of different knots can have isomorphic fundamental groups (cf. [Fox52]), but it is known that the *peripheral map*

$$\alpha : \pi_1(\partial(S^3 \setminus K)) \rightarrow \pi_1(S^3 \setminus K) \quad (1.1)$$

is a complete knot invariant. More precisely, a theorem of Waldhausen [Wal68] implies that the peripheral map determines the knot complement, and a theorem of Gordon and Luecke [GL89] shows that knots in  $S^3$  are determined by their complements.

The peripheral map  $\alpha$  is quite complicated, so it is natural to simplify (1.1) by replacing fundamental groups with their linear representations. To this end, fix a complex reductive algebraic group  $G$  and let  $\text{Rep}(\pi, G)$  be the set of group representations from a (discrete) group  $\pi$  into  $G$ . The algebraic structure of  $G$  gives the set  $\text{Rep}(\pi, G)$  the structure of an affine scheme and  $G$  acts on this scheme by conjugation. We write  $\text{Char}(\pi, G) := \text{Rep}(\pi, G) // G$  for the algebro-geometric quotient and  $\mathcal{O}\text{Char}(\pi, G)$  for the corresponding coordinate ring (which is the

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Received 20 June 2014, accepted in final form 15 October 2015, published online 1 April 2016.

*2010 Mathematics Subject Classification* 57M25, 20C08 (primary).

*Keywords:* double affine Hecke algebras, knots, skein modules, quantum torus, Kauffman bracket, Jones polynomial, character variety.

The work of the first author was partially supported by NSF grant DMS 09-01570.

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ring of  $G$ -invariant regular functions on  $\text{Rep}(\pi, G)$ ). This construction is functorial, so a map of groups  $f : \pi \rightarrow \pi'$  induces a map of commutative algebras  $f_* : \mathcal{O}\text{Char}(\pi, G) \rightarrow \mathcal{O}\text{Char}(\pi', G)$ .

The boundary of the complement of (a neighborhood of) a knot is a torus  $S^1 \times S^1$  and, if  $\mathbb{T} \subset G$  is a maximal torus, we have the following natural map:

$$\mathbb{T} \times \mathbb{T} = \text{Rep}(\mathbb{Z}^2, \mathbb{T}) \hookrightarrow \text{Rep}(\mathbb{Z}^2, G) \twoheadrightarrow \text{Char}(\mathbb{Z}^2, G).$$

In [Tha01], it was shown that this map induces a bijection between  $(\mathbb{T} \times \mathbb{T})/W$  and the connected component  $\text{Char}^\circ(\mathbb{Z}^2, G)$  of  $\text{Char}(\mathbb{Z}^2, G)$  containing the trivial character (here  $W$  is the Weyl group of  $G$  acting diagonally on  $\mathbb{T} \times \mathbb{T}$ ). In [Ric79], Richardson showed that if  $G$  is simply connected, then  $\text{Char}(\mathbb{Z}^2, G)$  is connected, i.e.  $\text{Char}^\circ(\mathbb{Z}^2, G) = \text{Char}(\mathbb{Z}^2, G)$ . Finally, in [Sik14], it was shown that for classical groups the bijection  $(\mathbb{T} \times \mathbb{T})/W \rightarrow \text{Char}^\circ(\mathbb{Z}^2, G)$  is an isomorphism of schemes. Therefore, for a simply connected (classical)  $G$  we have an isomorphism of commutative algebras

$$\mathcal{O}(\mathbb{T} \times \mathbb{T})^W \cong \mathcal{O}\text{Char}(\mathbb{Z}^2, G). \tag{1.2}$$

The commutative algebra  $\mathcal{O}(\mathbb{T} \times \mathbb{T})^W$  admits interesting (noncommutative) deformations, which have been studied extensively in recent years. These deformations can be described in the following way. Denote by  $P$  and  $P^\vee$  the weight and co-weight lattices of the Lie algebra  $\mathfrak{g}$  of  $G$ , and write  $\mathbb{C}[P \oplus P^\vee]$  for the group algebra of their direct sum. The diagonal action of  $W$  on  $P \oplus P^\vee$  extends by linearity to  $\mathbb{C}[P \oplus P^\vee]$ , and we can thus define the semidirect product  $\mathbb{C}[P \oplus P^\vee] \rtimes W$ . The algebra  $\mathcal{O}(\mathbb{T} \times \mathbb{T})^W$  is canonically isomorphic to the invariant subalgebra  $\mathbb{C}[P \oplus P^\vee]^W$ , which embeds (nonunitally) in  $\mathbb{C}[P \oplus P^\vee] \rtimes W$  via the map  $a \mapsto ae = eae$  (where  $e = \sum_w w/|W|$  is the symmetrizing idempotent of  $W$ ). The image of this last map is called the *spherical subalgebra*. In this way, we get an identification

$$\mathcal{O}(\mathbb{T} \times \mathbb{T})^W \cong e(\mathbb{C}[P \oplus P^\vee] \rtimes W)e.$$

Now, for  $G$  as above, Cherednik [Che95] (see also [Che05]) defined the *double affine Hecke algebra (DAHA)*  $H_{q,t}$  of type  $G$  as a two-parameter family of deformations of  $\mathbb{C}[P \oplus P^\vee] \rtimes W$ , depending on  $q \in \mathbb{C}^*$  and  $t \in (\mathbb{C}^*)^r$ , where  $r$  is the number of conjugacy classes of reflections in  $W$ . (If  $G$  is simple, then  $r \in \{1, 2\}$ .) The symmetrizing idempotent of  $W$  deforms to a distinguished idempotent  $e = e_{q,t} \in H_{q,t}$ , and the spherical subalgebra of  $\mathbb{C}[P \oplus P^\vee] \rtimes W$  thus deforms to the spherical subalgebra of  $H_{q,t}$ , which we denote  $\text{SH}_{q,t} := eH_{q,t}e$ . In particular, when  $q = t = 1$ , there is a natural algebra isomorphism between the spherical subalgebra  $\text{SH}_{1,1}$  and the commutative algebra  $\mathcal{O}(\mathbb{T} \times \mathbb{T})^W$ .

Summarizing the above discussion, for each knot  $K \subset S^3$ , we have a map of commutative algebras

$$\alpha_* : \text{SH}_{1,1} \rightarrow \mathcal{O}\text{Char}(\pi_1(S^3 \setminus K), G). \tag{1.3}$$

This leads us to propose the following natural questions.

*Question 1.* Let  $K \subset S^3$  be a knot and  $N = \mathcal{O}\text{Char}(\pi_1(S^3 \setminus K), G)$ , viewed as an  $\text{SH}_{1,1}$ -module via the map  $\alpha_*$ .

- (i) Is there a canonical  $\text{SH}_{q,t}$ -module  $N_{q,t}$  which is a deformation of  $N$ ?
- (ii) What knot invariants can be extracted from  $N_{q,t}$ ?

We remark that one’s initial inclination might be to deform  $\alpha_*$  as an algebra homomorphism, but this is too restrictive. When  $q$  is not a root of unity, it is known that  $\text{SH}_{q,t}$  is a simple algebra, and  $\mathcal{O}\text{Char}(\pi_1(S^3 \setminus K), G)$  is ‘small’ (in particular, the image of  $\text{Char}(\pi_1(S^3 \setminus K), G)$  inside  $\text{Char}(\mathbb{Z}^2, G)$  is Lagrangian). Therefore, a deformation of  $\alpha_*$  as an algebra homomorphism would necessarily have a nontrivial kernel, which is impossible by the simplicity of  $\text{SH}_{q,t}$ .

To the best of our knowledge, this question has only been raised for  $q$ -deformations (i.e. for  $t = 1$ ) and has only been answered when  $\mathfrak{g} = \mathfrak{sl}_2$  and  $t = 1$ . (There is partial progress for  $t = 1$  and  $\mathfrak{g} = \mathfrak{sl}_n$  in [Sik05]; see also [CKM14].) When  $\mathfrak{g} = \mathfrak{sl}_2$ , we can identify  $P \cong P^\vee \cong \mathbb{Z}$  and  $W \cong \mathbb{Z}_2$ , so that  $\mathbb{C}[P \oplus P^\vee] \rtimes W \cong \mathbb{C}[X^{\pm 1}, Y^{\pm 1}] \rtimes \mathbb{Z}_2$ , where  $\mathbb{Z}_2$  acts by simultaneously inverting  $X$  and  $Y$ . Then  $\mathbb{Z}_2$  acts in the same way on the quantum torus  $A_q$ , which is the deformation of the algebra  $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$  with  $X, Y$  satisfying the relation  $XY = q^2 YX$ . We then have isomorphisms  $H_{q,1} \cong A_q \rtimes \mathbb{Z}_2$  and  $\text{SH}_{q,1} \cong A_q^{\mathbb{Z}_2}$ .

The connection to representation varieties comes from the Kauffman bracket skein module. This is a topologically defined vector space  $K_q(M)$  associated to an oriented 3-manifold  $M$  and the parameter  $q \in \mathbb{C}^*$  that has three important properties.

- (i) For a surface  $F$ , the vector space  $K_q(F \times [0, 1])$  is an algebra (typically noncommutative).
- (ii) If  $\partial M = F$ , then  $K_q(M)$  is a module over  $K_q(F \times [0, 1])$ .
- (iii) If  $q = \pm 1$ , then  $K_{q=\pm 1}(M)$  is a commutative algebra (for any  $M$ ).

In [PS00], Przytycki and Sikora showed that  $K_{q=-1}(M)$  is naturally isomorphic to the commutative algebra  $\mathcal{O}\text{Char}(\pi_1(M), \text{SL}_2(\mathbb{C}))$  (see also [Bul97]). By a theorem of Frohman and Gelca in [FG00],  $K_q((S^1)^2 \times [0, 1])$  is isomorphic to the algebra  $A_q^{\mathbb{Z}_2}$ . Combining these two theorems shows that  $N_{q,t=1} := K_q(S^3 \setminus K)$  is an  $A_q^{\mathbb{Z}_2}$ -module, which gives a positive answer to the first part of Question 1 (when  $\mathfrak{g} = \mathfrak{sl}_2$  and  $t = 1$ ).

At this point, we pause to remark that the knot invariant  $K_q(S^3 \setminus K)$  is different from many other knot invariants in a fundamental way. Roughly, many knot invariants are defined combinatorially, in the sense that they assign certain data to each crossing in a diagram of  $K$  and then combine these data to produce an invariant that does not depend on the choice of diagram. In contrast, the definition of the module  $K_q(S^3 \setminus K)$  depends on the global topology of the complement  $S^3 \setminus K$ , and this makes it difficult to prove general statements about  $K_q(S^3 \setminus K)$ . In particular, one may ask what facts are known about  $K_q(S^3 \setminus K)$  for all  $q$  and for all knots  $K$  and, to the best of our knowledge, there are two such statements:  $K_q(S^3 \setminus K)$  is a module over  $A_q^{\mathbb{Z}_2}$  and  $K_q(S^3 \setminus K)$  determines the colored Jones polynomials of  $K$  (see below). However, we believe that the calculations in § 4 give some evidence that these modules are not as intractable as they might seem. In examples, these modules are  $q$ -analogues of smooth holonomic  $D$ -modules, i.e. vector bundles with flat connections over  $\mathbb{C}^*$ . (Precisely, they are  $\mathbb{Z} \rtimes \mathbb{Z}_2$ -equivariant vector bundles over  $\mathbb{C}^*$ .)

The skein module  $K_q(S^3 \setminus K)$  is known to be closely related to other ‘quantum’ knot invariants. In particular, in [RT90], Reshetikhin and Turaev defined a polynomial invariant  $J_V(\mathfrak{g}, K; q) \in \mathbb{C}[q^{\pm 1}]$  for each finite-dimensional representation  $V$  of the quantum group  $\mathcal{U}_q(\mathfrak{g})$ . If  $\mathfrak{g} = \mathfrak{sl}_2$  and  $V$  is the defining representation of  $\mathcal{U}_q(\mathfrak{sl}_2)$ , then  $J_V(\mathfrak{sl}_2, K; q)$  is the famous Jones polynomial and, if  $V_n$  is the  $n$ -dimensional irreducible representation of  $\mathcal{U}_q(\mathfrak{sl}_2)$ , then the polynomials  $J_n(K; q) := J_{V_n}(\mathfrak{sl}_2, K; q)$  are called the colored Jones polynomials. The second

part of Question 1 is answered by a theorem of Kirby and Melvin. The embedding  $S^3 \setminus K \hookrightarrow S^3$  induces a  $\mathbb{C}[q^{\pm 1}]$ -linear map  $\epsilon : K_q(S^3 \setminus K) \rightarrow K_q(S^3) = \mathbb{C}[q^{\pm 1}]$ , and it was shown in [KM91] that

$$J_n(K; q) = \epsilon(S_{n-1}(L) \cdot \emptyset), \tag{1.4}$$

where  $S_{n-1}(L)$  is the  $(n - 1)^{\text{st}}$  Chebyshev polynomial evaluated at the longitude  $L$  of  $K$  and applied to the empty link  $\emptyset$ . (The Chebyshev polynomials correspond to characters of irreducible representations of  $\mathfrak{sl}_2$ .)

Our main goal in this paper is to introduce the Hecke parameter  $t$  into this story. In fact, in the rank 1 case, more deformation parameters are available: there is a family of algebras  $\mathcal{H}_{q,\underline{t}}$  depending on a parameter  $q \in \mathbb{C}^*$  and four additional parameters  $\underline{t} \in (\mathbb{C}^*)^4$ . This family is a nontrivial deformation of  $A_q \rtimes \mathbb{Z}_2$  (see, e.g., [Sah99]), which is actually the universal (i.e. ‘maximum possible’) deformation (see [Ob104]). The algebra  $\mathcal{H}_{q,\underline{t}}$  is the double affine Hecke algebra associated to the (nonreduced) root system of type  $C^\vee C_1$ , and it was introduced by Sahi [Sah99] (see also [NS04] and [Sto03]) to study the Askey–Wilson polynomials, which are generalizations of the famous Macdonald polynomials. As an abstract algebra,  $\mathcal{H}_{q,\underline{t}}$  is generated by elements  $T_0, T_1, T_0^\vee, T_1^\vee$  subject to the relations

$$\begin{aligned} (T_0 - t_1)(T_0 + t_1^{-1}) &= 0, \\ (T_0^\vee - t_2)(T_0^\vee + t_2^{-1}) &= 0, \\ (T_1 - t_3)(T_1 + t_3^{-1}) &= 0, \\ (T_1^\vee - t_4)(T_1^\vee + t_4^{-1}) &= 0, \\ T_1^\vee T_1 T_0 T_0^\vee &= q. \end{aligned}$$

For  $\mathfrak{g} = \mathfrak{sl}_2$ , Cherednik’s DAHA  $H_{q,t}$  is isomorphic to  $\mathcal{H}_{q,1,1,t^{-1},1}$  (see Remark 2.19).

If  $q \in \mathbb{C}^*$  is not a root of unity, then  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2}$  are Morita equivalent algebras; in other words, the categories of modules over  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2}$  are equivalent via the projection functor  $N \mapsto \mathbf{e}N$ . This implies that there is a unique  $A_q \rtimes \mathbb{Z}_2$ -module  $\hat{K}_q(S^3 \setminus K)$  such that  $\mathbf{e}\hat{K}_q(S^3 \setminus K)$  and  $K_q(S^3 \setminus K)$  are isomorphic  $A_q^{\mathbb{Z}_2}$ -modules. Explicitly,

$$\hat{K}_q(S^3 \setminus K) := A_q \otimes_{A_q^{\mathbb{Z}_2}} K_q(S^3 \setminus K).$$

We call  $\hat{K}_q(S^3 \setminus K)$  the nonsymmetric skein module: by definition it is a module over  $A_q \rtimes \mathbb{Z}_2 = \mathcal{H}_{q,(1,1,1,1)}$ . We can now reformulate Question 1 as follows.

*Question 2.* Is there a canonical deformation of the module  $\hat{K}_q(S^3 \setminus K)$  to a family of modules over  $\mathcal{H}_{q,\underline{t}}$ ?

When  $\underline{t} = (t_1, t_2, 1, 1)$ , we (conjecturally) give a positive answer to this question using an approach inspired by a construction of shift functors for rational double affine Hecke algebras developed in [BC11] (see also [BS12]). Let  $D_q$  be the localization of the algebra  $A_q \rtimes \mathbb{Z}_2$  obtained by inverting all nonzero polynomials in  $X$ . For every  $\underline{t} \in (\mathbb{C}^*)^4$ , there is a natural embedding of  $\mathcal{H}_{q,\underline{t}}$  into  $D_q$ :

$$\Theta : \mathcal{H}_{q,\underline{t}} \hookrightarrow D_q, \tag{1.5}$$

whose image is the subalgebra generated by  $X, X^{-1}$ , and the following operators (see [Sah99, NS04]):

$$T_0 = t_1 s Y - \frac{q \bar{t}_1 X + \bar{t}_2}{q^{-1} X^{-1} - q X} (1 - s Y), \quad T_1 = t_3 s + \frac{\bar{t}_3 X^{-1} + \bar{t}_4}{X^{-1} - X} (1 - s),$$

where  $\bar{t}_i = t_i - t_i^{-1}$ . The operator  $T_1$  is usually called the *Demazure–Lusztig operator* (see [Lus89]), while  $T_1 T_0$  is called the *Cherednik–Dunkl operator* of type  $C^\vee C_1$ ; cf. [NS04]. If  $\hat{K}_q^{\text{loc}}(S^3 \setminus K)$  is the localization of  $\hat{K}_q(S^3 \setminus K)$  at nonzero polynomials in  $X$ , then  $\Theta$  gives  $\hat{K}_q^{\text{loc}}(S^3 \setminus K)$  the structure of a module over  $\mathcal{H}_{q,t}$ . We can now state our main conjecture.

CONJECTURE 1. For all knots  $K \subset S^3$ , the natural action of  $\mathcal{H}_{q,t_1,t_2,1,1}$  on  $\hat{K}_q^{\text{loc}}(S^3 \setminus K)$  preserves the subspace<sup>1</sup>  $\hat{K}_q(S^3 \setminus K)$ .

We remark that this conjecture implies that  $\hat{K}_q(S^3 \setminus K)$  is naturally a module over the three-parameter algebra  $\mathcal{H}_{q,t_1,t_2}$ , even though only one of these parameters appears in the definition of the skein module. It is natural to ask whether Conjecture 1 can be extended to the full double affine Hecke algebra  $\mathcal{H}_{q,t_1,t_2,t_3,t_4}$  depending on all five parameters. The simplest example shows that this is not possible: if  $t_3 \neq 1$  or  $t_4 \neq 1$ , the operator  $T_1$  does not preserve the skein module of the unknot. However, we believe that this is the *only* obstruction to a canonical extension of the action of  $\mathcal{H}_{q,t_1,t_2}$  on  $\hat{K}_q(S^3 \setminus K)$  to all five parameters.<sup>2</sup> More precisely, we have the following two conjectures.

CONJECTURE 2. For all knots, there is an embedding of left  $A_q^{\mathbb{Z}_2}$ -modules

$$K_q(S^3 \setminus \text{unknot}) \hookrightarrow K_q(S^3 \setminus K).$$

CONJECTURE 3. Assume Conjecture 2 and let  $\bar{K}_q(S^3 \setminus K)$  be the quotient of  $\hat{K}_q(S^3 \setminus K)$  by the image of the skein module of the unknot. Then the action of  $\mathcal{H}_{q,t_1,t_2,t_3,t_4}$  on  $\bar{K}_q^{\text{loc}}(S^3 \setminus K)$  preserves the subspace  $\bar{K}_q(S^3 \setminus K) \subset \bar{K}_q^{\text{loc}}(S^3 \setminus K)$ .

We provide some evidence for these conjectures with the following theorem (see Theorems 6.1, 4.1, and Corollary 3.11).

THEOREM 1. *Conjectures 1–3 hold in the following cases:*

- (i) when  $K \subset S^3$  is the unknot, a  $(2, 2p + 1)$  torus knot, or the figure eight knot;
- (ii) when  $q = -1$  and  $K$  is any 2-bridge knot.

Strictly speaking, for 2-bridge knots we prove the symmetric versions of Conjectures 1–3 (see § 3.3). The proof involves calculations which may appear somewhat miraculous and only applicable to 2-bridge knots. However, in a forthcoming paper, using a different technique, we extend these calculations to other families of knots (including all torus knots and certain pretzel knots). More importantly, we show that if Conjecture 1 is true for two knots, then it is true for their connect sum; thus, for  $q = -1$ , it suffices to prove Conjecture 1 for prime knots.

In § 3.4, we also show that the restriction  $q = -1$  is almost unnecessary: more precisely, we conjecturally identify the  $q = -1$  limit of the nonsymmetric skein module for 2-bridge knots (see Conjecture 3.16), and we show that if this identification is correct, Conjecture 1 holds for all 2-bridge knots with no restriction on  $q$  (see Corollary 3.18).

We now provide some remarks about these conjectures. First, Conjecture 1 is equivalent to the statement that the operator  $(1 - q^2 X^2)^{-1}(1 - s\hat{y})$  preserves the nonsymmetric skein module.

<sup>1</sup> Technically, part of this conjecture is that the localization map  $\hat{K}_q(S^3 \setminus K) \rightarrow \hat{K}_q^{\text{loc}}(S^3 \setminus K)$  is injective.

<sup>2</sup> In examples, five-parameter deformations of  $\hat{K}_q(S^3 \setminus K)$  can be produced ‘by hand’ (see § 7), but these are not canonical in general, unlike the deformations arising from Conjecture 1.

In Lemma 2.25, we show that  $H_{q,t}$  can be embedded in  $D_q$  using this operator. In particular, Conjecture 1 implies that  $H_{q,t}$  acts on the nonsymmetric skein module of a knot complement. We therefore expect that Question 1 has a positive answer for  $\mathfrak{sl}_n$ , and for reductive  $\mathfrak{g}$  when  $q = 1$ , at least up to a similar twist by an automorphism.

Second, Conjecture 1 can also be stated directly in terms of skein modules without using Morita theory; see Remark 6.3. When specialized to  $q = -1$ , this interpretation implies that the rational function

$$F(\rho) = \frac{\text{Tr}(\rho(ml)) - \text{Tr}(\rho(ml^{-1}))}{\text{Tr}(\rho(m))^2 - 4}$$

on  $\text{Char}(T^2)$  pulls back to a regular function on (each irreducible component of)  $\text{Char}(S^3 \setminus K)$ . (We will discuss geometric implications of Conjecture 1 in future work.)

We next provide two applications of Conjecture 1. First, the existence of a natural  $\mathcal{H}_{q,t_1,t_2}$ -module structure on the nonsymmetric skein module  $\hat{K}_q(S^3 \setminus K)$  allows us to define three-variable polynomial knot invariants  $J_n(K; q, t_1, t_2) \in \mathbb{C}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$  that specialize to the colored Jones polynomials when  $t_1 = t_2 = 1$ . To this end, we modify the Kirby–Melvin formula (1.4):

$$J_n(K; q, t_1, t_2) := \epsilon(S_{n-1}(L_{t_1,t_2}) \cdot \emptyset), \tag{1.6}$$

where we replaced the longitude  $L$  (viewed as an operator on  $K_q(S^3 \setminus K)$ ) by its natural  $(t_1, t_2)$ -deformation  $L_{t_1,t_2} := T_1 T_0 + T_0^{-1} T_1^{-1}$ , which is called the *Askey–Wilson operator* (cf. [AW85] and [NS04, Proposition 5.8]). By definition,  $L_{1,1} = L$ , and this combined with (1.4) shows that  $J_n(K; q, t_1, t_2)$  specializes to the classical colored Jones polynomial. The Askey–Wilson operator has a denominator involving the meridian: the key point of Conjecture 1 is that these denominators cancel with the structure constants of the skein module of the knot complement. In Proposition 6.9, we show that if  $\bar{K}$  is the mirror of the knot  $K$ , then  $J_n(\bar{K}; q, t_1, t_2) = J_n(K; q^{-1}, t_1^{-1}, t_2^{-1})$ , which generalizes the well-known symmetry for the classical colored Jones polynomials. This provides some evidence that definition (1.6) is natural.

We remark that a strong form of the so-called ‘AJ conjecture’ (see, e.g., [Lê06, Conjecture 3]) states that the submodule of  $K_q(S^3 \setminus K)$  generated by the empty link is determined by the colored Jones polynomials  $J_n(K; q)$ . However, for general knots it is not clear whether the action of  $\mathcal{H}_{q,t_1,t_2}$  preserves the  $A_q \rtimes \mathbb{Z}_2$ -submodule generated by the empty link. Hence, the polynomials  $J_n(K; q, t_1, t_2)$  may contain more information about the skein module  $K_q(S^3 \setminus K)$  over  $\mathbb{C}[q^{\pm 1}]$  than the classical Jones polynomials. This motivates the following question.<sup>3</sup>

*Question 3.* Is there an algorithm for computing  $J_n(K; q, t_1, t_2)$  for a fixed  $n$  that does not require computing the skein module  $K_q(S^3 \setminus K)$ ? Is there an interpretation of  $J_n(K; q, t_1, t_2)$  in terms of representation theory of the quantum group  $\mathcal{U}_q(\mathfrak{sl}_2)$ ?

One may also ask whether there is a purely topological construction of our deformations of skein modules, or of the corresponding polynomial knot invariants. One approach to this question (and its relation to Cherednik’s two-variable polynomials for torus knots [Che13]) is discussed in [Sam14].

As a second application, we deduce from Conjecture 1 some algebraic properties of the classical (colored) Jones polynomials, which, to the best of our knowledge, have not appeared in the earlier literature. Namely, we prove the following theorem (see Theorem 5.12).

<sup>3</sup> In future work we will clarify the relation between the  $J_n(K; q, t_1, t_2)$  and the classical colored Jones polynomials.



**THEOREM 2.** *If Conjecture 1 holds for  $K$ , then the following rational function is a Laurent polynomial:*

$$P_j(K; n; q) := \frac{(q^2 - 1)[J(n + j) + J(n - 1 - j)]}{q^{4n-2} - 1}.$$

In this theorem, we have used the notation  $J(n) := J_n(K; q)$  and the convention  $J(-n) = -J(n)$  (for our choice of normalization of  $J(n)$ , see Remark 2.15). We also remark that the proof of Theorem 2 suggests that the sequence  $J(n)$  of colored Jones polynomials satisfies a recursion relation governed by the algebra  $\mathcal{H}_{q,\underline{t}}$  (see Question 5.11). We hope to address this relation in later work.

As further evidence for Conjecture 1, we follow a suggestion of Garoufalidis and use Habiro’s cyclotomic expansion of the colored Jones polynomials to prove the following theorem (see Theorem 5.14).

**THEOREM 3.** *The rational function  $P_j(K; n; q) \in \mathbb{C}(q)$  is a Laurent polynomial for all knots  $K \subset S^3$ .*

The result of Theorem 3 seems to be new; however, one of its implications (namely, the numerator of  $P_j(n; q)$  is zero when  $q = -e^{i\pi/(2n+1)}$ ) also follows from [CM15b, Proposition 2.1]. We also note that this theorem can be viewed as a congruence relation, and it is remarkably similar to several congruence relations for knot polynomials conjectured in [CLPZ14]. In fact, the proof of Theorem 3 extends almost verbatim to a proof of [CLPZ14, Conjecture 1.6]. We provide the details in § 5.5 (see Theorem 5.16).

We confirm Conjecture 2 for the figure eight knot and for all  $(2, 2p + 1)$  torus knots in Theorem 4.1. This statement has a conceptual explanation: it can be viewed as a quantization of the fact that  $L - 1$  always divides the  $A$ -polynomial of the knot  $K$ . (See Remarks 2.4 and 4.2 for further explanation.) We also point out that even in the simplest examples, this embedding is not obvious: in particular, the empty link in the skein module of the unknot is sent to a nontrivial element in the skein module of  $S^3 \setminus K$ . We expect that there is a topological interpretation of this embedding, but we will not address this here. As an application of Conjecture 2, we prove the following theorem (see Theorem 5.10).

**THEOREM 4.** *Suppose that  $f : K_q(S^3 \setminus \text{unknot}) \rightarrow K_q(S^3 \setminus K)$  is an  $A_q^{\mathbb{Z}_2}$ -module map such that  $\text{im}(f) \subset K_q(T^2) \cdot \emptyset$ . Then there exist two-variable Laurent polynomials  $c_k(-, -)$  and a sequence*

$$P(n) := \sum_k c_k(q, q^{2n})J(n + k)$$

*such that  $P(n) = P(0)$  for all  $n$ .*

We now summarize the contents of the paper. In § 2, we give an introduction to double affine Hecke algebras and Kauffman bracket skein modules. In § 3, we prove Conjectures 1–3 for 2-bridge knots (when  $q = -1$ ). In § 4, we use computations by Gelca and Sain to give complete descriptions of the skein modules of the  $(2, 2p + 1)$  torus knots and the figure eight knot, and we prove Conjecture 2 for these knots. In § 5, we prove Theorems 2 and 3, which involve divisibility properties for colored Jones polynomials. We also prove Theorem 4, which involves inhomogeneous recursion relations for colored Jones polynomials. In § 6, we prove Conjectures 1 and 3 for  $(2, 2p + 1)$  torus knots and the figure eight knot. In § 7, we construct noncanonical deformations of  $\hat{K}_q(S^3 \setminus \text{trefoil})$  to a module over  $\mathcal{H}_{q,\underline{t}}$  for arbitrary  $\underline{t} \in (\mathbb{C}^*)^4$ . In an appendix, we include example computations of three-variable polynomials specializing to colored Jones polynomials of the trefoil, the  $(5, 2)$  torus knot, and the figure eight knot.



## 2. Preliminaries

In this section, we provide the background necessary for the rest of the paper by discussing definitions and basic properties of the Kauffman bracket skein module and double affine Hecke algebras.

### 2.1 Knot groups and their character varieties

Recall that two maps  $f, g : M \rightarrow N$  of smooth manifolds are *ambiently isotopic* if they are in the same orbit of the identity component of the diffeomorphism group of  $N$ . This is an equivalence relation, and a *knot* in a 3-manifold  $M$  is the equivalence class of a smooth embedding  $K : S^1 \hookrightarrow M$ . If  $N_K \subset M$  is an (open) tubular neighborhood of  $K$ , then the complement  $M \setminus N_K$  has a torus boundary.

For an oriented knot  $K \subset S^3$ , there is a canonical identification  $T = S^1 \times S^1 \xrightarrow{\sim} \partial(S^3 \setminus K)$ . More precisely, let  $N_K \subset S^3$  be a closed tubular neighborhood of  $K$  and let  $N_c$  be the closure of its complement. Then the following lemma provides a unique (up to isotopy) identification of  $N_K \cap N_c$  with  $S^1 \times S^1$  (see [BZ03, Theorem 3.1]).

LEMMA 2.1. *There is a unique (up to isotopy) pair of simple loops (the meridian  $m$  and longitude  $l$ ) in  $T$  subject to the following conditions:*

- (i)  $m$  is nullhomotopic in  $N_K$ ;
- (ii)  $l$  is nullhomotopic in  $N_c$ ;
- (iii)  $m, l$  intersect once in  $T$ ;
- (iv) in  $S^3$ , the linking numbers  $(m, K)$  and  $(l, K)$  are 1 and 0, respectively.

Therefore, to an oriented knot  $K \subset S^3$ , one can associate the data  $(\pi_1(S^3 \setminus K), m, l)$ , where  $m, l \in \pi_1(S^3 \setminus K)$  are the elements corresponding to the meridian and longitude defined above. Since the meridian and longitude are well defined up to (base-point-free) isotopy, the elements  $m, l$  are well defined up to inner automorphism. The following theorem (see [BZ03, Theorem 3.15]) shows that this data is a complete invariant of the knot.

THEOREM 2.2 (Waldhausen). *Two knots  $K, K' \subset S^3$  are ambiently isotopic if and only if there is an isomorphism  $\phi : \pi_1(S^3 \setminus K) \rightarrow \pi_1(S^3 \setminus K')$  that satisfies  $\phi(m) = m'$  and  $\phi(l) = l'$ .*

2.1.1 *Character varieties and the A-polynomial of a knot.* If  $\pi$  is a finitely generated (discrete) group and  $G$  is an algebraic group, the set  $\text{Rep}(\pi, G) := \text{Hom}(\pi, G)$  has a natural affine scheme structure. Informally, one way to define this structure is to pick generators  $g_1, \dots, g_n$  for  $\pi$ , so that a representation  $\rho : \pi \rightarrow G$  is completely determined by the images of the  $g_i$ . This realizes  $\text{Rep}(\pi, G)$  as a subscheme of  $G^n$ , where the ideal defining this subscheme is given by the relations between the  $g_i$ . It is well known that  $\text{Rep}(-, G)$  is functorial with respect to group homomorphisms  $f : \pi \rightarrow \pi'$ . In particular, the scheme structure on  $\text{Rep}(\pi, G)$  is independent of the choices made (see, e.g., [LM85]).

There is a natural action of  $G$  on  $\text{Rep}(\pi, G)$  (by conjugation), and this induces an action of  $G$  on the corresponding commutative algebra  $\mathcal{O}(\text{Rep}(\pi, G))$ . We denote the subalgebra of invariant functions by

$$\mathcal{O}\text{Char}(\pi, G) := \mathcal{O}(\text{Rep}(\pi, G))^G.$$

The *character variety* is the spectrum of this algebra:

$$\text{Char}(\pi, G) := \text{Spec}(\mathcal{O}\text{Char}(\pi, G)).$$

This scheme parameterizes closed  $G$  orbits on  $\text{Rep}(\pi, G)$ .

We now assume that  $\pi = \pi_1(M)$  for a manifold  $M$  and specialize to  $G = \text{SL}_2(\mathbb{C})$ . To shorten notation, we will write  $\mathcal{O}\text{Char}(M) := \mathcal{O}\text{Char}(\pi_1(M), \text{SL}_2(\mathbb{C}))$ , etc. If  $K \subset S^3$  is a knot and  $M := S^3 \setminus K$  is the knot complement, then the inclusion  $T^2 = \partial M \subset M$  induces a map of algebras  $\iota : \mathcal{O}\text{Char}(T^2) \rightarrow \mathcal{O}\text{Char}(M)$ .

We may identify the algebra  $\mathcal{O}\text{Char}(T^2)$  with  $\mathcal{O}(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_2}$ , where the generator of  $\mathbb{Z}_2$  acts via  $(a, b) \mapsto (a^{-1}, b^{-1})$ . The algebra  $\mathcal{O}(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_2}$  is generated by the functions  $x(a, b) = a + a^{-1}$ ,  $y(a, b) = b + b^{-1}$ , and  $z(a, b) = ab + (ab)^{-1}$ . Under the identification with  $\mathcal{O}\text{Char}(T^2)$ , these functions correspond to  $(A, B) \mapsto \text{tr}(A)$ ,  $(A, B) \mapsto \text{tr}(B)$ , and  $(A, B) \mapsto \text{tr}(AB)$ , respectively (cf. Theorem 2.10).

We recall the definition of the  $A$ -polynomial, which was originally introduced in [CCG<sup>+</sup>94]. If  $M$  is a knot complement, we have the diagram

$$\mathbb{C}^* \times \mathbb{C}^* \rightarrow (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2 \cong \text{Char}(T^2) \leftarrow \text{Char}(M).$$

We let  $X_M \subset \mathbb{C}^* \times \mathbb{C}^*$  be the union of the one-dimensional components of the preimage of the Zariski closure of the image of  $\text{Char}(M)$ . A theorem of Thurston says that  $X_M$  is nonempty, which allows the following definition.

DEFINITION 2.3. The  $A$ -polynomial is the polynomial  $A(m, l) \in \mathbb{C}[m^{\pm 1}, l^{\pm 1}]$  that defines the curve  $X_M$ .

(Here we have used Lemma 2.1 to pick generators  $m, l$  for  $\mathcal{O}(\mathbb{C}^* \times \mathbb{C}^*)$ .)

Remark 2.4. The abelianization of the fundamental group of a knot complement is isomorphic to  $\mathbb{Z}$ , and it is well known that the set of representations factoring through the abelianization map is a component of the character variety. This implies that  $l - 1$  always divides the  $A$ -polynomial. If  $K$  is the unknot, then  $\pi_1(S^3 \setminus K) = \mathbb{Z}$ , which implies that the  $A$ -polynomial of the unknot divides the  $A$ -polynomial of an arbitrary knot. If we write  $A_K$  and  $A_U$  for the  $A$ -polynomials of a knot  $K$  and the unknot  $U$ , then  $A_K = B_K A_U$ , and we have a map of  $\mathbb{C}[m^{\pm 1}, l^{\pm 1}]$ -modules (which is *not* a map of algebras):

$$\phi : \mathbb{C}[m^{\pm 1}, l^{\pm 1}]/A_U \rightarrow \mathbb{C}[m^{\pm 1}, l^{\pm 1}]/A_K, \quad f \mapsto B_K f. \tag{2.1}$$

### 2.2 Kauffman bracket skein modules

A *framed link* in an oriented 3-manifold  $M$  is an embedding of a disjoint union of annuli  $S^1 \times [0, 1]$  into  $M$ . (The framing refers to the  $[0, 1]$  factor and is a technical detail that will be suppressed when possible.) We will consider framed links to be equivalent if they are ambiently isotopic. In what follows, the letter  $q$  will denote either an element of  $\mathbb{C}^*$  or the generator of the ring  $\mathbb{C}[q, q^{-1}]$  (we will specify which when it matters and when it is not clear from context).

Let  $\mathcal{L}(M)$  be the vector space spanned by the set of ambient isotopy classes of framed unoriented links in  $M$  (including the empty link). Let  $\mathcal{L}'(M)$  be the smallest subspace of  $\mathcal{L}(M)$  containing the skein expressions  $L_+ - qL_0 - q^{-1}L_\infty$  and  $L \sqcup \bigcirc + (q^2 + q^{-2})L$ . The links  $L_+$ ,  $L_0$ , and  $L_\infty$  are identical outside of a small 3-ball (embedded as an oriented manifold), and inside the 3-ball they appear as in Figure 1. (All pictures drawn in this paper will have blackboard framing. In other words, a line on the page represents a strip  $[0, 1] \times [0, 1]$  in a tubular neighborhood of the page, and the strip is always perpendicular to the paper (the intersection with the paper is  $[0, 1] \times \{0\}$ .)

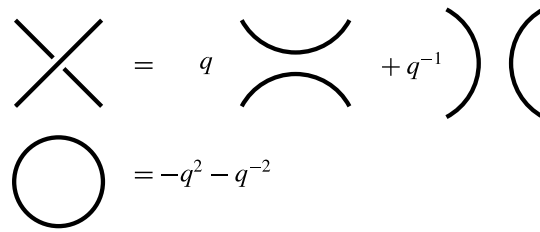


FIGURE 1. Kauffman bracket skein relations.

DEFINITION 2.5 [Prz91]. The *Kauffman bracket skein module* is the vector space  $K_q(M) := \mathcal{L}/\mathcal{L}'$ . It contains a canonical element  $\emptyset \in K_q(M)$  corresponding to the empty link.

Remark 2.6. To shorten the notation, if  $M = F \times [0, 1]$  for a surface  $F$ , we will often write  $K_q(F)$  for the skein module  $K_q(F \times [0, 1])$ .

Example 2.7. One original motivation for defining  $K_q(M)$  is the isomorphism

$$\mathbb{C}[q, q^{-1}] \xrightarrow{\sim} K_q(S^3), \quad 1 \mapsto \emptyset.$$

Kauffman proved that this map is an isomorphism and that the inverse image of a link is the Jones polynomial of the link. The map is surjective because the skein relations allow one to remove all crossings and loops in a diagram of any link, but showing that it is injective is (essentially) equivalent to showing that the Jones polynomial of a link is well defined, which is a nontrivial theorem.

In general  $K_q(M)$  is just a vector space; however, if  $M$  has extra structure, then  $K_q(M)$  also has extra structure. In particular:

- (i) if  $M = F \times [0, 1]$  for some surface  $F$ , then  $K_q(M)$  is an algebra, where the multiplication is given by ‘stacking links’;
- (ii) if  $M$  is a manifold with boundary, then  $K_q(M)$  is a module over  $K_q(\partial M)$ . The multiplication is given by ‘pushing links from the boundary into the manifold’;
- (iii) an oriented embedding  $M \hookrightarrow N$  of 3-manifolds induces a linear map  $K_q(M) \rightarrow K_q(N)$ . Therefore,  $K_q(-)$  can be considered as a functor on the category whose objects are oriented three-dimensional manifolds and whose morphisms are oriented embeddings;<sup>4</sup>
- (iv) if  $q = \pm 1$ , then  $K_q(M)$  is a commutative algebra (for any oriented 3-manifold  $M$ ). The multiplication is given by ‘disjoint union of links’, which makes sense because when  $q = \pm 1$ , the skein relations allow strands to ‘pass through’ each other.

Remark 2.8. The third property is stated in [Prz91, Proposition 4] and can be proved using the isotopy extension theorem. The first two properties are consequences of the third: for example, there is an obvious map  $F \times [0, 1] \sqcup F \times [0, 1] \rightarrow F \times [0, 1]$ , and the product structure of  $K_q(F \times [0, 1])$  comes from the application of the functor  $K_q(-)$  to this map.

<sup>4</sup> To be pedantic,  $K_q(-)$  is functorial with respect to maps  $M \rightarrow N$  that are oriented embeddings when restricted to the interior of  $M$ . In particular, if we identify a surface  $F$  with a boundary component of  $M$  and  $N$ , then the gluing map  $M \sqcup N \rightarrow M \sqcup_F N$  induces a linear map  $K_q(M) \otimes_{\mathbb{C}} K_q(N) \rightarrow K_q(M \sqcup_F N)$ .

*Example 2.9.* Let  $M = (S^1 \times [0, 1]) \times [0, 1]$  be the solid torus. If  $u$  is the nontrivial loop, then the map  $\mathbb{C}[u] \rightarrow K_q((S^1 \times [0, 1]) \times [0, 1])$  sending  $u^n$  to  $n$  parallel copies of  $u$  is surjective (because all crossings and trivial loops can be removed using the skein relations). This is clearly an algebra map and it also injective (see, e.g., [SW07]).

**2.2.1 Skein modules and representation varieties.** Here we recall a theorem of Przytycki and Sikora [PS00] (see also Bullock [Bul97]) that identifies the commutative algebra  $K_{q=-1}(M)$  with the algebra  $\mathcal{O}\text{Char}(M)$  of functions on the  $\text{SL}_2(\mathbb{C})$ -character variety of  $\pi_1(M)$ .

An unbased loop  $\gamma : S^1 \rightarrow M$  determines a conjugacy class in  $\pi_1(M)$  and, since the trace of a matrix is invariant on conjugacy classes, we can define  $\text{Tr}(\gamma) \in \mathcal{O}\text{Char}(M)$  via

$$\text{Tr}(\gamma)(\rho) := \text{Tr}[\rho(\gamma)].$$

**THEOREM 2.10** [PS00, Bul97]. *The assignment  $\gamma \mapsto -\text{Tr}(\gamma)$  extends to an algebra isomorphism*

$$K_{q=-1}(M) \xrightarrow{\sim} \mathcal{O}\text{Char}(M).$$

The key observation behind this theorem is that for  $q = -1$ , the skein relation becomes

$$\text{Tr}(A)\text{Tr}(B) = \text{Tr}(AB) + \text{Tr}(AB^{-1}).$$

(This formula is a simple consequence of the Hamilton–Cayley identity in  $\text{SL}_2(\mathbb{C})$ .)

**2.2.2 The Kauffman bracket skein module of the torus.** We recall that the *quantum torus* is the algebra

$$A_q := \frac{\mathbb{C}\langle X^{\pm 1}, Y^{\pm 1} \rangle}{XY - q^2 YX},$$

where  $q \in \mathbb{C}^*$  is a parameter. Note that  $\mathbb{Z}_2$  acts by algebra automorphisms on  $A_q$  by inverting  $X$  and  $Y$ .

We now recall a beautiful theorem of Frohman and Gelca in [FG00] that gives a connection between skein modules and the invariant subalgebra  $A_q^{\mathbb{Z}_2}$ . First, we introduce some notation. Let  $T_n \in \mathbb{C}[x]$  be the Chebyshev polynomials defined by  $T_0 = 2$ ,  $T_1 = x$ , and the relation  $T_{n+1} = xT_n - T_{n-1}$ . If  $m, l$  are relatively prime, write  $(m, l)$  for the  $m, l$  curve on the torus (the simple curve wrapping around the torus  $l$  times in the longitudinal direction and  $m$  times in the meridian direction). It is clear that the links  $(m, l)^n$  span  $K_q(T^2)$  and it follows from [SW07] that this set is actually a basis. However, a more convenient basis is given by the elements  $(m, l)_T := T_d((m/d, l/d))$  (where  $d = \text{gcd}(m, l)$ ). Define  $e_{r,s} = q^{-rs} X^r Y^s \in A_q$ , which form a linear basis for the quantum torus  $A_q$  and satisfy the relations

$$e_{r,s} e_{u,v} = q^{rv-us} e_{r+u, s+v}.$$

**THEOREM 2.11** [FG00]. *The map  $K_q(T^2) \rightarrow A_q^{\mathbb{Z}_2}$  given by  $(m, l)_T \mapsto e_{m,l} + e_{-m,-l}$  is an isomorphism of algebras.*

*Remark 2.12.* As explained in §2.1, if  $K$  is an oriented knot, then there is a canonical identification of  $S^1 \times S^1$  with the boundary of  $S^3 \setminus K$ . If the orientation of  $K$  is reversed, this identification is twisted by the ‘hyper-elliptic involution’ of  $S^1 \times S^1$  (which negates both components). However, this induces the identity isomorphism on  $K_q(T^2 \times [0, 1])$ , so the  $A_q^{\mathbb{Z}_2}$ -module structure on  $K_q(S^3 \setminus K)$  is canonical and does not depend on the choice of orientation of  $K$ .

We also recall another presentation of this algebra that will be useful for computations. Let  $x, y, z \in K_q(T^2)$  be the meridian, longitude, and  $(1, 1)$  curve, respectively.

**THEOREM 2.13** [BP00]. *The algebra  $K_q(T^2)$  is generated by  $x, y, z$  with relations*

$$[x, y]_q = (q^2 - q^{-2})z, \quad [z, x]_q = (q^2 - q^{-2})y, \quad [y, z]_q = (q^2 - q^{-2})x, \tag{2.2}$$

and the additional cubic relation

$$q^2x^2 + q^{-2}y^2 + q^2z^2 - qxyz = 2(q^2 + q^{-2}). \tag{2.3}$$

Combining this presentation with the isomorphism  $K_q(T^2) \cong A_q^{\mathbb{Z}_2}$ , we have

$$x \mapsto X + X^{-1}, \quad y \mapsto Y + Y^{-1}, \quad z \mapsto q^{-1}(XY + X^{-1}Y^{-1}).$$

**2.2.3 A topological pairing.** Let  $K \subset S^3$  be a knot. There is a natural pairing

$$K_q(D^2 \times S^1) \otimes_{\mathbb{C}} K_q(S^3 \setminus K) \rightarrow \mathbb{C},$$

which is used to compute colored Jones polynomials. Informally, this pairing is induced by gluing a solid torus  $D^2 \times S^1$  to the complement of a tubular neighborhood of a knot to obtain  $S^3$ .

Let  $N_K \subset S^3$  be a closed tubular neighborhood of  $K$  and let  $N_c$  be the closure of its complement. Then  $N_K \cap N_c$  is a torus  $T$ , and we let  $N_T$  be a closed tubular neighborhood of  $T$ . By Remark 2.12, both  $K_q(N_K)$  and  $K_q(N_c)$  have canonical  $A_q^{\mathbb{Z}_2}$ -module structure. More precisely, if we identify  $N_T$  with  $T \times [0, 1]$ , then the embedding  $T \times [0, 1] \hookrightarrow S^3$  gives  $K_q(N_c)$  and  $K_q(N_K)$  a left and a right  $A_q^{\mathbb{Z}_2}$ -module structure,<sup>5</sup> respectively.

It is easy to see that  $K_q(N_K \sqcup N_c) \cong K_q(N_K) \otimes_{\mathbb{C}} K_q(N_c)$ , and the embedding property (iii) above shows that this induces a map  $\langle -, - \rangle : K_q(N_K) \otimes_{\mathbb{C}} K_q(N_c) \rightarrow K_q(S^3)$ . If  $\alpha \subset N_T$  is a link, it can be isotoped to a link inside  $N_K$  or a link inside  $N_c$ , and inside  $S^3$  both these links are isotopic. Since these isotopies define the module structure of  $K_q(N_K)$  and  $K_q(N_c)$ , the pairing  $\langle -, - \rangle$  descends to

$$\langle -, - \rangle : K_q(D^2 \times S^1) \otimes_{K_q(T^2)} K_q(S^3 \setminus K) \rightarrow \mathbb{C}. \tag{2.4}$$

(To ease notation for later reference, in this formula we have identified  $N_K$  with the solid torus  $D^2 \times S^1$  and written  $S^3 \setminus K$  for  $N_c$ . We also used the isomorphism  $K_q(S^3) \cong \mathbb{C}$  described in Example 2.7.)

**2.2.4 The colored Jones polynomials.** The colored Jones polynomials  $J_n(K; q) \in \mathbb{C}[q^{\pm 1}]$  of a knot  $K \subset S^3$  were originally defined by Reshetikhin and Turaev in [RT90] using the representation theory of  $\mathcal{U}_q(\mathfrak{sl}_2)$ . (In fact, their definition works for any semisimple Lie algebra  $\mathfrak{g}$ , but we only deal with  $\mathfrak{g} = \mathfrak{sl}_2$ .) Here we recall a theory of Kirby and Melvin that shows that  $J_n(K; q)$  can be computed in terms of the pairing from the previous section.

If  $N_K$  is a tubular neighborhood of the knot  $K$ , then we identify  $K_q(N_K) \cong \mathbb{C}[u]$ , where  $u \in K_q(N_K)$  is the image of the (0-framed) longitude  $l \in K_q(\partial N_K)$ . Let  $S_n \in \mathbb{C}[u]$  be the Chebyshev polynomials of the second kind, which satisfy the initial conditions  $S_0 = 1$  and  $S_1 = u$  and the recursion relation  $S_{n+1} = uS_n - S_{n-1}$ .

---

<sup>5</sup> The asymmetry between left and right comes from the definition of multiplication in  $F \times [0, 1]$ : the product  $ab$  means ‘stack  $a$  on top of  $b$ ’. Since the tori  $T^2 \times \{0\}$  and  $T^2 \times \{1\}$  are glued to  $N_c$  and  $N_K$ , the spaces  $K_q(N_c)$  and  $K_q(N_K)$  are left and right  $K_q(T^2)$ -modules, respectively.

THEOREM 2.14 [KM91]. If  $\emptyset \in K_q(S^3 \setminus K)$  is the empty link, we have

$$J_n(K; q) = (-1)^{n-1} \langle \emptyset \cdot S_{n-1}(u), \emptyset \rangle.$$

Remark 2.15. The sign correction is chosen so that  $J_n(\text{unknot}; q) = (q^{2n} - q^{-2n}) / (q^2 - q^{-2})$ . Also, with this normalization,  $J_0(K; q) = 0$  and  $J_1(K; q) = 1$  for every knot  $K$ . These conventions agree with the convention of labelling irreducible representations of  $\mathcal{U}_q(\mathfrak{sl}_2)$  by their dimension.

### 2.3 The $C^\vee C_1$ double affine Hecke algebra

In this section, we define the five-parameter family of algebras  $\mathcal{H}_{q,\underline{t}}$  which was introduced by Sahi in [Sah99] (see also [NS04]). This is the universal deformation of the algebra  $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}] \rtimes \mathbb{Z}_2$  (see [Ob104]), and it depends on the parameters  $q \in \mathbb{C}^*$  and  $\underline{t} \in (\mathbb{C}^*)^4$ . The algebra  $\mathcal{H}_{q,\underline{t}}$  can be abstractly presented as follows: it is generated by the elements  $T_0, T_1, T_0^\vee$ , and  $T_1^\vee$  subject to the relations

$$\begin{aligned} (T_0 - t_1)(T_0 + t_1^{-1}) &= 0, \\ (T_0^\vee - t_2)(T_0^\vee + t_2^{-1}) &= 0, \\ (T_1 - t_3)(T_1 + t_3^{-1}) &= 0, \\ (T_1^\vee - t_4)(T_1^\vee + t_4^{-1}) &= 0, \\ T_1^\vee T_1 T_0 T_0^\vee &= q. \end{aligned} \tag{2.5}$$

Remark 2.16. Comparing our notation to [NS04], our  $q^{-2}$  is their  $q$ , and our parameters  $(t_1, t_2, t_3, t_4)$  are their  $(k_0, u_0, k_1, u_1)$ . These parameters relate to the original parameters  $a, b, c, d$  of Askey and Wilson [AW85] via  $a = t_3 t_4$ ,  $b = -t_3 t_4^{-1}$ ,  $c = q^{-1} t_1 t_2$ , and  $d = -q^{-1} t_1 t_2^{-1}$ .

Remark 2.17. The algebra  $\mathcal{H}_{q,\underline{t}}$  is a flat deformation of the fundamental group algebra of an orbifold Riemann surface. Recall that if  $X$  is a simply connected Riemann surface and  $\Gamma$  is a cocompact lattice (i.e. a Fuchsian subgroup) in  $\text{Aut}(X)$ , the quotient  $\Sigma = X/\Gamma$  is defined as an orbifold and  $\Gamma$  is isomorphic to the (orbifold) fundamental group of  $\Sigma$  (see, e.g., [Sco83, § 2]):

$$\pi_1^{\text{orb}}(\Sigma, *) = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n \mid c_i^{n_i} = 1, \prod_{i=1}^g [a_i, b_i] c_1 \cdots c_n = 1 \right\rangle,$$

where the  $c_i$  are generators corresponding to loops around special points of  $X$  with stabilizers  $\mathbb{Z}/n_i\mathbb{Z}$  and  $n_i > 1$ . In the case when  $X = \mathbb{C}$  and  $\Gamma = (\mathbb{Z} \oplus i\mathbb{Z}) \rtimes \mathbb{Z}_2$  is acting on  $X$  by translation–reflections, we recover from the isomorphism  $G \cong \pi_1^{\text{orb}}(\Sigma, *)$  the presentation

$$\Gamma = \langle c_1, c_2, c_3, c_4 \mid c_i^2 = 1, c_1 c_2 c_3 c_4 = 1 \rangle,$$

where the  $c_i$  are loops around four special points  $\{0, 1/2, 1/2 + i/2, i/2\} \in \mathbb{C}$ . Thus,  $\mathcal{H}_{1,1,1,1,1} \cong \mathbb{C}[\Gamma]$ . For other interesting examples of Hecke algebras associated to Fuchsian groups, see [EOR07].



For  $t = \underline{1}$ , this algebra is isomorphic to  $A_q \rtimes \mathbb{Z}_2$  (see Remark 2.19). As mentioned in the introduction, this algebra can also be realized as a subalgebra of a certain localization of  $A_q \rtimes \mathbb{Z}_2$ . More precisely, we recall that  $A_q \rtimes \mathbb{Z}_2$  is generated by  $X$ ,  $\hat{y}$ , and  $s$  (all invertible), which satisfy the relations

$$sX = X^{-1}s, \quad s\hat{y} = \hat{y}^{-1}s, \quad s^2 = 1, \quad X\hat{y} = q^2\hat{y}X.$$

Let  $D_q$  be the localization of  $A_q \rtimes \mathbb{Z}_2$  obtained by inverting all nonzero polynomials in  $X$ , and define the following operators in  $D_q$ :

$$T_0 = t_1s\hat{y} - \frac{q^2\bar{t}_1X^2 + q\bar{t}_2X}{1 - q^2X^2}(1 - s\hat{y}),$$

$$T_1 = t_3s + \frac{\bar{t}_3 + \bar{t}_4X}{1 - X^2}(1 - s).$$

(We have slightly abused notation by giving these operators the same names as the abstract generators of  $\mathcal{H}_{q,t}$ .) The following Dunkl-type embedding is defined using these operators (see [NS04, Theorem 2.22]).

PROPOSITION 2.18 [Sah99]. *The assignments*

$$T_i \mapsto T_i, \quad T_0^\vee \mapsto qT_0^{-1}X, \quad T_1^\vee \mapsto X^{-1}T_1^{-1} \tag{2.6}$$

extend to an injective algebra homomorphism  $\mathcal{H}_{q,t} \rightarrow D_q$ .

We recall that the standard polynomial representation  $V = \mathbb{C}(X)$  of  $D_q$  is isomorphic as a  $\mathbb{C}[X^{\pm 1}]$ -module to rational functions in  $X$ , with the action of  $s$  and  $\hat{y}$  given by

$$s \cdot f(X) = f(X^{-1}), \quad \hat{y} \cdot f(X) = f(q^{-2}X).$$

Under this action, it is easy to check that both operators  $T_0$  and  $T_1$  preserve the subspace  $\mathbb{C}[X^{\pm 1}] \subset \mathbb{C}(X)$ , which shows that the action of  $\mathcal{H}_{q,t}$  on  $\mathbb{C}(X)$  preserves  $\mathbb{C}[X^{\pm 1}] \subset \mathbb{C}(X)$ . In other words,  $\mathbb{C}[X^{\pm 1}]$  is an  $\mathcal{H}_{q,t}$ -module, which is called the *polynomial representation*.

Remark 2.19. The algebra  $\mathcal{H}_{q,t}$  is also generated by the elements  $X^{\pm 1}$ ,  $Y := T_1T_0$ , and  $T := T_1$ . With this set of generators, the relations become the following (see [NS04, 2.21]):

$$\begin{aligned} XT &= T^{-1}X^{-1} - \bar{t}_4, \\ T^{-1}Y &= Y^{-1}T + \bar{t}_1, \\ T^2 &= 1 + \bar{t}_3T, \\ TXY &= q^2T^{-1}YX - q^2\bar{t}_1X - q\bar{t}_2 - \bar{t}_4Y, \end{aligned} \tag{2.7}$$

where  $\bar{t}_i = t_i - t_i^{-1}$ . This presentation shows that  $\mathcal{H}_{q,1,1,1,1} = A_q \rtimes \mathbb{Z}_2$  (as subalgebras of  $D_q$ ), since, under this specialization of the parameters, we have  $T = T_1 = s$  and  $Y = \hat{y}$ . Furthermore, Cherednik’s  $\mathfrak{sl}_2$  double affine Hecke algebra  $H_{q,t}$  is isomorphic to  $\mathcal{H}_{q,1,1,t^{-1},1}$ . Under this specialization, the presentation (2.7) becomes

$$TXT = X^{-1}, \quad TY^{-1}T = Y, \quad (T - t^{-1})(T + t) = 0, \quad XY = q^2T^{-2}YX. \tag{2.8}$$

The standard presentation of  $H_{q,t}$  (see [Che05]) replaces the last relation (2.8) with  $XY = q^2YXT^2$ . Under the map  $H_{q,t} \rightarrow \mathcal{H}_{q,1,1,t^{-1},1}$  given by  $X \mapsto X^{-1}$ ,  $Y \mapsto Y^{-1}$ , and  $T \mapsto T^{-1}$ , this becomes the last relation in (2.8).

The element  $\mathbf{e} := (T_1 + t_3^{-1})/(t_3 + t_3^{-1})$  is an idempotent in  $\mathcal{H}_{q,t}$ , and the algebra  $S\mathcal{H}_{q,t} := \mathbf{e}\mathcal{H}_{q,t}\mathbf{e}$  is called the *spherical subalgebra*. It is easy to check that  $\mathbf{e}$  commutes with  $X + X^{-1}$ , and this implies that the subspace  $\mathbf{e} \cdot \mathbb{C}[X^{\pm 1}]$  is equal to the subspace  $\mathbb{C}[X + X^{-1}]$  of symmetric polynomials. The spherical algebra therefore acts on  $\mathbb{C}[X + X^{-1}]$ , and this module is called the *symmetric polynomial representation*.

A presentation for the spherical subalgebra  $S\mathcal{H}_{q,t}$  has been given by Terwilliger in [Ter13]. (A less symmetric presentation was given in [Koo08].) We now recall this presentation in our notation. Define

$$\begin{aligned} x &= (X + X^{-1})\mathbf{e}, \\ y &= (Y + Y^{-1})\mathbf{e}, \\ z &= (T_1T_0^\vee + (T_1T_0^\vee)^{-1})\mathbf{e}, \\ \Omega &= -qyzx + q^2x^2 + q^2y^2 + q^{-2}z^2 - q\alpha x - q\beta y - q^{-1}\gamma z, \end{aligned}$$

where

$$\alpha := \bar{t}_1\bar{t}_2 + (\overline{qt_3})\bar{t}_4, \quad \beta := \bar{t}_2\bar{t}_4 + (\overline{qt_3})\bar{t}_1, \quad \gamma := \bar{t}_1\bar{t}_4 + (\overline{qt_3})\bar{t}_2.$$

Here and later we use the notation  $\overline{qt_3} := qt_3 - q^{-1}t_3^{-1}$ .

**THEOREM 2.20** [Ter13, Proposition 16.4]. *The spherical subalgebra is generated by  $x, y, z$  with relations*

$$\begin{aligned} [x, y]_q &= (q^2 - q^{-2})z - (q - q^{-1})\gamma, \\ [y, z]_q &= (q^2 - q^{-2})x - (q - q^{-1})\alpha, \\ [z, x]_q &= (q^2 - q^{-2})y - (q - q^{-1})\beta, \\ \Omega &= (\bar{t}_1)^2 + (\bar{t}_2)^2 + (\overline{qt_3})^2 + (\bar{t}_4)^2 - \bar{t}_1\bar{t}_2(\overline{qt_3})\bar{t}_4 + (q^2 + q^{-2})^2. \end{aligned}$$

*Remark 2.21.* To convert from Terwilliger’s notation to ours, we note that his generators  $(t_0, t_1, t_2, t_3)$  are our generators  $(-iT_1, -iT_0, -iT_0^\vee, -iT_1^\vee)$  (where  $i^2 = -1$ ). Furthermore, his  $(X, Y)$  are our  $(X^{-1}, Y)$ , and his  $(A, B, C)$  are our  $(-y, -x, -z)$ . His  $\Omega$  is our  $\Omega$ , and his  $(\alpha, \beta, \gamma)$  are our  $(-\beta, -\alpha, -\gamma)$ . Finally, his parameters  $(q, P_1, P_2, P_3)$  are our parameters  $(q^{-1}, i\bar{t}_1, i\bar{t}_2, i\bar{t}_4)$ .

*Proof.* Technically, [Ter13, Proposition 16.4] gives a presentation of the (unital) subalgebra of  $\mathcal{H}_{q,t}$  generated by  $x, y, z, T_1$ . However, all elements in this subalgebra commute with  $T_1$ , so when we multiply this subalgebra by the idempotent  $\mathbf{e}$  to obtain the (nonunital) spherical subalgebra, the generator  $T_1$  is absorbed by  $\mathbf{e}$  and becomes the constant  $t_3$ . Furthermore, all elements of the spherical subalgebra commute with  $T_1$ , so [Ter13, Theorem 13.6] shows that the spherical subalgebra is contained in the algebra generated by  $x, y, z, T_1$ . We therefore have the equality  $\mathbf{e}\mathcal{H}_{q,t}\mathbf{e} = \mathbf{e}\langle x, y, z, T_1 \rangle\mathbf{e}$  of (nonunital) subalgebras of  $\mathcal{H}_{q,t}$ . Finally, the Poincaré–Birkhoff–Witt theorem [Ter13, Proposition 12.13] for the algebra  $\langle x, y, z, T_1 \rangle$  shows that this algebra can be identified with the tensor product  $\langle x, y, z \rangle \otimes_{\mathbb{C}} \mathbb{C}[T_1]$ , which shows that the claimed presentation above is actually a presentation (i.e. that we have listed sufficiently many relations).  $\square$

*Remark 2.22.* Let  $t'_3 = qt_3$ . The above presentation makes it clear that if we specialize any three of the parameters  $\{t_1, t_2, t'_3, t_4\}$  to 1 and set the last equal to  $t$ , then the four spherical subalgebras obtained in this way are isomorphic. This is not *a priori* obvious. (We also remind the reader that our parameters  $q, t_3$  are the parameters  $q^{1/4}, t^{1/2}$  of the standard presentation of the DAHA of type  $A_1$  in [Che05].)

*Remark 2.23.* If  $q = 1$ , then the spherical subalgebras are commutative for any  $(t_1, t_2, t_3, t_4)$ . The corresponding varieties are affine cubic surfaces studied in detail in [Obl04] (and the presentation in [Obl04] agrees exactly with the one above, where our  $x, y, z$  are his  $X_1, X_2, X_3$ ). Also, if  $t_i = 1$ , a short computation with the commutation relations shows that the presentation of Theorem 2.20 agrees with the presentation of [BP00] for the skein algebra of the torus (see Theorem 2.13).

The operator  $L_{t_1, t_2} := Y + Y^{-1}$  is called the *Askey–Wilson operator*: it also commutes with  $\mathbf{e}$ , so it preserves the subspace of symmetric functions  $\mathbb{C}[X + X^{-1}]$ . If we write  $L_{t_1, t_2}^{\text{sym}}$  for the restriction of this operator to  $\mathbb{C}[X + X^{-1}]$ , then  $L_{t_1, t_2}^{\text{sym}}$  is diagonalizable with distinct eigenvalues (for generic parameters), and its eigenvectors are the *Askey–Wilson polynomials* (see, e.g., [Mac03]). In the following lemma, we use the notation  $(x)^+ = x + x^{-1}$  and  $(x)^- = x - x^{-1}$ .

LEMMA 2.24. *If  $t_3 = t_4 = 1$ , the Askey–Wilson operator can be written as follows:*

$$L_{t_1, t_2}^{\text{sym}} = \frac{1}{(X^2)^+ - (q^2)^+} [t_1(X^2\hat{y})^+ + t_1^{-1}(X^{-2}\hat{y})^+ - t_1^+(X^2)^+ - (t_1^{-1}q^2)^+(\hat{y}^+ - 2) + t_2^-(q(X\hat{y})^+ - q^{-1}(X^{-1}\hat{y})^+ - q^-X^+)] + t_1^+.$$

*Proof.* This follows from [NS04, Proposition 5.8] and a short calculation. □

(We remark that under the isomorphism of Theorem 2.11,  $(X\hat{y})^+ = q(1, 1)$ , where the right-hand side is the  $(1, 1)$  curve on the torus. This accounts for the apparent differences in the powers of  $q$  in the last terms of the above formula for  $L_{t_1, t_2}^{\text{sym}}$  and Remark 6.3.)

Let  $M$  be an  $A_q \rtimes \mathbb{Z}_2$ -module, let  $M^{\text{loc}}$  be its localization at nonzero polynomials in  $X$ , and suppose that  $M \rightarrow M^{\text{loc}}$  is injective. Let  $U_0 = (1 - q^2X^2)^{-1}(1 - \hat{s}\hat{y}) \in D_q$ .

LEMMA 2.25. *If  $U_0M \subset M$ , then  $H_{q,t}$  acts on  $M$ .*

*Proof.* Cherednik’s embedding of  $H_{q,t}$  into  $D_q$  is given by the following formulas (see [Che05]):

$$X \mapsto X, \quad T \mapsto T_1, \quad Y \mapsto \hat{y}sT_1,$$

with parameters  $(t_1, t_2, t_3, t_4) = (1, 1, t, 1)$ . We now twist this embedding by the automorphism of  $D_q$  given by  $X \mapsto qX$ ,  $\hat{y} \mapsto \hat{y}$ , and  $s \mapsto s\hat{y}$ . After this twist, the map  $H_{q,t} \rightarrow D_q$  is given by

$$X \mapsto qX, \quad T \mapsto ts\hat{y} + \bar{t}U_0, \quad Y \mapsto t\hat{y} + \bar{t}sU_0.$$

By assumption,  $U_0M \subset M$ , which implies that  $H_{q,t}M \subset M$ . □

### 3. Deformed skein modules of 2-bridge knots

In this section, we will prove Conjectures 1–3 (when  $q = -1$ ) for an arbitrary 2-bridge knot. To this end, we will use an algebraic construction of  $SL_2$  character varieties of finitely generated groups due to Brumfiel and Hilden [BH95]. We begin by recalling the results of [BH95] in the form that we need.

**3.1 The Brumfiel–Hilden construction**

Let  $\mathbf{Grp}$  be the category of (discrete) groups and let  $\mathbf{Alg}^*$  be the category of associative  $\mathbb{C}$ -algebras equipped with an anti-involution  $a \mapsto a^*$ . Assigning to a group  $\pi$  its complex group algebra  $\mathbb{C}[\pi]$  defines a functor  $\mathbb{C}[-] : \mathbf{Grp} \rightarrow \mathbf{Alg}^*$ , where the anti-involution on  $\mathbb{C}[\pi]$  is defined on the group elements by  $g^* := g^{-1}$  and is extended to  $\mathbb{C}[\pi]$  by linearity. The group algebra functor has an obvious right adjoint  $\mathbf{SU} : \mathbf{Alg}^* \rightarrow \mathbf{Grp}$  defined by  $\mathbf{SU}(A) := \{a \in A \mid aa^* = a^*a = 1\}$ .

Now, for a commutative  $\mathbb{C}$ -algebra  $B$ , the algebra  $\mathbb{M}_2(B)$  of  $2 \times 2$  matrices over  $B$  has an anti-involution given by the classical adjoint:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{3.1}$$

In this case,  $\mathbf{SU}[\mathbb{M}_2(B)] = \mathbf{SL}_2(B)$ . Thus, for any commutative  $\mathbb{C}$ -algebra  $B$ , we have a natural bijection

$$\mathbf{Hom}_{\mathbf{Alg}^*}(\mathbb{C}[\pi], \mathbb{M}_2(B)) = \mathbf{Hom}_{\mathbf{Grp}}(\pi, \mathbf{SL}_2(B)). \tag{3.2}$$

The representation scheme  $\mathbf{Rep}(\pi) := \mathbf{Rep}(\pi, \mathbf{SL}_2)$  is defined by its functor of points and, with the identification (3.2), this can be written as

$$\mathbf{Rep}(\pi) : \mathbf{CommAlg} \rightarrow \mathbf{Sets}, \quad B \mapsto \mathbf{Hom}_{\mathbf{Alg}^*}(\mathbb{C}[\pi], \mathbb{M}_2(B)).$$

As explained in § 2.1.1, this functor is represented by the commutative algebra  $\mathcal{O}\mathbf{Rep}(\pi) := \mathcal{O}(\mathbf{Rep}(\pi))$ . Let  $\rho_u : \mathbb{C}[\pi] \rightarrow \mathbb{M}_2(\mathcal{O}\mathbf{Rep}(\pi))$  be the algebra map corresponding to the universal representation  $\rho_u : \pi \rightarrow \mathbf{SL}_2(\mathcal{O}\mathbf{Rep}(\pi))$ . Geometrically, the scheme  $\mathbf{Rep}(\pi)$  parametrizes representations of  $\pi$  into  $\mathbf{SL}_2(\mathbb{C})$ , and  $\rho_u(g)$  is the section of the trivial bundle  $\mathbf{Rep}(\pi) \times \mathbb{M}_2(\mathbb{C})$  given by  $\rho \mapsto \rho(g)$ .

It is easy to see that for any  $\pi$ , the homomorphism  $\rho_u$  factors through the algebra

$$H[\pi] := \mathbb{C}[\pi] / \{[g, h + h^{-1}] \mid g, h \in \pi\},$$

which we call the *Brumfiel–Hilden algebra* of  $\pi$  (since it was introduced and studied in [BH95]). The algebra  $H[\pi]$  has a canonical (commutative) subalgebra  $H^+[\pi] := \{a \in H[\pi] \mid a^* = a\}$  (which is actually central in  $H[\pi]$ ). The meaning of these algebras is made clear by the following result proved in [BH95, Proposition 9.1].

**THEOREM 3.1.** *Assume that  $\pi$  is a finitely presented group.*

- (i) *The universal representation  $\rho_u : \mathbb{C}[\pi] \rightarrow \mathbb{M}_2(\mathcal{O}\mathbf{Rep}(\pi))$  factors through  $H[\pi]$ , and the induced map  $\bar{\rho}_u : H[\pi] \rightarrow \mathbb{M}_2(\mathcal{O}\mathbf{Rep}(\pi))$  is injective. The image of  $\bar{\rho}_u$  coincides with the subring of  $\mathbf{GL}_2(\mathbb{C})$ -invariants in  $\mathbb{M}_2(\mathcal{O}\mathbf{Rep}(\pi))$ , and we therefore have a canonical isomorphism of algebras*

$$\bar{\rho}_u : H[\pi] \xrightarrow{\sim} \mathbb{M}_2(\mathcal{O}\mathbf{Rep}(\pi))^{\mathbf{GL}_2(\mathbb{C})}.$$

- (ii) *The invariant subring  $H^+[\pi]$  is mapped bijectively by  $\bar{\rho}_u$  onto  $\mathcal{O}\mathbf{Rep}(\pi)^{\mathbf{GL}_2(\mathbb{C})}$ .*

Summarizing, we have the following commutative diagram.

$$\begin{array}{ccccc} H^+[\pi] & \hookrightarrow & H[\pi] & \longleftarrow & \mathbb{C}[\pi] \\ \cong \downarrow & & \cong \downarrow & & \downarrow \rho_u \\ \mathcal{O}\mathbf{Rep}(\pi)^{\mathbf{GL}_2(\mathbb{C})} & \hookrightarrow & \mathbb{M}_2(\mathcal{O}\mathbf{Rep}(\pi))^{\mathbf{GL}_2(\mathbb{C})} & \hookrightarrow & \mathbb{M}_2(\mathcal{O}\mathbf{Rep}(\pi)) \end{array}$$

Geometrically, this theorem says that  $H[\pi]$  is the algebra of  $GL_2(\mathbb{C})$ -equivariant matrix-valued functions on the representation scheme, and  $H^+[\pi]$  is the (commutative) algebra of  $GL_2(\mathbb{C})$ -invariant scalar functions. In other words,  $H^+[\pi]$  is the ring of functions on the  $SL_2$  character variety of  $\pi$ .

Now suppose that  $M$  is a manifold with boundary  $\partial M$ . The inclusion  $\partial M \rightarrow M$  induces a map of groups  $\alpha : \pi_1(\partial M) \rightarrow \pi_1(M)$ . By functoriality of the Brumfiel–Hilden construction, we have the following commutative diagram of algebra homomorphisms.

$$\begin{array}{ccccc}
 H^+[\pi_1(\partial M)] & \hookrightarrow & H[\pi_1(\partial M)] & \longleftarrow & \mathbb{C}[\pi_1(\partial M)] \\
 \alpha_*^+ \downarrow & & \bar{\alpha} \downarrow & & \alpha \downarrow \\
 H^+[\pi_1(M)] & \hookrightarrow & H[\pi_1(M)] & \longleftarrow & \mathbb{C}[\pi_1(M)]
 \end{array}$$

If  $M = S^3 \setminus K$  is the complement of a nontrivial knot in  $S^3$ , then  $\partial M = T^2$  and the map  $\alpha$  is an embedding (see, e.g., [BZ03, Proposition 3.17]) (however, the induced map  $\bar{\alpha}$  is not injective). Also, in this case the map  $\mathbb{C}[\pi_1(T^2)] \rightarrow H[\pi_1(T^2)]$  is an isomorphism because the fundamental group of  $T^2$  is abelian. With the identifications of Theorem 3.1, the leftmost arrow is precisely the peripheral map discussed in § 2.1.1.

### 3.2 The Brumfiel–Hilden algebra of a 2-bridge knot

Theorem 3.1 reduces the problem of describing the character variety  $\text{Char}(\pi, SL_2)$  for a finitely presented group to that of describing the algebra  $H[\pi]$ . In many interesting cases,  $H[\pi]$  can be computed explicitly (see [BH95, ch. 3]). If  $K \subset S^3$  is a 2-bridge knot, the fundamental group  $\pi_1(S^3 \setminus K)$  is generated by two elements (meridians) subject to one relation; in this case,  $H^+[\pi]$  is isomorphic to the ring of regular functions on a plane curve and  $H[\pi]$  has the structure of a generalized quaternion algebra over  $H^+[\pi]$ . We will briefly describe this structure below, and refer the reader to [BH95, Appendix A4\*] for proofs and more details. We begin with the standard presentation of  $\pi_1(S^3 \setminus K)$  in the case of 2-bridge knots.

3.2.1 *The fundamental group.* Recall that the 2-bridge knots can be indexed by pairs  $(p, q)$  of relatively prime odd integers with  $p > 0$  and  $0 < q < p$ . The pairs  $(p, q)$  and  $(p', q')$  define the same knot if and only if  $p = p'$  and  $qq' \equiv \pm 1 \pmod{p}$ . For a knot  $K = K(p, q)$ , the group  $\pi := \pi_1(S^3 \setminus K(p, q))$  has a presentation

$$\pi = \langle a, b \mid aw = wb \rangle,$$

where  $a$  and  $b$  are meridians around two trivial strands contained in one of the balls in a 2-bridge decomposition of  $S^3 \setminus K$ . The element  $w$  can be expressed in terms of  $a^{\pm 1}$  and  $b^{\pm 1}$  as a product of two words:

$$w = v\bar{v},$$

which are images of each other under the anti-involution of the free group  $F\langle a, b \rangle$  that switches  $a$  and  $b$ . The word  $v$  is given by

$$v := b^{e_1} a^{e_2} \dots (a \text{ or } b)^{e_d},$$

where  $d = (p - 1)/2$  and the exponents  $e_n \in \{\pm 1\}$  are computed by the rule  $\text{sign}(e_n) = \text{sign}(k_n)$ , with  $k_n$  defined by the conditions

$$k_n \equiv nq \pmod{2p}, \quad -p < k_n < p, \quad k_n \neq 0.$$

The peripheral map  $\alpha : \pi_1(T^2) \rightarrow \pi$  is defined by the assignments

$$\alpha(m) = a, \quad \alpha(l) = w\tilde{w}a^{-s},$$

where  $\tilde{w}$  is the word  $w$  written backwards (without switching  $a$  and  $b$ ) and  $s = 4 \sum_{n=1}^d e_n$ .

*Example 3.2.* Let  $p = 5$  and  $q = 3$ . The corresponding knot  $K = K(5, 3)$  is the figure eight knot. In this case,  $d = (5 - 1)/2 = 2$ ,  $k_1 = 3$ , and  $k_2 = -4$ , so that  $e_1 = 1$ ,  $e_2 = -1$ , and  $s = e_1 + e_2 = 0$ . We therefore have  $w = ba^{-1}b^{-1}a$ , and the fundamental group has the following presentation:

$$\pi_1(K) = \langle a, b \mid aba^{-1}b^{-1}a = ba^{-1}b^{-1}ab \rangle = \langle a, b \mid aba^{-1}ba = bab^{-1}ab \rangle.$$

(The second presentation is standard, and the relation in the first is conjugate to the relation in the second.) In this notation, the peripheral map  $\alpha : \pi_1(T^2) \rightarrow \pi_1(K)$  is given by

$$m \mapsto a, \quad l \mapsto ba^{-1}b^{-1}a^2b^{-1}a^{-1}b.$$

**3.2.2 The Brumfiel–Hilden algebra.** For a 2-bridge knot, the algebra  $H[\pi]$  has the following structure (see [BH95, Proposition A.4\*.9]).

**THEOREM 3.3.** *Let  $\pi$  be the knot group of a 2-bridge knot of type  $(p, q)$ . There is a polynomial  $Q \in \mathbb{C}[I, J]$  of degree  $d = (p - 1)/2$  such that  $H[\pi]$  admits an  $H^+[\pi]$ -module decomposition*

$$H[\pi] = H^+[\pi] \oplus \bar{H}^+[\pi]\mathbf{i} \oplus H^+[\pi]\mathbf{j} \oplus \bar{H}^+[\pi]\mathbf{k},$$

where

$$H^+[\pi] := \frac{\mathbb{C}[x, I, J]}{\langle IQ, I + J - 4(x^2 - 1) \rangle}, \quad \bar{H}^+[\pi] := \frac{\mathbb{C}[x, I, J]}{\langle Q, I + J - 4(x^2 - 1) \rangle}.$$

The multiplication in  $H[\pi]$  is determined by the (generalized) quaternion relations

$$\mathbf{i}^2 = I, \quad \mathbf{j}^2 = J, \quad \mathbf{ij} = -\mathbf{j}\mathbf{i} = \mathbf{k}.$$

The canonical projection  $\mathbb{C}[\pi] \rightarrow H[\pi]$  is given by the equations

$$a^{\pm 1} \mapsto x \pm \frac{1}{2}(\mathbf{i} + \mathbf{j}), \quad b^{\pm 1} \mapsto x \mp \frac{1}{2}(\mathbf{i} - \mathbf{j}).$$

Finally, the canonical anti-involution  $*$  :  $H[\pi] \rightarrow H[\pi]$  is given by

$$x^* = x, \quad \mathbf{i}^* = -\mathbf{i}, \quad \mathbf{j}^* = -\mathbf{j}, \quad \mathbf{k}^* = -\mathbf{k}.$$

*Remark 3.4.* The image of the polynomial  $Q$  in  $H^+[\pi]$  determines a curve of conjugacy classes of mostly irreducible  $SL_2$  representations of  $\pi$ ; more precisely, the classes of irreducible representations  $\rho : \pi \rightarrow SL_2(\mathbb{C})$  correspond to the algebra homomorphisms  $\varphi : H^+[\pi] \rightarrow \mathbb{C}$  such that  $\varphi(Q) = 0$  and  $\varphi(I) \neq 0$ . It is shown in [BH95] that  $Q$  actually has integral coefficients, i.e.  $Q \in \mathbb{Z}[I, J]$ . We give a formula for  $Q$  in Remark 3.7.

*Remark 3.5.* Under the identification of Theorem 3.1,  $H^+[\pi] \cong \mathcal{O}\text{Char}(\pi)$ , the generators  $x, I$ , and  $J$  correspond to the following functions:

$$\begin{aligned} x &\mapsto \frac{1}{2}\text{Tr}(a), \\ I &\mapsto \frac{1}{2}\text{Tr}(a)^2 + \frac{1}{2}\text{Tr}(a)\text{Tr}(b) - \text{Tr}(ab) - 2, \\ J &\mapsto \frac{1}{2}\text{Tr}(a)^2 - \frac{1}{2}\text{Tr}(a)\text{Tr}(b) + \text{Tr}(ab) - 2, \end{aligned} \tag{3.3}$$

where  $\text{Tr}(g)$  is the character function  $\rho \mapsto \text{Tr}(\rho(g))$ .



**3.3 The nonsymmetric skein module (at  $q = -1$ )**

In this section, we prove Conjectures 1–3 for all 2-bridge knots when  $q = -1$ . We fix a 2-bridge knot  $K = K(p, q)$  and write  $H := H[\pi]$  and  $H^+ := H^+[\pi]$  for the corresponding knot group  $\pi = \pi_1(S^3 \setminus K)$ . Let  $X \in H$  and  $Y \in H$  denote the images of the meridian  $m$  and longitude  $l$  under the (induced) peripheral map

$$\bar{\alpha} : H[T^2] = \mathbb{C}[m^{\pm 1}, l^{\pm 1}] \rightarrow H. \tag{3.4}$$

With the identification of Theorem 3.3, we have

$$X^{\pm 1} = x \pm \frac{1}{2}(i + j), \quad Y = w\tilde{w}X^{-s}. \tag{3.5}$$

Next, we let  $H^+[X^{\pm 1}]$  denote the subalgebra of  $H$  generated by  $H^+$  and  $X^{\pm 1}$ . Note that  $H^+[X^{\pm 1}]$  is commutative and the canonical anti-involution on  $H$  restricts to  $H^+[X^{\pm 1}]$  and maps  $X \mapsto X^{-1}$ . The following observation is implicit in [BH95, Appendix A.4\*].

**PROPOSITION 3.6.** *For any 2-bridge knot, the image of the peripheral map (3.4) is contained in  $H^+[X^{\pm 1}]$ .*

*Proof.* We need to prove that  $Y \in H^+[X^{\pm 1}]$ . Following [BH95], we introduce the algebra

$$\bar{H}[\langle a, b \rangle] := H[\langle a, b \rangle] / \langle a^+ - b^+ \rangle,$$

which is the quotient of the Brumfiel–Hilden algebra of the free group  $F\langle a, b \rangle$  on generators  $a, b$  modulo the relation  $a^+ = b^+$ , where  $a^+ = 1/2(a + a^{-1})$ . Note that, for any 2-bridge knot group  $\pi = \langle a, b \mid aw = wb \rangle$ , the natural projection  $H[\langle a, b \rangle] \rightarrow H[\pi]$  factors through  $\bar{H}[\langle a, b \rangle]$ , since the relation  $aw = wb$  implies that  $a^+ = b^+$  in  $H[\pi]$ . The advantage of  $\bar{H}[\langle a, b \rangle]$  over  $H[\langle a, b \rangle]$  is that  $\bar{H}$  is a free quaternion algebra:

$$\bar{H} = \bar{H}^+ \oplus \bar{H}^+i \oplus \bar{H}^+j \oplus \bar{H}^+k,$$

with  $\bar{H}^+$  being isomorphic to a free polynomial ring in two variables (see [BH95, Proposition 3.1]).

The algebra  $\bar{H}[\langle a, b \rangle]$  inherits many symmetries from  $H[\langle a, b \rangle]$ . In particular, we have the anti-automorphism  $\gamma : \bar{H} \rightarrow \bar{H}$  that fixes  $a$  and  $b$  and sends  $ab \mapsto ba$ , and the involution  $\sigma : \bar{H} \rightarrow \bar{H}$  that ‘switches  $a$  and  $b$ ’ (i.e.  $\sigma(a) = b$  and  $\sigma(b) = a$ ). It is easy to see that  $\gamma$  and  $\sigma$  act trivially on  $\bar{H}^+$ , while

$$\gamma : (i, j, k) \mapsto (i, j, -k), \quad \sigma : (i, j, k) \mapsto (-i, j, -k).$$

If we write  $v = D + Ei + Fj + Gk \in \bar{H}$  for some  $D, E, F, G \in \bar{H}^+$ , then

$$\bar{v} = \sigma\gamma(v) = D - Ei + Fj + Gk.$$

It follows that

$$w = v\bar{v} = L + Mj + Nk, \quad \bar{w} = \gamma(w) = L + Mj - Nk,$$

where

$$\begin{aligned} L &= D^2 - E^2I + F^2J - G^2IJ, \\ M &= 2DF + 2GEI, \\ N &= 2DG + 2EF. \end{aligned} \tag{3.6}$$

In  $\bar{H}$ , we therefore have

$$w\bar{w} = (L^2 + M^2J + N^2IJ) + 2MNJ\mathbf{i} + 2ML\mathbf{j}. \tag{3.7}$$

On the other hand,

$$ww^* = L^2 - M^2J + N^2IJ = 1, \tag{3.8}$$

$$aw - wb = (L - NJ)\mathbf{i}. \tag{3.9}$$

It follows from (3.7) and (3.8) that  $w\bar{w} = (1 + 2M^2J) + 2M(NJ\mathbf{i} + L\mathbf{j})$  in  $\bar{H}$ , while (3.9) implies that  $L\mathbf{i} = NJ\mathbf{i}$  in  $H[\pi]$ . Hence, projecting onto  $H[\pi]$ , we get

$$Y = w\bar{w}X^{-s} = [(1 + 2M^2J) + 2ML(\mathbf{i} + \mathbf{j})]X^{-s} = [(1 + 2M^2J) + 2ML\delta]X^{-s}, \tag{3.10}$$

where  $\delta := X - X^{-1} = \mathbf{i} + \mathbf{j}$ ; see (3.5). This shows that  $Y \in H^+[X^{\pm 1}]$ , completing the proof of the proposition.  $\square$

*Remark 3.7.* In the notation of the previous proposition,  $Q = L - NJ$  in  $H^+[\pi]$ .

*Example 3.8.* In the case of the figure eight knot,  $K = K(5, 3)$  (cf. Example 3.2), we have

$$v = ba^{-1} = (x - \frac{1}{2}(\mathbf{i} - \mathbf{j}))(x - \frac{1}{2}(\mathbf{i} + \mathbf{j})) = (1 + \frac{1}{2}I) - x\mathbf{i} + \frac{1}{2}\mathbf{k}.$$

Hence, by (3.6),

$$L = 1 - \frac{1}{2}IJ, \quad M = -xI, \quad N = 1 + \frac{1}{2}I.$$

By Remark 3.7,  $Q = 1 - J - IJ$  and, by Theorem 3.3,

$$\mathcal{O}\text{Char}(\pi) = H^+[\pi] \cong \frac{\mathbb{C}[x, I, J]}{\langle I(1 - J - IJ), I + J - 4(x^2 - 1) \rangle}.$$

Finally, formula (3.10) gives

$$Y = (1 + I - IJ) - IJX^2 + IX^{-2}.$$

Next, we note that  $S = \{1, \delta, \delta^2, \dots\}$  is an Ore subset in  $H$ . We write  $H[\delta^{-1}]$  for the localization of  $H$  at  $S$  and define

$$M := H^+[X^{\pm 1}] + H^+[X^{\pm 1}]Q\delta^{-1} \subset H[\delta^{-1}], \tag{3.11}$$

where  $Q = Q(I, J)$  is the polynomial featured in Theorem 3.3. By construction,  $M$  is a module (actually, a fractional ideal) over  $H^+[X^{\pm 1}]$  and, by Proposition 3.6, it is a module over  $\mathbb{C}[m^{\pm 1}, l^{\pm 1}]$ . We extend this last module structure to  $\mathbb{C}[m^{\pm 1}, l^{\pm 1}] \rtimes \mathbb{Z}_2$ , letting  $s \in \mathbb{Z}_2$  act on  $M$  by the canonical involution  $X \mapsto X^{-1}$ .

Now, recall the double affine Hecke algebra from (2.5). We let  $q = -1$  and  $t_3 = t_4 = 1$  and define

$$\begin{aligned} T_0 &\mapsto -t_1sY + (\bar{t}_1X + \bar{t}_2)\delta^{-1}(1 + sY), \\ T_0^\vee &\mapsto T_0^{-1}X^{-1}, \\ T_1 &\mapsto s, \\ T_1^\vee &\mapsto Xs, \end{aligned} \tag{3.12}$$

where  $\bar{t}_i = t_i - t_i^{-1}$  for  $i = 1, 2$ . The main result of this section is the following theorem.

**THEOREM 3.9.** *For any 2-bridge knot, the assignment (3.12) extends to an action of  $\mathcal{H}_{-1,t_1,t_2}$  on  $M$ .*

*Remark 3.10.* The term  $(1 + sY)$  in formula (3.12) agrees with the term  $(1 - s\hat{y})$  in formula (2.6) because of the sign in Theorem 2.10 (a loop  $\gamma$  in the skein algebra at  $q = -1$  gets sent to the function  $\rho \mapsto -\text{Tr}(\rho(\gamma))$  on the character variety). (The change  $X \mapsto X^{-1}$  is inessential.)

*Proof.* Let  $U = \delta^{-1}(1 + sY) = \delta^{-1}(1 + Y^{-1}s) \in H[\delta^{-1}]$ . We need to check that  $UM \subset M$ . For this, it suffices to check that there exist  $f, g, h \in H^+[X^{\pm 1}]$  such that

$$Y = fQ + g\delta - 1, \tag{3.13}$$

$$YQ = (1 + h\delta)Q \tag{3.14}$$

or, equivalently, there exist  $f', g', h' \in H^+[X^{\pm 1}]$  such that

$$Y^{-1} = f'Q + g'\delta - 1, \tag{3.15}$$

$$Y^{-1}Q = (1 + h'\delta)Q. \tag{3.16}$$

Indeed, observe that

$$M = H^+[X^{\pm 1}] + H^+[X^{\pm 1}]Q\delta^{-1} = H^+ + H^+\delta + H^+Q\delta^{-1}, \tag{3.17}$$

so it suffices to check that  $U$  maps  $H^+$ ,  $H^+\delta$ , and  $H^+Q\delta^{-1}$  into  $M$ . Assuming (3.15) and (3.16), we have

$$\begin{aligned} U[H^+] &= \delta^{-1}(1 + Y^{-1}s)H^+ = \delta^{-1}(1 + Y^{-1})H^+ \\ &= \delta^{-1}(f'Q + g'\delta)H^+ = g'H^+ + f'H^+Q\delta^{-1}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} U[H^+Q\delta^{-1}] &= \delta^{-1}(1 + Y^{-1}s)H^+Q\delta^{-1} = \delta^{-1}(1 - Y^{-1})QH^+\delta^{-1} \\ &= \delta^{-1}(-h'\delta Q)H^+\delta^{-1} = -h'H^+Q\delta^{-1} \subset M, \\ U[H^+\delta] &= \delta^{-1}(1 + Y^{-1}s)H^+\delta = \delta^{-1}(1 - Y^{-1})H^+\delta \\ &= (1 - Y^{-1})H^+ \subset H^+[X^{\pm 1}] \subset M, \end{aligned}$$

where in the last line we used Proposition 3.6 (and the fact that  $X$  and  $Y$  commute).

To prove (3.13), we note that by Remark 3.7, we have  $L - JN = Q$ , and we also have  $I = 4(x^2 - 1) - J = \delta^2 - J$ . Hence, by (3.8) and (3.10), we have

$$\begin{aligned} Y &= X^{-s}(1 + 2M^2J + 2ML\delta) \\ &= X^{-s}(2L^2 + 2N^2J(\delta^2 - J) - 1 + 2ML\delta) \\ &= X^{-s}[2(L + NJ)(L - NJ) + 2(N^2J\delta + LM)\delta - 1] \\ &= 2X^{-s}(L + NJ)Q + 2X^{-s}(N^2J\delta + LM)\delta - X^{-s} \\ &= 2X^{-s}(L + NJ)Q + X^{-s}[2N^2J\delta + 2LM + A(X)]\delta - 1, \end{aligned}$$

where  $A(X) := X + X^3 + \dots + X^{s-1}$  with  $s = 4 \sum_{n=1}^d e_n$ .

Formula (3.14) follows from Lemma 3.12 below, which completes the proof of the theorem.  $\square$

As a consequence of Theorem 3.9, we deduce Conjecture 1 (at  $q = -1$ ) ‘at the symmetric level’.

COROLLARY 3.11. *The  $SL_2$ -character ring of any 2-bridge knot carries a natural action of the spherical subalgebra  $S\mathcal{H}_{-1,t_1,t_2}$  of the double affine Hecke algebra  $\mathcal{H}_{-1,t_1,t_2}$ .*

*Proof.* Since  $e\delta = 1/2(1 + s)\delta = 0$ , equation (3.17) shows that  $eM = H^+$ , while Theorem 3.1 shows that  $H^+ = \mathcal{O}\text{Char}(\pi, SL_2)$ . Then the action of  $\mathcal{H}_{-1,t_1,t_2,1,1}$  on  $M$  induces an action of  $S\mathcal{H}_{-1,t_1,t_2,1,1}$  on  $eM$ . □

To prove Conjecture 2, we need the following lemma, which refines formula (3.14) used in the proof of Theorem 3.9. We will keep the notation introduced earlier in this section.

LEMMA 3.12. *For any 2-bridge knot, the following identity holds in  $H[\pi]$ :*

$$YQ = Q. \tag{3.18}$$

*Proof.* We will actually prove a stronger identity:

$$\tilde{w}a^{-s/2}Q = Q, \tag{3.19}$$

where  $s = 4 \sum_{n=1}^d e_n$  and  $\tilde{w}$  is the word  $w$  written backwards. To see that (3.19) implies (3.18), first note that there is an automorphism<sup>6</sup> of the fundamental group  $\pi$  mapping  $a \mapsto a^{-1}$ ,  $b \mapsto b^{-1}$ . This induces an automorphism  $\beta$  of the Brumfiel–Hilden algebra  $H[\pi]$ , which fixes elements of  $H^+$  and maps  $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \mapsto (-\mathbf{i}, -\mathbf{j}, \mathbf{k})$ . Applying  $\beta$  to (3.19), we get  $w^*a^{s/2}Q = Q$ . Therefore,  $w^*a^{s/2}Q = \tilde{w}a^{-s/2}Q$ , which implies that  $Q = a^{-s/2}w\tilde{w}a^{-s/2}Q = w\tilde{w}a^{-s}Q = YQ$ , since  $w\tilde{w}$  commutes with  $a$ .

To prove (3.19), we assume (without loss of generality) that  $s \geq 0$  and

$$v = b^{e_1}a^{e_2} \dots b^{e_{d-1}}a^{e_d}, \quad w = v\bar{v}, \quad \tilde{w} = a^{e_1}b^{e_2} \dots b^{e_d}a^{e_d}b^{e_{d-1}} \dots b^{e_1}.$$

Next, we introduce the following notation: for  $e = \pm 1$ , we write  $x_e := x + (1/2)e\mathbf{j}$ , so that

$$a^{e_n} = x + \frac{1}{2}e_n(\mathbf{i} + \mathbf{j}) = x_{e_n} + \frac{1}{2}e_n\mathbf{i}, \quad b^{e_n} = x + \frac{1}{2}e_n(\mathbf{j} - \mathbf{i}) = x_{e_n} - \frac{1}{2}e_n\mathbf{i}$$

for  $n = 1, 2, \dots, d$ . Observe that we obviously have

$$x_{e_n}x_{e_m} = x_{e_m}x_{e_n}, \quad \mathbf{i}x_e = x_{-e}\mathbf{i}. \tag{3.20}$$

Using the commutator relations (3.20) and the fact that  $\mathbf{i}Q = Q\mathbf{i} = 0$  in  $H[\pi]$ , we compute

$$\begin{aligned} \tilde{w}a^{-s/2}Q &= (x_{e_1} + \frac{1}{2}e_1\mathbf{i})(x_{e_2} - \frac{1}{2}e_2\mathbf{i}) \dots (x_{e_d} - \frac{1}{2}e_d\mathbf{i})(x_{e_d} + \frac{1}{2}e_d\mathbf{i})(x_{e_{d-1}} - \frac{1}{2}e_{d-1}\mathbf{i}) \dots \\ &\quad \dots (x_{e_1} - \frac{1}{2}e_1\mathbf{i})(x_- - \frac{1}{2}\mathbf{i})^{s/2}Q \\ &= x_{e_1}x_{e_2} \dots x_{e_d}x_{e_d}x_{e_{d-1}} \dots x_{e_1}x_-^{s/2}Q \\ &= x_+^{2N_+}x_-^{2N_-}x_-^{s/2}Q \\ &= x_+^{2N_+}x_-^{2N_+}Q \\ &= (x_+x_-)^{2N_+}Q, \end{aligned}$$

where  $N_+$  is the number of  $(+1)$  among the  $\{e_1, \dots, e_d\}$  and  $N_-$  is the number of  $(-1)$ , so that  $N_+ - N_- = \sum_{n=1}^d e_n = s/4$ . Finally, note that

$$x_+x_- = (x + \frac{1}{2}\mathbf{j})(x - \frac{1}{2}\mathbf{j}) = x^2 - \frac{1}{4}J = 1 + \frac{1}{4}I.$$

This shows that  $(x_+x_-)Q = Q$  and therefore  $\tilde{w}a^{-s/2}Q = (x_+x_-)^{2N_+}Q = Q$ . □

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<sup>6</sup> The existence of this automorphism is a consequence of the fact that 2-bridge knots are invertible [BZ03, Proposition 3.19].

*Remark 3.13.* The proof of Lemma 3.12 shows that we also have  $va^{-s/4}Q = Q$  in  $H[\pi]$ .

To prove Conjecture 2, we note that by Theorem 3.3 (cf. also [BH95, Proposition A.4\*.10]) the element  $Q \in H[\pi]$  has the following form:

$$Q = a_0 + a_1I + \dots + a_{d-1}I^{d-1} + (-1)^dI^d,$$

where  $a_i \in \mathbb{C}[x^2] \subset H^+$ . Hence, every element of  $H^+[X^{\pm 1}] \cong \mathbb{C}[X^{\pm 1}, I]/(IQ)$  can be written (uniquely) in the form

$$u = u_0 + u_1I + \dots + u_dI^d \pmod{IQ},$$

where  $u_i \in \mathbb{C}[X^{\pm 1}]$ . It follows that  $H^+[X^{\pm 1}]Q\delta^{-1} = \mathbb{C}[X^{\pm 1}]Q\delta^{-1}$  in  $H[\delta^{-1}]$ . The module  $M = H^+[X^{\pm 1}] + H^+[X^{\pm 1}]Q\delta^{-1}$  from (3.11) is therefore free over  $\mathbb{C}[X^{\pm 1}]$  of rank  $d + 1$ ; for a basis in  $M$ , we can take  $\{1, I, \dots, I^{d-1}, Q\delta^{-1}\}$ . Now, by Lemma 3.12,  $M_0 := \mathbb{C}[X^{\pm 1}]Q\delta^{-1} \subset M$  is a  $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}] \rtimes \mathbb{Z}_2$ -submodule of  $M$  which is isomorphic to the sign representation. The corresponding  $A_1^{\mathbb{Z}_2}$ -module  $eM_0$  is thus isomorphic to the skein module of the unknot, which implies Conjecture 2 for two-bridge knots at the symmetric level (when  $q = -1$ ).

Now, for  $M_0$  as above, consider the quotient  $\bar{M} := M/M_0$  and identify

$$\bar{M} = \mathbb{C}[X^{\pm 1}]1 \oplus \dots \oplus \mathbb{C}[X^{\pm 1}]I^{d-1}, \tag{3.21}$$

so that  $\bar{M}^{\text{loc}} = \mathbb{C}(X^{\pm 1})1 \oplus \dots \oplus \mathbb{C}(X^{\pm 1})I^{d-1}$ . To prove Conjecture 3, we need to show that the operators

$$\begin{aligned} T_0 &= -t_1sY + (\bar{t}_1X + \bar{t}_2)\delta^{-1}(1 + sY), \\ T_1 &= t_3s + (\bar{t}_3X + \bar{t}_4)\delta^{-1}(1 - s), \end{aligned}$$

preserve the subspace  $\bar{M} \subset \bar{M}^{\text{loc}}$  for all values  $t_1, \dots, t_4$ . For this, it suffices to check that  $U = \delta^{-1}(1 + sY)$  and  $U_1 = \delta^{-1}(1 - s)$  preserve (3.21). The inclusion  $U\bar{M} \subset \bar{M}$  follows from (3.13) by the same argument as in the proof of Theorem 3.9, and  $U_1\bar{M} \subset \bar{M}$  follows from the fact that  $s$  acts trivially on the basis vectors  $\{1, I, \dots, I^{d-1}\}$ . Thus, Conjecture 3 follows.

We conclude this section by exhibiting an interesting relation between the polynomial  $Q$  and the classical Alexander polynomial  $\Delta_K(t)$  of a 2-bridge knot. First, we observe that for any knot group  $\pi = \pi_1(S^3 \setminus K)$  and for any complex reductive group  $G$ , there is a natural map

$$f : \mathbb{T} \xrightarrow{\sim} \text{Rep}(\pi, \mathbb{T}) \hookrightarrow \text{Rep}(\pi, G) \rightarrow \text{Char}(\pi, G), \tag{3.22}$$

where  $\mathbb{T} \subset G$  is a maximal torus of  $G$  (the first arrow in the definition of  $f$  is an isomorphism induced by the abelianization map  $\pi \rightarrow \pi/[\pi, \pi] = \mathbb{Z}$  of  $\pi$ ). It is easy to see that  $f$  factors through the quotient by  $W$ , so that  $f : \mathbb{T}/W \rightarrow \text{Char}(\pi, G)$ . Hence, by dualizing (3.22), we get a map of commutative algebras

$$f_* : \mathcal{O}\text{Char}(\pi, G) \rightarrow \mathcal{O}(\mathbb{T})^W \subset \mathcal{O}(\mathbb{T}). \tag{3.23}$$

If  $G = \text{SL}_2(\mathbb{C})$  and  $\pi$  is the fundamental group of a 2-bridge knot, we can identify  $\mathcal{O}\text{Char}(\pi, G)$  with  $\mathbb{C}[x, I, J]/(IQ, I + J - 4(x^2 - 1))$  and  $\mathcal{O}(\mathbb{T}) = \mathbb{C}[X^{\pm 1}]$  using (3.3). With this identification, the map (3.23) is given by

$$f_*(x) = \frac{1}{2}(X + X^{-1}), \quad f_*(J) = \frac{1}{4}(X - X^{-1})^2, \quad f_*(I) = 0.$$

A direct calculation (similar to the one in Lemma 3.12) shows that

$$f_*(Q) = X^{-s/2}[1 - X^{2e_1} + X^{2(e_1+e_2)} - X^{2(e_1+e_2+e_3)} + \dots + X^{2(e_1+\dots+e_{p-1})}],$$

where  $s = 4 \sum_{i=1}^d e_i$ . The expression in the right-hand side coincides with a known formula for the Alexander polynomial  $\Delta_K(t)$  of a 2-bridge knot evaluated at  $t = X^2$  (see, e.g., [Fuk05, Theorem 1.2(1)] or [Min82]). Thus, we conclude the following proposition.

PROPOSITION 3.14. *For any 2-bridge knot, the image of  $Q$  under the map (3.23) is equal to  $\Delta_K(X^2)$ .*

Remark 3.15. The relation between  $Q$  and the Alexander polynomial  $\Delta_K(t)$  was observed in [BH95] (see Example A.8\*.13 therein). However, this relation is stated in [BH95] in purely algebraic terms, without referring to the map (3.23), and the proof in [BH95] is quite different from ours.

**3.4 Conjecture 1 for 2-bridge knots for an arbitrary  $q$**

Technically, we proved Conjectures 1, 2, and 3 for the module  $M \subset H[\delta^{-1}]$  defined in (3.11), which imply symmetric versions of these conjectures for skein modules of 2-bridge knot complements (cf. Corollary 3.11). However, we believe that the following conjecture is true.

CONJECTURE 3.16. The module  $M$  defined in (3.11) is the  $q = -1$  specialization of the nonsymmetric skein module of the 2-bridge knot  $K$ .

Using the results of §4, it is easy to check that this conjecture is true for the trefoil and the figure eight knot. We now show that Conjecture 3.16 implies Conjecture 1 for arbitrary  $q$ . Let  $U = (1 - q^2 X^2)^{-1}(1 - s\hat{y}) \in D_q$ .

THEOREM 3.17. *Let  $N$  be a module over  $A_q \rtimes \mathbb{Z}_2$  which is free and finitely generated over  $\mathbb{C}[X^{\pm 1}]$ , and suppose that  $UN \subset N$  for  $q = \pm 1$ . Then  $UN \subset N$  for arbitrary  $q$ .*

Proof. Pick an identification of  $\mathbb{C}[X^{\pm 1}]$ -modules  $N \cong \mathbb{C}[X^{\pm 1}] \otimes_{\mathbb{C}} V$  for some finite-dimensional vector space  $V$ . The actions of  $\hat{y}$  and  $s$  on  $N$  are completely determined by the matrices  $A(X)$ ,  $B(X) \in \text{End}_{\mathbb{C}[X^{\pm 1}]}(N)$  defined by

$$A(X) \cdot v := \hat{y} \cdot (1 \otimes v), \quad B(X) \cdot v := s \cdot (1 \otimes v).$$

Define operators  $S, P : N \rightarrow N$  via the formulas  $S \cdot (f(X) \otimes v) = f(X^{-1}) \otimes v$  and  $P \cdot (f(X) \otimes v) = f(q^{-2}X) \otimes v$ . Then the actions of  $\hat{y}$  and  $s$  on  $N$  can be written in terms of the operators  $P, S$  as follows:

$$\hat{y} = A(X)P, \quad s = B(X)S,$$

where the equalities are inside  $\text{End}_{\mathbb{C}}(N)$ . Furthermore, the operators  $X$ ,  $\hat{y}$ , and  $s$  satisfy the relations of  $A_q \rtimes \mathbb{Z}_2$ , and the relation  $\hat{y}s\hat{y}s = 1$  implies the identity

$$B(X)A(X^{-1})B(q^{-2}X^{-1})A(q^2X) = \text{Id}. \tag{3.24}$$

Let  $C(q) \in \text{End}_{\mathbb{C}}(V)$  be the matrix  $C(q) = B(q^{-1})A(q)$ . By Remark 6.5, the condition  $UN \subset N$  is equivalent to the conditions  $C(q) = \text{Id}$  and  $C(-q) = \text{Id}$ .

Now, by assumption, we have  $C(1) = C(-1) = \text{Id}$  and, if we substitute  $X = q^{-1}$  into equation (3.24), we get  $C(q)^2 = \text{Id}$ . Now let  $q = e^z$  and write  $C(q) = C(e^z) = \sum_i C_i z^i$ . Expanding the equation  $C(q)^2 = \text{Id}$  in powers of  $z$ , we obtain  $C(e^z)^2 = C_0^2 + C_0 C_1 z + \dots = \text{Id}$ , and induction on powers of  $z$  shows that  $C(e^z) = \text{Id}$ . This shows that  $C(q) = \text{Id}$ , and a similar argument shows that  $C(-q) = \text{Id}$ , which completes the proof.  $\square$

COROLLARY 3.18. *Conjecture 3.16 implies Conjecture 1 for 2-bridge knots (for an arbitrary  $q$ ).*

Proof. Indeed, if  $M$  is the correct specialization of  $N = \hat{K}_q$  at  $q = -1$ , then the assumption  $UN \subset N$  of Theorem 3.17 holds by (the proof of) Theorem 3.9 for  $q = -1$ . A theorem of Barrett [Bar99] shows that the  $q = 1$  and  $q = -1$  skein modules are isomorphic, which shows that  $UN \subset N$  for  $q = 1$ . The fact that  $N$  is free and finitely generated over  $\mathbb{C}[X^{\pm 1}]$  was proved in [Lê06], and Theorem 3.17 therefore implies that  $UN \subset N$  for arbitrary  $q$ , which implies Conjecture 1.  $\square$



4. Examples of skein modules of knot complements

In this section, we give an explicit description of the nonsymmetric skein modules for the unknot,  $(2, 2p + 1)$  torus knots, and the figure eight knot. In the process, we prove Conjecture 2 for these knots.

**THEOREM 4.1.** *If  $K$  is the figure eight knot or any  $(2, 2p + 1)$  torus knot, then  $K_q(\text{unknot})$  is a submodule of  $K_q(S^3 \setminus K)$ .*

*Remark 4.2.* This theorem can be viewed as a quantization of the map  $\phi$  from (2.1). In particular, in examples the image of the empty link is a nontrivial element of the skein module of the knot complement, and this quantizes the fact that  $\phi(1) = B_K$  (see Remark 2.4).

We also remark that the natural algebra map  $\mathbb{C}[m^{\pm 1}, l^{\pm 1}]/A_K \rightarrow \mathbb{C}[m^{\pm 1}, l^{\pm 1}]/A_U$  from Remark 2.4 does not quantize for  $(2, 2p + 1)$  torus knots or the figure eight knot. In particular, for the trefoil, Lemma 4.10 shows that if  $q$  is not a root of unity, then the unknot submodule of  $M := K_q(\text{trefoil})$  is the unique submodule, which shows that  $M$  has a unique quotient. This quotient is clearly not isomorphic to the skein module of the unknot.

To simplify notation, we will divide the proof of Theorem 4.1 into separate subsections after first proving some useful technical lemmas. For the trefoil (i.e. the  $(2, 3)$  torus knot) and the figure eight knot, Gelca and Sain have given complete calculations of the symmetrized module  $K_q(S^3 \setminus K)$  in [Gel02] and [GS04], respectively. Using these calculations, we describe the corresponding  $A_q \rtimes \mathbb{Z}_2$ -modules explicitly. For the  $(2, 2p + 1)$  torus knots, Gelca and Sain gave only partial computations of the module structure of  $K_q(S^3 \setminus K)$  in [GS03]. We complete their computations to fully determine the module structure of the submodule of  $K_q(S^3 \setminus K)$  generated by the empty link and describe the corresponding  $A_q \rtimes \mathbb{Z}_2$ -module.

*Remark 4.3.* The calculations in this section are lengthy, and the reader might worry about errors with signs or powers of  $q$ . However, a strong ‘consistency check’ is available: one can use Lemma 5.6 together with the module structures described in this section to give explicit computations of the colored Jones polynomials, and then compare these to known results. For the  $(2, 2p + 1)$  knots this has been done in [Sam12, Lemma 6.4.7] for all  $n$ , and for the figure eight knot this has been done for many small  $n$ . (See also the explicit computations in the appendix.)

We now establish a few technical lemmas. We recall that if  $m, l \in K_q(T^2)$  are the meridian and longitude, respectively, and  $z$  is the  $(1, 1)$  curve, then, under the embedding  $K_q(T^2) \hookrightarrow A_q \rtimes \mathbb{Z}_2$ , we have

$$m \mapsto x := X + X^{-1}, \quad l \mapsto y := Y + Y^{-1}, \quad z \mapsto q^{-1}(XY + X^{-1}Y^{-1}).$$

As an algebra, the image of  $K_q(T^2)$  is generated by  $x, y$ , and  $z$ . We now prove a lemma that is useful for constructing isomorphisms of  $A_q^{\mathbb{Z}_2}$ -modules: it essentially says that the  $A_q^{\mathbb{Z}_2}$ -module structure of  $M$  is determined by the actions of  $y$  and  $z$  on a  $\mathbb{C}[x]$ -basis for  $M$ .

**LEMMA 4.4.** *Suppose that  $M$  and  $N$  are modules over  $A_q^{\mathbb{Z}_2}$ , and that as a  $\mathbb{C}[x]$ -module,  $M$  is generated by elements  $\{m_i\} \subset M$ . Furthermore, suppose that  $f : M \rightarrow N$  is an isomorphism of  $\mathbb{C}[x]$ -modules that satisfies  $f(y m_i) = y f(m_i)$  and  $f(z m_i) = z f(m_i)$ . Then  $f$  is an isomorphism of  $A_q^{\mathbb{Z}_2}$ -modules.*

*Proof.* From (2.2), the elements  $x, y, z \in A_q^{\mathbb{Z}_2}$  satisfy the commutation relations

$$[x, y]_q = (q^2 - q^{-2})z, \quad [z, x]_q = (q^2 - q^{-2})y, \quad [y, z]_q = (q^2 - q^{-2})x$$

(where we have used the notation  $[a, b]_q := qab - q^{-1}ba$ ). An arbitrary element of  $M$  can be written as  $m = \sum_{i=1}^n p_i(x)m_i$  and, using the  $\mathbb{C}[x]$ -linearity of  $f$  and the commutation relations, powers of  $x$  in the expressions  $ym$  and  $zm$  can inductively be moved to the left. This shows that  $f(ym) = yf(m)$  and  $f(zm) = zf(m)$  for arbitrary  $m \in M$ , which completes the proof.  $\square$

We also give a lemma which is useful for explicitly constructing modules over  $A_q \rtimes \mathbb{Z}_2$ . Let  $M = \mathbb{C}[X^{\pm 1}] \otimes_{\mathbb{C}} V$  and define operators  $S, P : M \rightarrow M$  via the formulas

$$S \cdot (f(X) \otimes v) := f(X^{-1}) \otimes v, \quad P \cdot (f(X) \otimes v) := f(q^{-2}X) \otimes v. \tag{4.1}$$

Then the operators  $X, S$ , and  $P$  satisfy the relations of  $A_q \rtimes \mathbb{Z}_2$  (and, with these operators,  $M$  is a direct sum of copies of the standard polynomial representation of  $A_q \rtimes \mathbb{Z}_2$ ). Furthermore, if  $A(X) \in \text{End}_{\mathbb{C}[X^{\pm 1}]}(M)$ , then  $SA(X) = A(X^{-1})S$  and  $PA(X) = A(q^{-2}X)P$  (where the equalities are inside  $\text{End}_{\mathbb{C}}(M)$ ).

LEMMA 4.5. *Suppose that  $A(X), B(X) \in \text{End}_{\mathbb{C}[X^{\pm 1}]}(M)$  satisfy*

$$B(X^{-1})B(X) = \text{Id}_M, \quad A(X)B(q^{-2}X)A(q^2X^{-1}) = B(X).$$

*Then the operators  $X, s := B(X)S$ , and  $Y := A(X)P$  endow  $M$  with the structure of an  $A_q \rtimes \mathbb{Z}_2$ -module.*

*Proof.* The relation  $XY = q^2YX$  follows from the fact that  $X$  commutes with  $A(X)$  and the fact that  $XP = q^2PX$ . Since  $X$  commutes with  $B(X)$ , we see that  $XsX = s$ . The relation  $s^2 = 1$  follows from the fact that  $B(X^{-1})B(X) = 1$ . For the final relation, we compute that  $YsY$  is equal to

$$A(X)PB(X)SA(X)P = A(X)B(q^{-2}X)A(q^2X^{-1})PSP = A(X)B(q^{-2}X)A(q^2X^{-1})S = s. \quad \square$$

In the following sections, we will frequently use the element

$$\delta := X - X^{-1} \in A_q \rtimes \mathbb{Z}_2.$$

We will also use the Chebyshev polynomials  $S_n, T_n \in \mathbb{C}[x]$ , which are defined by

$$\begin{aligned} S_0 &= 1, & S_1 &= x, & S_{n+1} &= xS_n - S_{n-1}, \\ T_0 &= 2, & T_1 &= x, & T_{n+1} &= xT_n - T_{n-1}. \end{aligned}$$

LEMMA 4.6. *The Chebyshev polynomials satisfy the identities*

$$(X - X^{-1})S_n(X + X^{-1}) = X^{n+1} - X^{-n-1} \quad \text{and} \quad T_n(X + X^{-1}) = X^n + X^{-n}.$$

**4.1 The unknot**

Let  $K \subset S^3$  be the unknot, so that  $S^3 \setminus K$  is a solid torus. Then  $K_q(S^3 \setminus K) \cong \mathbb{C}[u]1_K$ , where  $1_K \in K_q(S^3 \setminus K)$  is the empty link. The action of  $K_q(T^2)$  on  $K_q(S^3 \setminus K)$  is given by

$$\begin{aligned} x \cdot f(u)1_K &= uf(u)1_K, \\ y \cdot 1_K &= (-q^2 - q^{-2})1_K, \\ z \cdot 1_K &= -q^{-3}u1_K. \end{aligned} \tag{4.2}$$

(The image of the longitude inside the solid torus is contractible, and the  $-q^{-3}$  factor in the third formula comes from the framing of the image of the  $(1, 1)$  curve inside the solid torus.)

We give the  $\mathbb{C}[X^{\pm 1}]$ -module  $\hat{M} := \mathbb{C}[X^{\pm 1}]$  the structure of an  $A_q \rtimes \mathbb{Z}_2$ -module via the formulas

$$Y \cdot f(X) := -f(q^{-2}X), \quad s \cdot f(X) := -f(X^{-1}). \tag{4.3}$$

The module  $\hat{M}$  is called the *sign representation* of  $A_q \rtimes \mathbb{Z}_2$ . As a  $\mathbb{Z}_2$ -module, we have the decomposition  $\hat{M} \cong \mathbb{C}[x] \oplus \mathbb{C}[x]\delta$ . Since  $s \cdot 1 = -1$ , we see that  $\mathbf{e}\hat{M} = \mathbb{C}[x]\delta$  as a  $\mathbb{C}[x]$ -module.

LEMMA 4.7. *The  $\mathbb{C}[x]$ -isomorphism  $f : \mathbf{e}\hat{M} \rightarrow K_q(S^3 \setminus K)$  defined by  $f(\delta) = 1_K$  is an isomorphism of  $A_q^{\mathbb{Z}_2}$ -modules. In particular, the skein module of the unknot is the (symmetric) sign representation.*

*Proof.* By Lemma 4.4, the following computations show the claim:

$$\begin{aligned} y \cdot \delta &= (Y + Y^{-1})(X - X^{-1}) = -(q^2 - q^{-2})\delta, \\ z \cdot \delta &= q^{-1}(XY + X^{-1}Y^{-1})(X - X^{-1}) = -q^{-3}(X + X^{-1})\delta. \end{aligned} \quad \square$$

**4.2 The trefoil**

Let  $M = K_q(S^3 \setminus K)$  be the skein module of the complement of the trefoil knot. In [Gel02], Gelca showed that  $M$  is free and finitely generated as a module over the meridian subalgebra  $\mathbb{C}[x]$ . (This was also shown in [Lê06] and [BLF05].) Gelca’s generators are  $w', 1_K \in M$ , where  $1_K$  is the empty link and  $w'$  is the loop labelled  $d$  in Figure 2. However, the vector  $w := w' + q^{-2}1_K$  generates a proper submodule of  $M$ , so it is easiest to describe the  $A_q^{\mathbb{Z}_2}$ -module structure in terms of the basis  $w, 1_K$ . Translating his formulas for the actions of  $y$  and  $z$  into this basis, we get

$$\begin{aligned} y \cdot w &= -(q^2 + q^{-2})w, \\ z \cdot w &= -q^{-3}S_1(x)w, \\ y \cdot 1_K &= (q^6S_4(x) - q^2)w + q^6T_6(x)1_K, \\ z \cdot 1_K &= q^5S_3(x)w + q^5T_5(x)1_K. \end{aligned}$$

(This follows from [Gel02], Lemmas 3 and 7 for  $q = 0$  and  $q = 1$ . The parameter  $q$  in [Gel02] is an integer, unrelated to our  $q$ , and the parameter  $t$  is our  $q$ .)

Remark 4.8. From these formulas, it is clear that  $w$  generates an  $A_q^{\mathbb{Z}_2}$ -submodule of  $M$ . Comparing to formula (4.2), we see that this submodule is isomorphic to the skein module of the unknot. (The isomorphism is determined by sending the empty link to  $w \in M$ .)

To describe the nonsymmetric module, we first define a  $\mathbb{C}[X^{\pm 1}]$ -module  $\hat{M}$  via

$$\hat{M} \cong \mathbb{C}[X^{\pm 1}]u \oplus \mathbb{C}[X^{\pm 1}]v. \tag{4.4}$$

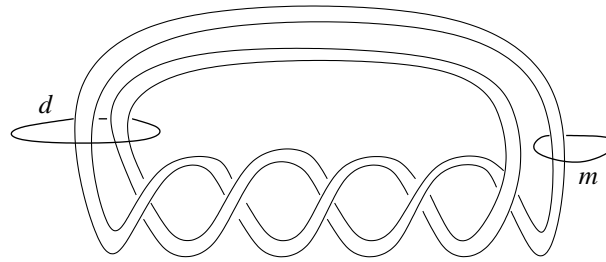


FIGURE 2. Generators for the  $(2, 2p + 1)$  knot.

(With this notation,  $v$  will be identified with the empty link.) Let  $P, S : \hat{M} \rightarrow \hat{M}$  be the operators given by (4.1). We then define an  $A_q \rtimes \mathbb{Z}_2$ -module structure on  $\hat{M}$  via the following matrices (written with respect to the ordered basis  $u, v$ ):

$$s = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} S, \quad Y = \begin{bmatrix} -1 & q^2 X^{-1} - q^6 X^{-5} \\ 0 & q^6 X^{-6} \end{bmatrix} P. \tag{4.5}$$

It is easy to check that these matrices satisfy the conditions of Lemma 4.5, which implies that they define a representation of  $A_q \rtimes \mathbb{Z}_2$ . Explicitly, the action of  $Y$  on  $\hat{M}$  is given by the formulas

$$Y \cdot u = -u, \quad Y \cdot v = (q^2 X^{-1} - q^6 X^{-5})u + q^6 X^{-6}v. \tag{4.6}$$

Since  $s$  acts diagonally in the basis  $(u, v)$ , there is a  $\mathbb{C}[x]$ -module isomorphism  $\mathbf{e}\hat{M} = \mathbb{C}[x]\delta u \oplus \mathbb{C}[x]v$ .

LEMMA 4.9. *The  $\mathbb{C}[x]$  isomorphism  $f : \mathbf{e}\hat{M} \rightarrow M$  determined by  $f(\delta u) = w$  and  $f(v) = 1_K$  is an isomorphism of  $A_q^{\mathbb{Z}_2}$ -modules.*

*Proof.* Lemma 4.4 reduces this to several straightforward computations. For example,

$$\begin{aligned} y \cdot v &= (Y + Y^{-1})v = (Y + sYs)v = (1 + s)Yv \\ &= (1 + s)[(q^2 X^{-1} - q^6 X^{-5})u + q^6 X^{-6}v] \\ &= [q^2(X^{-1} - X) - q^6(X^{-5} - X^5)]u + q^6(X^{-6} + X^6)v \\ &= [-q^2 + q^6 S_4(x)]\delta u + q^6 T_6(x)v. \end{aligned}$$

(In the last step we used Lemma 4.6.) □

The next lemma describes the structure of  $M$  in terms of standard (induced) modules of  $A_q \rtimes \mathbb{Z}_2$ . Let  $\tau : A_q \rtimes \mathbb{Z}_2 \rightarrow A_q \rtimes \mathbb{Z}_2$  denote the automorphism

$$\tau(X) = X, \quad \tau(s) = s, \quad \tau(Y) = q^{-1}XY.$$

Let  $V^-$  be the sign representation of  $A_q \rtimes \mathbb{Z}_2$  (i.e. the nonsymmetric skein module of the unknot) and let  $V^+$  be the standard polynomial representation.

LEMMA 4.10. *The module  $M$  admits a decomposition into a nonsplit exact sequence*

$$0 \rightarrow V^- \rightarrow M \rightarrow \tau^{-6}(V^+) \rightarrow 0,$$

where  $\tau^N(V^+)$  is the twist of  $V^+$  by  $\tau^N$ . If  $q$  is not a root of unity, then  $V^-$  is the unique nontrivial submodule of  $M$ .

*Proof.* The existence of this short exact sequence is clear because in our chosen basis (4.4) the operators  $S, P, s, Y,$  and  $X$  all act by upper-triangular matrices. We have already identified the submodule with  $V^-$ , and the identification of the quotient is clear by (4.5). If this sequence were split, there would exist an  $m \in M$  with  $m = v + f(x)u$  and  $(q^6X^{-6} - Y) \cdot m = 0$  (since this equation holds in the quotient). However, this equation implies that  $q^6X^{-6}f(X) + f(q^{-2}X) = q^6X^{-5} - q^2X^{-1}$ , and this is impossible because the total degree of the left-hand side is at least 6, while the total degree of the right-hand side is 4.

If  $q$  is not a root of unity, then in the standard polynomial representation  $V^+$  the element  $X^k$  is a  $\mathbb{C}$ -basis for the kernel of the operator  $Y - q^{-2k}$ . This implies that  $V^+$  is simple, which implies that  $V^-$  and  $\tau(V^+)$  are simple. Then the final claim follows from the general fact that a nonsplit extension of two simple modules has a unique nontrivial submodule.  $\square$

### 4.3 (2, 2p + 1) torus knots

In this subsection, we recall the calculations of Gelca and Sain [GS03] for the  $(2, 2p + 1)$  torus knot  $K_p$ . We also extend their calculations to completely determine the module structure of the submodule of  $K_q(S^3 \setminus K_p)$  generated by the empty link  $1_K \in K_q(S^3 \setminus K_p)$ , and we give an explicit presentation of the nonsymmetric version of this module.

In [GS03], the authors proved that there is an isomorphism of  $\mathbb{C}[x]$ -modules

$$K_q(S^3 \setminus K_p) \cong \bigoplus_{i=0}^p \mathbb{C}[x]v^i.$$

Here  $v$  is the loop labelled  $d$  in Figure 2 and  $v^i$  is  $i$  parallel copies of  $v$  (and  $v^0 = 1_K$  is the empty link, by convention). We define the element

$$w := S_{p-1}(v) + q^{-2}S_p(v) \in K_q(S^3 \setminus K_p).$$

Then Gelca and Sain proved the following lemma.

LEMMA 4.11 [GS03, Proposition 4.4].

$$\begin{aligned} y \cdot 1_K &= q^{4p+2}T_{4p+2}1_K + (-1)^{p+1}q^{2p+2}(q^2S_{2p+2} - q^{-2}S_{2p-2})w, \\ z \cdot 1_K &= q^{4p+1}T_{4p+1}1_K + (-1)^{p+1}q^{2p+2}(qS_{2p+1} - q^{-3}S_{2p-3})w. \end{aligned} \tag{4.7}$$

We extend their computations with the following lemma.

LEMMA 4.12. *The action of  $K_q(T^2)$  on  $w \in K_q(S^3 \setminus K_p)$  is determined by*

$$\begin{aligned} y \cdot w &= -(q^2 + q^{-2})w, \\ z \cdot w &= -q^{-3}xw. \end{aligned} \tag{4.8}$$

*Proof.* (In [GS03], the convention for the isomorphism of Theorem 2.11 is different from ours. In particular, in their convention the image of  $(m, n)_T \in K_q(T^2)$  in  $A_q^{\mathbb{Z}_2}$  is  $q^{mn}(X^mY^{-n} + X^{-m}Y^n)$ . In this proof only, we will follow their convention.) We first note that proving equations (4.8) is equivalent to proving the following:

$$\begin{aligned} (1, -4p - 2)_T \cdot w &= -q^{4p-2}(q^{-2}S_{4p+2} - q^2S_{4p})w, \\ (1, -4p - 1)_T \cdot w &= -q^{4p-1}(q^{-2}S_{4p+1} - q^2S_{4p-1})w. \end{aligned} \tag{4.9}$$

To see this, we first note that the proof of Lemma 4.15 (along with a short calculation) shows that both sets of equations (4.8) and (4.9) hold inside the  $A_q^{\mathbb{Z}_2}$ -module  $\mathbf{e}\hat{M}$  (which is defined via

the operators in (4.11)). Then an appropriate Dehn twist of  $T^2$  provides an automorphism of  $A_q^{\mathbb{Z}_2}$  that fixes  $x$  and sends  $y, z$  to the elements  $(1, -4p-1)_T$  and  $(1, -4p-2)_T$ , respectively. Therefore, the proof of Lemma 4.4 shows that either set of equations (together with Lemma 4.11) completely determines the  $A_q^{\mathbb{Z}_2}$ -module structure of  $\mathbf{e}\hat{M}$ . (We note that  $\mathbf{e}\hat{M}$  is actually an  $A_q^{\mathbb{Z}_2}$ -module by Lemma 4.14.)

To prove the first equation of (4.9), we follow the strategy of [GS03, Proposition 4.1]. Namely, if we remove from  $S_3 \setminus K_p$  a regular neighborhood of the Möbius band that is bounded by the knot, then the resulting 3-manifold is a solid torus which contains both the  $(1, -4p-2)_T$  curve and the element  $w$  in its interior. Therefore, the left-hand side of the first equation in (4.9) can be simplified inside the skein module of the solid torus as follows, where we abbreviate  $S_k := S_k(v)$ :

$$(2p+1, -2)_T \cdot [S_{p-1} + q^{-2}S_p] = q^{-4p-2}[q^{-4p}S_{3p} + q^{-4p-6}S_{3p+1} - q^{4p}S_p - q^{4p+2}S_{p-1}]. \tag{4.10}$$

(The image in the solid torus of the  $(1, -4p-2)_T$  curve on the original torus is the same as the image of the  $(2p+1, -2)_T$  curve on the torus which bounds the solid torus, and the  $v$  curve in the original knot complement is the image of the longitude of the boundary of the solid torus.) Then the right-hand side of the first equation in (4.9) can be simplified using [GS03, Theorem 3.1], and this agrees with the right-hand side of equation (4.10). (To make the powers of  $q$  match exactly, note that the rightmost parenthesized expression of [GS03, Theorem 3.1] is  $qw$ .) This completes the proof of the first equation of (4.9).

The proof of the second equation in (4.9) is more lengthy, so we include a sketch and leave the details for the interested reader. The strategy is to follow the proof of [GS03, Proposition 4.3]. To prove this, the authors define two sequences of skeins  $a_k, b_k \in K_q(S^3 \setminus K_p)$  so that  $q^{-6p}a_{2p+1} = (1, -4p-1)_T \cdot 1_K$ . These sequences of skeins can be modified in a straightforward way to obtain sequences  $a'_k, b'_k$  with  $q^{-6p}a'_{2p+1} = (1, -4p-1)_T \cdot w$ . The authors then show that the sequence  $b_k$  satisfies a second-order recurrence that can be solved explicitly, and the sequence  $a_k$  satisfies a first-order recurrence with an inhomogeneous term depending on  $b_k$ , which can also be solved explicitly. Then [GS03, Theorem 3.1] allows the simplification of this explicit expression to obtain the second formula of Lemma 4.11. In a similar way, the sequences  $a'_k$  and  $b'_k$  can be written explicitly to obtain the second formula of (4.9).  $\square$

We define the submodule  $M \subset K_q(S^3 \setminus K_p)$  by

$$M := K_q(T^2) \cdot 1_K.$$

**COROLLARY 4.13.** *We have equality of subspaces  $M = K_q(T^2) \cdot 1_K = \mathbb{C}[x]1_K + \mathbb{C}[x]w$ .*

*Proof.* This is straightforward from the formulas (4.7) and (4.8).  $\square$

We now describe the  $A_q \rtimes \mathbb{Z}_2$ -module  $\hat{M}$  that satisfies  $\mathbf{e}\hat{M} \cong M$ . As in the case of the trefoil, we define the  $\mathbb{C}[X^{\pm 1}]$ -module structure first:

$$\hat{M} := \mathbb{C}[X^{\pm 1}]u \oplus \mathbb{C}[X^{\pm 1}]v.$$

(In this notation, the empty link is identified with  $v$ .) We define operators  $P, S : \hat{M} \rightarrow \hat{M}$  using the formula (4.1). Then we define the actions of  $Y$  and  $s$  via the following operators (which are written with respect to the ordered basis  $(u, v)$ ):

$$s = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} S, \quad Y = \begin{bmatrix} -1 & (-1)^p q^{2p+4} (X^{-2p-3} - q^{-4} X^{-2p+1}) \\ 0 & q^{2(2p+1)} X^{-2(2p+1)} \end{bmatrix} P. \tag{4.11}$$

LEMMA 4.14. *The formulas (4.11) give  $\hat{M}$  an  $A_q \rtimes \mathbb{Z}_2$ -module structure.*

*Proof.* This follows from a straightforward calculation and Lemma 4.5. □

Explicitly, the action of  $Y$  on  $\hat{M}$  is given by

$$Y \cdot u = -u, \quad Y \cdot v = [(-1)^p q^{2p+4}(X^{-2p-3} - q^{-4}X^{-2p+1})]u + q^{2(2p+1)}X^{-2(2p+1)}v.$$

As before,  $s$  acts diagonally, which gives a decomposition  $e\hat{M} = \mathbb{C}[x]\delta u \oplus \mathbb{C}[x]v$ .

LEMMA 4.15. *The  $\mathbb{C}[x]$ -module isomorphism  $f : e\hat{M} \rightarrow M$  given by  $f(\delta u) = w$  and  $f(v) = 1_K$  is an isomorphism of  $A_q^{\mathbb{Z}_2}$ -modules.*

*Proof.* The element  $u \in \hat{M}$  generates a proper submodule of  $\hat{M}$ , and it is clear from Lemma 4.7 that the restriction of  $f$  to this submodule is an isomorphism. We then compute

$$\begin{aligned} y \cdot v &= (Y + sYs)v = (1 + s)Yv \\ &= (1 + s) [(-1)^p q^{2p+4}(X^{-2p-3} - q^{-4}X^{-2p+1})u + q^{4p+2}X^{-4p-2}v] \\ &= (-1)^p q^{2p+4}(X^{-2p-3} - X^{2p+3} + q^{-4}X^{2p-1} - q^{-4}X^{-2p+1})u + q^{4p+2}(X^{-4p-2} + X^{4p+2})v \\ &= (-1)^{p+1} q^{2p+4}(-q^{-4}S_{2p-2} + S_{2p+2})\delta u + q^{4p+2}T_{4p+2}v. \end{aligned}$$

(In the last step we have used Lemma 4.6.) This shows that  $f(y \cdot v) = y \cdot f(v)$ . A similar computation shows that  $f(z \cdot v) = z \cdot f(v)$ , and an application of Lemma 4.4 completes the proof. □

#### 4.4 The figure eight knot

Let  $M = K_q(S^3 \setminus K)$  be the skein module of the complement of the figure eight knot. First, we recall some facts from [GS04] (translated into our notation). As  $\mathbb{C}[x]$ -modules, we have an isomorphism

$$M \cong \mathbb{C}[x]u \oplus \mathbb{C}[x]v \oplus \mathbb{C}[x]w.$$

Under this identification, the empty link is  $u \in M$  and, if  $v', w' \in M$  are the loops labelled  $y$  and  $z$  (respectively) in Figure 3, then  $v = q^2v' + u$  and  $w = q^{-2}w' + u$ . Gelca and Sain then give the following formulas to describe the module structure.

LEMMA 4.16 [GS04]. *The actions of  $y$  and  $z$  on  $M$  are determined by the formulas*

$$\begin{aligned} y \cdot u &= (q^2 + q^{-2})S_2u + (q^2S_2 + q^{-2})v + (q^2 + q^{-2}S_2)w, \\ y \cdot v &= (-q^6S_4 + q^2)u + (-q^6S_4 + q^2S_2)v + (-q^6S_2 - q^2)w, \\ y \cdot w &= (q^{-2} - q^{-6}S_4)u + (-q^{-2} - q^{-6}S_2)v + (q^{-2}S_2 - q^{-6}S_4)w, \\ z \cdot u &= (qS_1 + q^{-3}S_3)u + (q + q^{-3})S_1v + q^{-3}S_3w, \\ z \cdot v &= -q^5S_3u + (qS_1 - q^5S_3)v + (-q^5 - q)S_1w, \\ z \cdot w &= (q^{-3}S_1 - q^{-7}S_5)u - q^{-7}S_3v + (q^{-3}S_3 - q^{-7}S_5)w. \end{aligned}$$

As in the case of the trefoil, there is a proper submodule of  $M$  which is isomorphic to the skein module of the unknot. In this case it is generated by the element

$$p := (x^2 - 3)u + v + w.$$



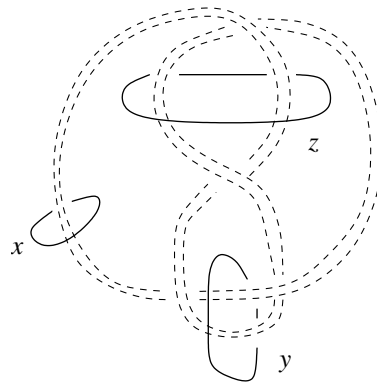


FIGURE 3. Generators for the figure eight knot.

LEMMA 4.17. We have equalities

$$y \cdot p = (-q^2 - q^{-2})p, \quad z \cdot p = -q^{-3}xp.$$

*Proof.* This is an entertaining but lengthy computation, which we omit. □

As before, it is convenient to describe the module structure of  $M$  using the  $\mathbb{C}[x]$ -basis  $p, u, v$ .

LEMMA 4.18. As  $\mathbb{C}[x]$ -modules, we have an isomorphism  $M \cong \mathbb{C}[x]p \oplus \mathbb{C}[x]u \oplus \mathbb{C}[x]v$ , and the actions of  $y$  and  $z$  in this basis are given by

$$\begin{aligned} y \cdot p &= -(q^2 + q^{-2})p, \\ z \cdot p &= -q^{-3}xp, \\ y \cdot u &= (q^2 + q^{-2}S_2)p + [-q^{-2}T_4 + q^{-2}T_2 + q^2T_0]u + (q^2 - q^{-2})T_2v, \\ z \cdot u &= q^{-3}S_3p + (-q^{-3}T_5 + q^{-3}T_3 + qT_1)u + (-q^{-3}T_3 + qT_1)v, \\ y \cdot v &= (-q^6S_2 - q^2)p + (q^2 - q^6)T_2u + (-q^6T_4 + q^2T_2 + q^2T_0)v, \\ z \cdot v &= (-q^5 - q)S_1p + (qT_3 - q^5T_1)u + (-q^5T_3 + 2qT_1)v. \end{aligned}$$

*Proof.* This is a straightforward calculation. □

We now define the  $A_q \rtimes \mathbb{Z}_2$ -module  $\hat{M}$  that satisfies  $\mathbf{e}\hat{M} \cong M$ . As a  $\mathbb{C}[X^{\pm 1}]$ -module, we define

$$\hat{M} := \mathbb{C}[X^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{C}\{p', u, v\}.$$

(We have slightly abused notation by reusing the letters  $u, v$ , but this is justified by the fact that the inclusion  $K_q(S^3 \setminus K) \rightarrow \hat{M}$  identifies  $u, v \in K_q(S^3 \setminus K)$  with  $u, v \in \hat{M}$ . In particular, with this notation, the empty link is identified with  $u$ .) We use formula (4.1) to define operators  $S, P : \hat{M} \rightarrow \hat{M}$ . As before, we will define the  $A_q \rtimes \mathbb{Z}_2$ -module structure on  $\hat{M}$  using matrices (with respect to the ordered basis  $p', u, v$ ). To write these matrices in a more compact form, we define the following polynomials in  $\mathbb{C}[X^{\pm 1}]$ :

$$a := -q^{-2}X^4 + q^{-2}X^2 + q^2, \quad b := -q^{-2}X^2 + q^2X^{-2}, \quad c := q^{-2}X^3 - q^2X^{-1}.$$

To shorten notation further in what follows, if  $f \in \mathbb{C}[X^{\pm 1}]$ , we write  $f'(X) := f(q^2X^{-1})$  (note that  $f'' = f$ ). We then define  $s, Y \in \text{End}_{\mathbb{C}}(\hat{M})$  as follows:

$$s := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S, \quad Y := \begin{bmatrix} -1 & c & q^2c' \\ 0 & a & q^4b' \\ 0 & b & a' \end{bmatrix} P. \tag{4.12}$$

LEMMA 4.19. *The operators in (4.12) define a representation of  $A_q \rtimes \mathbb{Z}_2$ .*

*Proof.* Let  $B$  and  $A(X)$  be the matrices in (4.12). The first condition of Lemma 4.5 clearly holds, so we are reduced to checking that  $A(X)BA(q^2X^{-1}) = B$ . We compute

$$\begin{bmatrix} -1 & c & q^2c' \\ 0 & a & q^4b' \\ 0 & b & a' \end{bmatrix} B \begin{bmatrix} -1 & c' & q^2c \\ 0 & a' & q^4b \\ 0 & b' & a \end{bmatrix} = \begin{bmatrix} -1 & c' + a'c + q^2b'c' & q^2(c + q^2bc + ac') \\ 0 & aa' + q^4b'b' & q^4(ab + ab') \\ 0 & a'b + a'b' & q^4bb + aa' \end{bmatrix}.$$

The fact that the matrix on the right is the matrix  $B$  follows from the identities

$$b' = -b, \quad c' = -q^2X^{-2}c, \quad aa' - q^4bb' = 1, \quad a'c + q^2b'c' + c' = 0. \quad \square$$

Explicitly, the action of  $Y$  on  $\hat{M}$  is given by

$$\begin{aligned} Y \cdot p' &= -p', \\ Y \cdot u &= [q^{-2}X^3 - q^2X^{-1}]p' + [-q^{-2}X^4 + q^{-2}X^2 + q^2]u + [-q^{-2}X^2 + q^2X^{-2}]v, \\ Y \cdot v &= [q^6X^{-3} - q^2X]p' + [q^2X^2 - q^6X^{-2}]u + [-q^6X^{-4} + q^2X^{-2} + q^2]v. \end{aligned}$$

Again, since  $s$  acts diagonally, we see that  $\mathbf{e}\hat{M} = \mathbb{C}[x]\{\delta p', w, v\}$ .

LEMMA 4.20. *The  $\mathbb{C}[x]$ -isomorphism  $f : \mathbf{e}\hat{M} \rightarrow M$  defined by  $f(\delta p') = p$ ,  $f(u) = u$ , and  $f(v) = v$  is an isomorphism of  $A_q^{\mathbb{Z}_2}$ -modules.*

*Proof.* This is a straightforward computation, which is quite similar to the proof of Lemma 4.9. □

*Remark 4.21.* The lower-right  $2 \times 2$  block of the matrix defining the action of  $Y$  in formula (4.12) appeared in [CM15a, Proposition 4.5], where it was used to describe an inhomogeneous recursion relation satisfied by the sequence of colored Jones polynomials for the figure eight knot.

*Remark 4.22.* The module  $\hat{M}$  fits into a short exact sequence

$$0 \rightarrow N \rightarrow \hat{M} \rightarrow Q \rightarrow 0,$$

where  $N$  is the unknot submodule and  $Q$  is the quotient. It is not too difficult to see that  $Q$  is simple, as long as  $q$  is generic. Furthermore,  $\hat{M}$  is a nontrivial extension, so, since  $N$  is also simple,  $\hat{M}$  is generated by the empty link  $u$  (as long as  $q$  is generic).

### 5. Divisibility and recursion relations of colored Jones polynomials

If  $q^4 - 1$  is invertible, then  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2}$  are Morita equivalent, so there is a unique  $A_q \rtimes \mathbb{Z}_2$ -module  $\hat{K}_q(S^3 \setminus K)$  such that  $\mathbf{e}\hat{K}_q(S^3 \setminus K)$  is isomorphic to  $K_q(S^3 \setminus K)$  as an  $A_q^{\mathbb{Z}_2}$ -module. We call  $\hat{K}_q(S^3 \setminus K)$  the ‘nonsymmetric skein module’, and we give a formula (5.2) for the colored Jones polynomials in terms of  $\hat{K}_q(S^3 \setminus K)$ .

Garoufalidis and Lê [GL05] defined an action of  $A_q \rtimes \mathbb{Z}_2$  on the space  $\mathbf{H} := \text{Hom}(\mathbb{Z}, \mathbb{C}[q^{\pm 1}])$  of sequences of Laurent polynomials (see (5.3)). If we define  $J_{-n}(q) = -J_n(q)$ , then we can consider  $J_n(q)$  as an element of  $\mathbf{H}$ , and the main theorem of [GL05] is that the annihilator of  $J_n(q)$  in  $A_q$  is nonzero. In other words, the sequence  $J_n(q)$  satisfies a (generalized) recurrence relation.

Lemmas 5.6 and 5.7 give a close relationship between the  $A_q \rtimes \mathbb{Z}_2$ -modules  $\hat{K}_q(D^2 \times S^1)$  and  $\mathbf{H}$ : the latter is the linear dual of the former (up to a twist by an automorphism; see Remark 5.9).

We use this observation to give two applications of our conjectures. First, Conjecture 2 states that  $K_q(S^3 \setminus K)$  contains  $K_q(S^3 \setminus \text{unknot})$  as a submodule. (In particular, the action of the longitude on the skein module of the knot complement has an eigenvector with eigenvalue  $-q^2 - q^{-2}$ .) In Theorem 5.10, we show that this conjecture implies that the colored Jones polynomials of a knot satisfy a (generalized) inhomogeneous recursion relation. These generalized inhomogeneous recursion relations have been found in examples, but when this paper was written it had not been proved that they exist for all knots.

Second, the algebra  $\mathcal{H}_{q,t_1,t_2,1,1}$  acts on the space  $\mathbf{H}(q) := \text{Hom}(\mathbb{Z}, \mathbb{C}(q))$  of sequences of rational functions. We show that our conjectured action of  $\mathcal{H}_{q,t_1,t_2,1,1}$  on the nonsymmetric skein module of the complement of a knot  $K$  implies that  $a \cdot (J_n(q)) \in \mathbb{C}(q)\mathbf{H} \subset \mathbf{H}(q)$  for any  $a \in \mathcal{H}_{q,t_1,t_2,1,1}$ . In other words, if  $P_n(q)$  is a sequence of rational functions obtained by multiplying  $J_n(q)$  by an element of  $\mathcal{H}_{q,t_1,t_2,1,1}$ , then there is a uniform common denominator for the rational functions in the sequence  $P_n(q)$ . (This is a nontrivial statement, because the action of  $\mathcal{H}_{q,t_1,t_2,1,1}$  does not preserve the subspace  $\mathbb{C}(q)\mathbf{H} \subset \mathbf{H}(q)$ .)

As a corollary, we show a divisibility property for the colored Jones polynomials: the rational function

$$\frac{(q - q^{-1})(J_{n+j}(q) + J_{n-1-j}(q))}{1 - q^{4n-2}}$$

is actually a Laurent polynomial. Following a suggestion of Garoufalidis, we use Habiro’s cyclotomic expansion of the colored Jones polynomials to give a proof of this statement that does not rely on Conjecture 1.

### 5.1 Nonsymmetric pairings

In this section, we show that the topological pairing (2.4) lifts via the Morita equivalence between  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2}$ , and we give a formula for the colored Jones polynomials in terms of this nonsymmetric pairing.

5.1.1 *Pairings and Morita equivalence.* We first give a lemma that gives a sufficient condition for  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2}$  to be Morita equivalent.

LEMMA 5.1. *Suppose that  $q^4 - 1$  is invertible. Then  $A_q \rtimes \mathbb{Z}_2$  and  $A_q^{\mathbb{Z}_2}$  are Morita equivalent.*

*Proof.* Recall that  $\mathbf{e} = (1 + s)/2 \in A_q \rtimes \mathbb{Z}_2$  and that  $A_q^{\mathbb{Z}_2}$  is isomorphic to  $\mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e}$  via the map  $a \mapsto a\mathbf{e}$ . Standard Morita theory shows that  $A_q \rtimes \mathbb{Z}_2$  and  $\mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e}$  are Morita equivalent if the two-sided ideal generated by  $\mathbf{e}$  contains 1. Then the following computation completes the proof:

$$[Y, s[X, 1+s]]_q Y^{-1} X = q^{-1} - q^3. \quad \square$$

We now give a lemma showing that pairings lift via Morita equivalences. Let  $A$  be an algebra with an idempotent  $\mathbf{e}^2 = \mathbf{e} \in A$ .

LEMMA 5.2. *Suppose that  $A\mathbf{e}A = A$  and let  $M\mathbf{e}$  and  $\mathbf{e}N$  be left and right  $\mathbf{e}A\mathbf{e}$ -modules, respectively. Then the natural map of vector spaces  $M\mathbf{e} \otimes_{\mathbf{e}A\mathbf{e}} \mathbf{e}N \rightarrow M \otimes_A N$  is an isomorphism.*

*Proof.* Since  $AeA = A$ , the functors  $Ae \otimes_{eAe} -$  and  $eA \otimes_A -$  are inverse equivalences, so the natural map  $Ae \otimes_{eAe} eA \otimes_A A \rightarrow A$  given by  $ae \otimes eb \otimes c \mapsto aebc$  is an isomorphism. We then have the following isomorphisms of vector spaces:

$$\begin{aligned} Me \otimes_{eAe} eN &\cong (M \otimes_A Ae) \otimes_{eAe} (eA \otimes_A N) \\ &\cong M \otimes_A (Ae \otimes_{eAe} eA) \otimes_A N \\ &\cong M \otimes_A A \otimes_A N \\ &\cong M \otimes_A N. \end{aligned}$$

Under this chain of isomorphisms, we have  $me \otimes en \mapsto me \otimes en$ . □

**COROLLARY 5.3.** *If  $q^4 - 1$  is invertible, then there is a unique (nonsymmetric) pairing*

$$\langle -, - \rangle : \hat{K}_q(D^2 \times S^1) \otimes_{A_q \rtimes \mathbb{Z}_2} \hat{K}_q(S^3 \setminus K) \rightarrow \mathbb{C},$$

which lifts the topological pairing of § 2.2.3.

5.1.2 *The nonsymmetric skein module of the solid torus.* Recall that under the isomorphism  $K_q(T^2) \rightarrow A_q^{\mathbb{Z}_2}$ , the meridian and longitude are sent to  $X + X^{-1}$  and  $Y + Y^{-1}$ , respectively, and the  $(1, 1)$  curve is sent to  $q^{-1}(XY + X^{-1}Y^{-1})$ . If  $N_K \subset S^3$  is a closed tubular neighborhood of a knot  $K \subset S^3$ , then  $N_K$  is diffeomorphic to  $D^2 \times S^1$  and  $K_q(N_K)$  is a right  $K_q(T^2)$ -module. Let  $u = 1_K \cdot (Y + Y^{-1}) \in K_q(N_K)$ , where  $1_K$  is the empty link in  $N_K$ .

**LEMMA 5.4.** *As vector spaces,  $K_q(N_K) \cong \mathbb{C}[u]$ . The right action of  $K_q(T^2)$  is determined by the formulas*

$$\begin{aligned} 1_K \cdot f(Y + Y^{-1}) &= f(u), \\ 1_K \cdot (X + X^{-1}) &= -(q^2 + q^{-2})1_K, \\ 1_K \cdot q^{-1}(XY + X^{-1}Y^{-1}) &= -q^{-3}u. \end{aligned}$$

*Proof.* The identification  $K_q(N_K) \cong \mathbb{C}[u]$  and the first claimed formula follow from § 2.2.4. The second formula follows from the fact that the meridian is contractible inside  $D^2 \times S^1$  and 0-framed inside  $S^3$  under the inclusion  $D^2 \times S^1 \hookrightarrow S^3$ . The third formula follows from the fact that inside of the solid torus, the  $(1, 1)$  curve is isotopic to the longitude with a framing twist (which accounts for the factor of  $q$ ). The statement that these formulas completely determine the module structure of  $K_q(N_K)$  follows from Lemma 4.4. □

Let  $V = \mathbb{C}[U^{\pm 1}]$  and give  $V$  a right  $A_q \rtimes \mathbb{Z}_2$ -module structure via

$$f(U) \cdot g(Y) = f(U)g(U^{-1}), \quad f(U) \cdot X = -f(q^2U), \quad f(U) \cdot s = -f(U^{-1}).$$

Because of the sign in the action of  $s$ , we have  $Ve = (U - U^{-1})\mathbb{C}[U + U^{-1}] \subset V$ .

**LEMMA 5.5.** *The  $\mathbb{C}$ -linear isomorphism  $K_q(N_K) \rightarrow Ve$  given by  $f(u) \mapsto (U - U^{-1})f(U + U^{-1})$  is an isomorphism of right  $A_q^{\mathbb{Z}_2}$ -modules.*

*Proof.* The isomorphism is linear over  $\mathbb{C}[Y + Y^{-1}]$  because  $Y + Y^{-1}$  commutes with  $s$ . We then compute

$$\begin{aligned} (U - U^{-1}) \cdot (X + X^{-1}) &= -(q^2 + q^{-2})(U - U^{-1}), \\ (U - U^{-1}) \cdot q^{-1}(XY + X^{-1}Y^{-1}) &= -q^{-3}(U - U^{-1}) \cdot (Y + Y^{-1}). \end{aligned}$$

Then the claim follows from Lemma 4.4. □

5.1.3 *The colored Jones polynomials from the nonsymmetric pairing.* Let  $\hat{K}_q(S^3 \setminus K)$  be the nonsymmetric skein module of the complement of a knot  $K$ . From Corollary 5.3 and Lemma 5.5, we see that if  $q^4 - 1$  is invertible, then the topological pairing lifts to a nonsymmetric pairing

$$\langle -, - \rangle : V \otimes_{A_q \rtimes \mathbb{Z}_2} \hat{K}_q(S^3 \setminus K) \rightarrow \mathbb{C}. \tag{5.1}$$

(The fact that we have used the same notation for the topological pairing and the nonsymmetric pairing is justified by the last sentence of the proof of Lemma 5.2.) From Theorem 2.14, we have the equality

$$J_n(K; q) = (-1)^{n-1} \langle S_{n-1}(u), \emptyset \rangle.$$

In the nonsymmetric setting, this formula simplifies substantially.

LEMMA 5.6. *We have the equality*

$$J_n(K; q) = (-1)^n 2 \langle 1_V \cdot Y^n, \emptyset \rangle. \tag{5.2}$$

*Proof.* Under the isomorphism  $K_q(D^2 \times S^1) \cong V\mathbf{e}$ , the empty link is identified with the element  $U - U^{-1}$ . Combining this with Lemma 4.6 gives

$$S_{n-1}(u) = (U - U^{-1})S_{n-1}(U + U^{-1}) = U^n - U^{-n} \in V.$$

Then the following computation completes the proof:

$$\langle U^n - U^{-n}, \emptyset \rangle = -\langle 1_V Y^n(1+s), \emptyset \rangle = -\langle 1_V Y^n, 2\mathbf{e}\emptyset \rangle = -2\langle 1_V Y^n, \emptyset \rangle. \quad \square$$

5.1.4 *The action of  $A_q \rtimes \mathbb{Z}_2$  on the sequence  $J_n(K; q)$ .* If we define  $\mathbf{H} := \text{Hom}(\mathbb{Z}, \mathbb{C}[q^{\pm 1}])$  and fix a knot  $K$ , then  $J(n) := J_n(K; q)$  is an element of  $\mathbf{H}$  (after extending the colored Jones polynomials to negative integers via  $J_{-n}(K; q) := -J_n(K; q)$ ). In [GL05], Garoufalidis and Lê defined a left action of  $A_q \rtimes \mathbb{Z}_2$  on  $\mathbf{H}$  as follows:

$$(Xf)(n) := -q^{-2n} f(n), \quad (Yf)(n) := -f(n+1), \quad (sf)(n) := -f(-n). \tag{5.3}$$

(Actually, their action is a twist of this action by an automorphism of  $A_q \rtimes \mathbb{Z}_2$ ; we have chosen this twist so that Lemma 5.7 holds.) We now relate this action to the formula for the colored Jones polynomials from Lemma 5.6.

LEMMA 5.7. *For  $a \in A_q \rtimes \mathbb{Z}_2$ , we have the equality*

$$(a \cdot J)(n) = (-1)^n \langle Y^n, a \cdot 1_M \rangle. \tag{5.4}$$

*Proof.* It suffices to show the claim for generators of  $A_q \rtimes \mathbb{Z}_2$ , and these are straightforward computations. For example,

$$(-1)^n \langle Y^n, Y \cdot 1_M \rangle = (-1)^n \langle Y^{n+1}, 1_M \rangle = -J(n+1) = (Y \cdot J)(n). \quad \square$$

Remark 5.8. This lemma has appeared in the literature for  $a \in A_q^{\mathbb{Z}_2}$ . However, the extension of the lemma to all of  $A_q \rtimes \mathbb{Z}_2$  gives an interpretation of the appearance of the action (5.3). Also, the fact that  $V$  is the sign representation (i.e.  $1_V s = -1_V$ ) gives a skein-theoretic interpretation of the sign in the definition  $J(n) := -J(-n)$ .

Remark 5.9. If  $V$  is the nonsymmetric skein module of the solid torus, then the choice of basis  $\{U^n\}$  for  $V$  gives a linear isomorphism  $V^* \rightarrow \mathbf{H}$ , where  $V^*$  is the linear dual of  $V$ . Since  $V$  is a right  $A_q \rtimes \mathbb{Z}_2$ -module, its dual  $V^*$  is a left  $A_q \rtimes \mathbb{Z}_2$ -module and, in this language, Lemma 5.7 can be interpreted as the statement that the map  $V^* \rightarrow \mathbf{H}$  is an isomorphism of left  $A_q \rtimes \mathbb{Z}_2$ -modules.

**5.2 Inhomogeneous recursion relations**

We will say that a sequence  $f(n) \in \mathbf{H}$  satisfies an *inhomogeneous recursion relation* if there is a nonzero  $a \in A_q \rtimes \mathbb{Z}_2$  such that the sequence<sup>7</sup>  $P(n) := af(n)$  satisfies  $P(n) = P(0)$ . Let  $K_\emptyset \subset K_q(S^3 \setminus K)$  be the  $A_q^{\mathbb{Z}_2}$ -submodule generated by the empty link  $\emptyset$ .

**THEOREM 5.10.** *Suppose that there is a nonzero map  $K_q(S^3 \setminus \text{unknot}) \rightarrow K_\emptyset$  of  $A_q^{\mathbb{Z}_2}$ -modules. Then the sequence  $J(n) := J_n(K; q)$  satisfies an inhomogeneous recursion relation.*

*Proof.* Let  $V = \hat{K}_q(S^3 \setminus \text{unknot})$  be the nonsymmetric skein module of the unknot described in (4.3). Since Morita equivalence is functorial, we have a nonzero map  $V \rightarrow \hat{K}_\emptyset \subset \hat{K}_q(S^3 \setminus K)$ . Let  $p \in \hat{K}_\emptyset$  be the image of  $1_V$  under this map (which is *not* the image of the empty link). The surjective map  $A_q^{\mathbb{Z}_2} \twoheadrightarrow K_\emptyset$  lifts via the Morita equivalence to a surjective map  $(A_q \rtimes \mathbb{Z}_2)\mathbf{e} \twoheadrightarrow \hat{K}_\emptyset$ , which implies that  $\hat{K}_\emptyset$  is generated (as an  $A_q \rtimes \mathbb{Z}_2$ -module) by the empty link  $\emptyset$ . Therefore, there exists  $a \in A_q \rtimes \mathbb{Z}_2$  such that  $p = a \cdot \emptyset$ . Formula (4.3) shows that  $1_V$  is an eigenvector of  $Y$  with eigenvalue  $-1$ , so we have  $Ya\emptyset = -a\emptyset$ . We define

$$P(n) := a \cdot J(n).$$

We then compute

$$P(n+1) = -Ya \cdot J(n) = (-1)^n \langle Y^n, Ya\emptyset \rangle = (-1)^{n-1} \langle Y^n, a\emptyset \rangle = P(n). \quad \square$$

**5.3 Divisibility properties of  $J_n(K; q)$**

Recall that  $D_q$  is the localization of  $A_q \rtimes \mathbb{Z}_2$  at the multiplicative set consisting of all nonzero polynomials in  $X$ . If we define  $\mathbf{H}(q) := \text{Hom}(\mathbb{Z}, \mathbb{C}(q))$  to be the space of sequences of rational functions, then the action of  $A_q \rtimes \mathbb{Z}_2$  on  $\mathbf{H}$  extends to an action of  $D_q$  on  $\mathbf{H}(q)$  via the formulas

$$\left( \frac{F(X)}{G(X)} \cdot f \right)(n) := \frac{F(-q^{-2n})}{G(-q^{-2n})} f(n), \quad (Y \cdot f)(n) := -f(n+1), \quad (s \cdot f)(n) := -f(-n). \quad (5.5)$$

The double affine Hecke algebra  $\mathcal{H}_{q,t}$  can be viewed as a subalgebra of  $D_q$ , and this gives  $\mathbf{H}(q)$  the structure of an  $\mathcal{H}_{q,t}$ -module. Garoufalidis and Lê [GL05] showed that the  $A_q \rtimes \mathbb{Z}_2$ -module map defined by  $a \mapsto a \cdot J(n)$  has a nontrivial kernel. This leads to the following question, which we hope to address in future work.

*Question 5.11.* Does the map  $\mathcal{H}_{q,t} \rightarrow \mathbf{H}(q)$  defined by  $a \mapsto a \cdot J(n)$  have a nontrivial kernel?

In this section, we relate the conjectured action of  $\mathcal{H}_{q,t_1,t_2,1,1}$  on the nonsymmetric skein module to the action of  $\mathcal{H}_{q,t_1,t_2,1,1}$  on  $\mathbf{H}(q)$ . We recall from Proposition 2.18 that under the standard embedding,  $\mathcal{H}_{q,t_1,t_2,1,1}$  is the subalgebra of  $D_q$  generated by the elements  $X$ ,  $s$ , and

$$Y_{k,u} := t_1Y - (q^2\bar{t}_1X^{-2} + q\bar{t}_2X^{-1})sU,$$

where  $U$  is the operator

$$U := \frac{1}{1 - q^2X^2}(1 - sY). \quad (5.6)$$

<sup>7</sup> In the definition of inhomogeneous recursion relation, we do not require  $P(n)$  to be nonzero. If  $P(n) = 0$ , then  $f(n)$  satisfies an inhomogeneous recursion relation that happens to be homogeneous.

Let  $M := (A_q \rtimes \mathbb{Z}_2) \cdot 1_K \subset \hat{K}_q(S^3 \setminus K)$  be the submodule of the nonsymmetric skein module of a knot  $K$  generated by the empty link. If  $M^{\text{loc}}$  is the localization of  $M$  at all nonzero polynomials in  $X$ , then both  $D_q$  and its subalgebra  $\mathcal{H}_{q,t_1,t_2,1,1}$  naturally act on  $M^{\text{loc}}$ . We recall that Conjecture 1 states that the natural map  $M \rightarrow M^{\text{loc}}$  is injective and that the action of  $\mathcal{H}_{q,t_1,t_2,1,1}$  preserves the subspace  $M \subset M^{\text{loc}}$ . It is clear that this statement implies that  $UY^j 1_M \in M$  for all  $j \in \mathbb{Z}$ . We then define

$$P_j(n) := (-1)^{n+j} \frac{J(n+j) + J(n-j-1)}{q^{4n-2} - 1}. \tag{5.7}$$

**THEOREM 5.12.** *If Conjecture 1 holds for  $K \subset S^3$ , then  $(q^2 - 1)P_j(n)$  is a Laurent polynomial in  $\mathbb{C}[q^{\pm 1}]$ .*

*Proof.* We use Lemma 5.7 to compute the quantity  $\langle Y^n, UY^j 1_M \rangle$  in terms of the pairing:

$$\begin{aligned} \langle Y^n, UY^j 1_M \rangle &= (-1)^n UY^j J(n) \\ &= (-1)^n \frac{1}{1 - q^2 X^2} (1 - sY) Y^j J(n) \\ &= (-1)^{n+j} \frac{1}{1 - q^2 X^2} (1 - sY) J(n+j) \\ &= \frac{(-1)^j}{1 - q^2 X^2} [(-1)^n J(n+j) + (-1)^{-n+1} J(-n+1+j)] \\ &= \frac{(-1)^{n+j} [J(n+j) + J(n-1-j)]}{q^{4n-2} - 1} \\ &= P_j(n). \end{aligned}$$

Since the nonsymmetric pairing exists whenever  $q^4 - 1$  is invertible, the rational function  $P_j(n)$  can only have poles when  $q^4 - 1 = 0$ . However, the colored Jones polynomials are Laurent polynomials, and the denominator of  $P_j(n)$  has simple zeros when  $q^2 - 1 = 0$  and does not have zeros when  $q^2 + 1 = 0$ . Therefore,  $(q^2 - 1)P_j(n)$  is a Laurent polynomial, which shows the claim.  $\square$

**5.4 Habiro’s cyclotomic expansion**

In this section, we use Habiro’s cyclotomic expansion [Hab08, §6] (see also [GL11, (12)]) of the colored Jones polynomials to prove that for any knot, the rational function  $(q^2 - 1)P_j(n)$  from (5.7) is actually a Laurent polynomial. We first recall this expansion in our normalization conventions (see Remark 2.15). Define the polynomials

$$c_{n,k} := \prod_{j=1}^k (q^{4n} + q^{-4n} - q^{4j} - q^{-4j}).$$

By definition,  $c_{n,0} = 1$ ,  $c_{n,n} = 0$ , and  $c_{n,k+1} = (q^{4n} + q^{-4n} - q^{4k+4} - q^{-4k-4})c_{n,k}$ .

**THEOREM 5.13 [Hab08].** *There exist (unique)  $H_k \in \mathbb{Z}[q^{\pm 1}]$ , independent of  $n$ , such that*

$$J(n) = \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}} \sum_{k=0}^{n-1} c_{n,k} H_k.$$



Note that  $H_0 = 1$  for all knots. Since  $c_{n,n} = 0$ , we may take the upper limit of this sum to be infinity. Also, as an example of the theorem, for the figure eight knot we have  $H_k = 1$  for all  $k$ , and for the unknot we have  $H_0 = 1$  and  $H_k = 0$  for  $k > 1$ . Finally, since we use the convention  $J(n) = -J(-n)$ , the theorem is also true for negative  $n$  if we define  $H_{-k} = H_k$ .

**THEOREM 5.14.** *The following rational function is actually a Laurent polynomial:*

$$(q^2 - 1)P_j(n) = (q^2 - 1) \frac{J(n + j) + J(n - j - 1)}{q^{4n-2} - 1}.$$

*Proof.* (In the statement of the theorem we have ignored the sign  $(-1)^{n+j}$ .) Since Theorem 5.13 is true for both positive and negative  $n$ , we are free to assume that  $n + j \geq 0$  and  $n - j - 1 \geq 0$ . If we shorten notation by writing  $a = (q^{2n+2j} - q^{-2n-2j})$  and  $b = (q^{2n-2j-2} - q^{2+2j-2n})$ , we then have

$$\begin{aligned} (q^2 - 1)P_j(n) &= \frac{1}{(q + q^{-1})(q^{4n-2} - 1)} \sum_{k=0}^{\infty} H_k [ac_{n+j,k} + bc_{n-j-1,k}] \\ &=: \sum_{k=0}^{\infty} H_k \frac{s_k}{(q + q^{-1})(q^{4n-2} - 1)}. \end{aligned}$$

We prove that  $s_k$  is divisible by  $(q + q^{-1})(q^{4n-2} - 1)$  by induction on  $k$ . Since  $c_{n,0} = 1$ , we have

$$\begin{aligned} s_{k=0} &= (q^{2n+2j} - q^{-2n-2j}) + (q^{2(n-j-1)} - q^{2(1+j-n)}) \\ &= q^{2n+2j} - q^{2(1+j-n)} - q^{-2n-2j} + q^{2(n-j-1)} \\ &= q^{2j}(q^{2n} - q^{-2n+2}) + q^{-2j}(q^{2n-2} - q^{-2n}) \\ &\equiv q^{2j-2n}(q^{-2} + q^{-4j}) \pmod{q^{4n-2} - 1}. \end{aligned}$$

Since the expression on the final line is divisible by  $q^{4j-2} + 1$ , it is divisible by  $q^2 + 1$ , and this proves the claim for  $k = 0$ . For the inductive step, we will show that  $s_k \equiv s_{k-1} \pmod{q^{4n-2} - 1}$ . We first compute

$$\begin{aligned} s_k &= ac_{n+j,k} + bc_{n-j-1,k} \\ &= a(q^{4(n+j)} + q^{-4(n+j)} - q^{4k} - q^{-4k})c_{n+j,k-1} \\ &\quad + b(q^{4(n-j-1)} + q^{-4(n-j-1)} - q^{4k} - q^{-4k})c_{n-j-1,k-1}. \end{aligned}$$

We now split this into four terms, each of which can be dealt with similarly. For example,

$$\begin{aligned} aq^{4(n+j)}c_{n+j,k-1} + bq^{-4(n-j-1)}c_{n-j-1,k-1} &= q^{4j}(aq^{4n}c_{n+j,k-1} + bq^{4-4n}c_{n-j-1,k-1}) \\ &\equiv q^{4j}(aq^{4n}c_{n+j,k-1} + bq^{4n}c_{n-j-1,k-1}) \\ &\equiv q^{4j+4n}s_{k-1} \pmod{q^{4n-2} - 1}. \quad \square \end{aligned}$$

*Remark 5.15.* This proof actually shows that the following rational function is a Laurent polynomial:

$$\frac{(q^2 - 1)(J(n + j) + J(n - j - 1))}{(q^{4n-2} - 1)(q^{4j-2} + 1)}.$$

**5.5 More divisibility properties**

In this section, we prove [CLPZ14, Conjecture 1.6] of Chen *et al.* using the techniques of the previous section (and, in particular, Habiro’s theorem). Their conjecture is stated using a different normalization of the colored Jones polynomials, so in this section we use their normalization. In particular, their  $q$  is our  $q^2$ , and they used the normalized colored Jones polynomials  $\bar{J}_n(K)$ , which are related to ours via  $\bar{J}_n(K) = J_{n+1}(K)/J_{n+1}(\text{unknot})$ . Let  $[n] := q^n - q^{-n}$ . We prove the following theorem, which implies [CLPZ14, Conjecture 1.6].

**THEOREM 5.16.** *For any knot, the following congruence holds:*

$$\bar{J}_{n-1}(q) - \bar{J}_{k-1}(q) \equiv 0 \pmod{[n - k][n + k]}. \tag{5.8}$$

*Proof.* First, we define

$$d_{n,j} := \prod_{k=1}^j (q^{2n} + q^{-2n} - q^{2k} - q^{-2k})$$

(where  $d_{n,0} = 1$  by convention). Habiro’s theorem in the normalization conventions of [CLPZ14] says that there exist polynomials  $H_k \in \mathbb{Z}[q^{\pm 1}]$ , independent of  $n$ , such that

$$\bar{J}_{n-1}(q) = \sum_{j=0}^{\infty} d_{n,j} H_j.$$

(This sum is finite because  $d_{n,n} = 0$ .) We can therefore write the left-hand side of (5.8) as follows:

$$\bar{J}_{n-1}(q) - \bar{J}_{k-1}(q) = \sum_{j=0}^{\infty} (d_{n,j} - d_{k,j}) H_j.$$

We prove by induction on  $j$  that each coefficient  $a_j := (d_{n,j} - d_{k,j})$  is congruent to 0 modulo  $[n - k][n + k]$ . The base case  $j = 0$  is trivial since  $d_{n,0} = 1$ . Now assume that  $a_{j-1} \equiv 0$ . We first compute

$$\begin{aligned} d_{n,j} &= (q^{2n} + q^{-2n} - q^{2j} - q^{-2j})d_{n,j-1} \\ &\equiv (-[n - k][n + k] + q^{2n} + q^{-2n} - q^{2j} - q^{-2j})d_{n,j-1} \\ &= (q^{2k} + q^{-2k} - q^{2j} - q^{-2j})d_{n,j-1}. \end{aligned}$$

We then have the following congruences:

$$\begin{aligned} a_j &= (q^{2n} + q^{-2n} - q^{2j} - q^{-2j})d_{n,j-1} - d_{k,j} \\ &\equiv (q^{2k+2} + q^{-2k-2} - q^{2j} - q^{-2j})d_{n,j-1} - d_{k,j} \\ &= (q^{2k+2} + q^{-2k-2} - q^{2j} - q^{-2j})(d_{n,j-1} - d_{k,j-1}) \\ &\equiv 0. \end{aligned}$$

This completes the proof of the theorem. □

*Remark 5.17.* The difference in the signs in (5.8) and the numerator of Remark 5.15 comes from the differences in normalization between  $J_n$  and  $\bar{J}_n$ .

### 6. Canonical three-parameter deformations

In this section, we discuss deformations of (nonsymmetric) skein modules to the double affine Hecke algebra  $\mathcal{H}_{q,t_1,t_2,t_3,t_4} = \mathcal{H}_{q,\underline{t}}$  of type  $C^\vee C_1$  introduced by Sahi in [Sah99] (see also [NS04]). To reduce confusion, in this section we write  $X, \hat{y}$ , and  $s$  for the generators of  $A_q \rtimes \mathbb{Z}_2$ . As before, we let  $D_q$  be the algebra obtained from  $A_q \rtimes \mathbb{Z}_2$  by inverting all nonzero polynomials in  $X$ . Then  $\mathcal{H}_{q,\underline{t}}$  is the subalgebra of  $D_q$  generated by  $X, X^{-1}$ , and the following operators in  $D_q$ :

$$T_0 = t_1 s \hat{y} - \frac{q^2 \bar{t}_1 X^2 + q \bar{t}_2 X}{1 - q^2 X^2} (1 - s \hat{y}),$$

$$T_1 = t_3 s + \frac{\bar{t}_3 + \bar{t}_4 X}{1 - X^2} (1 - s).$$

If  $M$  is an  $A_q \rtimes \mathbb{Z}_2$ -module, we write  $M^{\text{loc}} := D_q \otimes_{A_q \rtimes \mathbb{Z}_2} M$  for the localization of  $M$  at all nonzero polynomials  $X$ . If  $M$  is free over the subalgebra  $\mathbb{C}[X^{\pm 1}]$  (which is the case in all our examples), then the natural map  $M \rightarrow M^{\text{loc}}$  is injective. Since  $\mathcal{H}_{q,\underline{t}}$  is a subalgebra of  $D_q$ , it acts naturally on  $M^{\text{loc}}$ .

We now prove Conjectures 1 and 3 for our example knots.

**THEOREM 6.1.** *Let  $M$  be the  $A_q \rtimes \mathbb{Z}_2$ -module which is the nonsymmetric skein module of the unknot, a  $(2, 2p + 1)$  torus knot, or the figure eight knot, and let  $M'$  be the quotient of  $M$  by the unknot submodule.*

- (i) *The action of  $\mathcal{H}_{q,t_1,t_2,1,1}$  preserves the subspace  $M \subset M^{\text{loc}}$ .*
- (ii) *The action of  $\mathcal{H}_{q,t_1,t_2,t_3,t_4}$  preserves the subspace  $M' \subset (M')^{\text{loc}}$ .*

*Remark 6.2.* For the knots listed in the theorem, the results of §4 make it clear that there is a unique map from the skein module of the unknot to the skein module of  $K$ , so the quotient  $M'$  in the second statement of the theorem is well defined.

*Proof.* We define the operators

$$U_0 := \frac{1}{1 - q^2 X^2} (1 - s \hat{y}), \quad U_1 := \frac{1}{1 - X^2} (1 - s). \tag{6.1}$$

To prove the first statement, it suffices to show that  $U_0$  preserves  $M \subset M^{\text{loc}}$ . Once this is proved, the second statement is implied by the statement  $U_1 M' \subset M'$ . To avoid confusion of notation, after proving the technical Lemma 6.4, we will divide the proof into separate subsections (one for each knot). □

*Remark 6.3.* Before continuing with the proof of the theorem, we remark that if  $\mathbf{e}M = K_q(S^3 \setminus K)$ , the conjecture that  $U_0$  preserves the nonsymmetric skein module  $M \subset M^{\text{loc}}$  implies that  $q^{-1} \mathbf{e} X^{-1} U_0 \mathbf{e}$  preserves  $\mathbf{e}M$  inside the localization of  $\mathbf{e}M$  at polynomials in  $x = X + X^{-1}$ . Under the identification  $A_q^{\mathbb{Z}_2} = K_q(T^2 \times [0, 1])$ , we have

$$q^{-1} \mathbf{e} X^{-1} U_0 \mathbf{e} = \frac{1}{(x - (q + q^{-1}))(x + (q + q^{-1}))} [(q - q^{-1})x - q^{-2}(1, -1) + q^2(1, 1)].$$

(Here  $x$  is the meridian  $(1, 0)$ , and the notation  $(m, l)$  refers to the  $(m, l)$  curve on  $T^2$ .) We note that if  $K$  is the unknot, then the operator  $q^{-1} \mathbf{e} X^{-1} U_0 \mathbf{e}$  annihilates the empty link.

We first give a technical lemma that provides conditions that imply that  $U_i M \subset M$  for  $i = 0, 1$ . We recall the notation of Lemma 4.5. In particular, suppose that  $M = \mathbb{C}[X^{\pm 1}] \otimes_{\mathbb{C}} V$  and define operators  $S, P \in \text{End}_{\mathbb{C}}(M)$  using (4.1). Furthermore, suppose that  $A(X), B(X) \in \text{End}_{\mathbb{C}[X^{\pm 1}]}(M)$  satisfy

$$B(X)B(X^{-1}) = 1, \quad A(X)B(X)A(q^2 X^{-1}) = B(X).$$

We then define  $s, \hat{y} : M \rightarrow M$  by  $\hat{y} = A(X)P$  and  $s = B(X)S$ , and Lemma 4.5 shows that the operators  $X, s, \hat{y} : M \rightarrow M$  define a representation of  $A_q \rtimes \mathbb{Z}_2$ .

LEMMA 6.4. *Assume the notation in the previous paragraph, and suppose that the following condition holds:*

$$(1 - B(X)A(X^{-1}))M \subset (1 - q^2 X^2)M.$$

*Then the operator  $U_0$  from formula (6.1) preserves  $M \subset M^{\text{loc}}$ . Also, the statement  $U_1 M' \subset M'$  is implied by the following condition:*

$$(1 - B(X))M' \subset (1 - X^2)M'.$$

*Proof.* Since  $(SP)^2 = 1$ , the elements  $\mathbf{f}_{\pm} := (1 \pm SP)/2$  are idempotents that satisfy  $(SP)\mathbf{f}_{\pm} = \pm \mathbf{f}_{\pm}$ . (These are not the standard idempotents that have been used previously.) We can then write

$$\begin{aligned} (1 - s\hat{y}) &= (1 - B(X)SA(X)P)(\mathbf{f}_+ + \mathbf{f}_-) \\ &= (1 - B(X)A(X^{-1})SP)(\mathbf{f}_+ + \mathbf{f}_-) \\ &= (1 - B(X)A(X^{-1}))\mathbf{f}_+ + (1 + B(X)A(X^{-1}))\mathbf{f}_-. \end{aligned}$$

Since  $(1 - SP) \cdot (f(X) \otimes v) = (f(X) - f(q^{-2} X^{-1})) \otimes v$ , we see that  $\mathbf{f}_- \cdot M \subset (1 - q^2 X^2)M$  and, since  $B(X)$  and  $A(X)$  are  $\mathbb{C}[X^{\pm 1}]$ -linear, this implies that

$$(1 + B(X)A(X^{-1}))\mathbf{f}_- M \subset (1 - q^2 X^2)M.$$

By assumption,  $(1 - B(X)A(X^{-1}))\mathbf{f}_+ M \subset (1 - q^2 X^2)M$ , and this shows that  $U_0 M \subset M$ . The second statement follows by a similar argument, which we omit.  $\square$

Remark 6.5. The first condition in Lemma 6.4 is equivalent to the conditions  $B(q^{-1})A(q) = \text{Id}$  and  $B(-q^{-1})A(-q) = \text{Id}$ , where both equalities hold in  $M_n(\mathbb{C})$ . We remark that the conditions in Lemma 4.5 hold *a priori*, and when specialized to  $X = q^{-1}$  they become  $(B(q^{-1})A(q))^2 = \text{Id}$  (and similar for  $X = -q$ ).

### 6.1 The unknot

We recall that the module structure of the (nonsymmetric) skein module  $M$  of the unknot is given by

$$M \cong \mathbb{C}[X^{\pm 1}], \quad s \cdot 1 = -1, \quad \hat{y} \cdot 1 = -1.$$

We then compute

$$(1 - s\hat{y})X^n = X^n - q^{-2n} X^{-n}.$$

The right-hand side is clearly divisible by  $(1 - q^2 X^2)$  and, since  $\{X^n\}$  is a basis for  $M$ , we see that  $U_0$  preserves  $M \subset M^{\text{loc}}$ . The second claim of Theorem 6.1 is tautological in this case.

**6.2 (2, 2p + 1) torus knots**

Let  $M$  be the nonsymmetric skein module of the  $(2, 2p + 1)$  knot  $L_p$ . (More precisely,  $M$  is the lift of the  $K_q(T^2)$  submodule of  $K_q(S^3 \setminus L_p)$  generated by the empty link.) We recall that  $M = \mathbb{C}[X^{\pm 1}]\{u, v\}$  as a  $\mathbb{C}[X^{\pm 1}]$ -module, with the action of  $s, \hat{y}$  given by

$$s = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} S, \quad \hat{y} = A(X)P := \begin{bmatrix} -1 & (-1)^p q^{2p+4}(X^{-2p-3} - q^{-4}X^{-2p+1}) \\ 0 & q^{2(2p+1)}X^{-2(2p+1)} \end{bmatrix} P.$$

Since  $u$  generates the unknot submodule, we may abuse notation and identify  $M' = \mathbb{C}[X^{\pm 1}]v$ , with the action of  $s, \hat{y}$  given by

$$s = B'S = \text{Id } S, \quad \hat{y} = [q^{2(2p+1)}X^{-2(2p+1)}]P.$$

LEMMA 6.6. *The operator  $U_0$  preserves  $M \subset M^{\text{loc}}$ , and the operator  $U_1$  preserves  $M' \subset (M')^{\text{loc}}$ .*

*Proof.* To show that the condition of Lemma 6.4 holds, we compute

$$1 - BA(X^{-1}) = \begin{bmatrix} 0 & (-1)^{p+1}q^{2p+4}(X^{2p+3} - q^{-4}X^{2p-1}) \\ 0 & 1 - q^{2(2p+1)}X^{2(2p+1)} \end{bmatrix}.$$

Since all the entries are divisible by  $1 - q^2X^2$ , this shows the first claim. The second claim follows from the fact that  $B'$  is the identity matrix. □

*Example 6.7.* The action of the generators  $X, s, Y$  of  $\mathcal{H}_{q,t_1,t_2,1,1}$  on the generators  $u, v$  can be computed explicitly. For example, for the trefoil (i.e.  $p = 1$ ),  $X$  acts by multiplication, and the actions of  $s$  and  $Y$  are given by the following:

$$\begin{aligned} s \cdot u &= -u, \\ s \cdot v &= v, \\ Y \cdot u &= -t_1u, \end{aligned}$$

$$\begin{aligned} Y \cdot v &= [t_1(q^2X^{-1} - q^6X^{-5}) - (q^2\bar{t}_1X^{-2} + q\bar{t}_2X^{-1})(q^4X^{-3} + q^2X^{-1})]u \\ &\quad + [t_1q^6X^{-6} - (q^2\bar{t}_1X^{-2} + q\bar{t}_2X^{-1})(q^4X^{-4} + q^2X^{-2} + 1)]v. \end{aligned}$$

Here we have written  $\bar{t}_i = t_i - t_i^{-1}$ . It is clear that when  $t_1 = t_2 = 1$ , these formulas specialize to the operator  $Y$  in (4.6).

**6.3 The figure eight**

Let  $M$  be the nonsymmetric skein module of the figure eight knot. We recall that as a  $\mathbb{C}[X^{\pm 1}]$ -module, we have  $M = \mathbb{C}[X^{\pm 1}] \otimes_{\mathbb{C}} (\mathbb{C}p' \oplus \mathbb{C}u \oplus \mathbb{C}v)$ . We recall the polynomials  $a, b, c \in \mathbb{C}[X^{\pm 1}]$ :

$$a = -q^{-2}X^4 + q^{-2}X^2 + q^2, \quad b = -q^{-2}X^2 + q^2X^{-2}, \quad c = q^{-2}X^3 - q^2X^{-1}.$$

Finally,  $s$  and  $\hat{y}$  are defined by matrices (with respect to the ordered basis  $\{p', u, v\}$ ):

$$s = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S, \quad \hat{y} = \begin{bmatrix} -1 & c(X) & q^2c(q^2X^{-1}) \\ 0 & a(X) & q^4b(q^2X^{-1}) \\ 0 & b(X) & a(q^2X^{-1}) \end{bmatrix} P.$$

Since  $p'$  generates the unknot submodule, we may write the action of  $s$  on the quotient  $M'$  as

$$s = B'S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S.$$

LEMMA 6.8. *The operator  $U_0$  preserves  $M \subset M^{\text{loc}}$ , and the operator  $U_1$  preserves  $M' \subset (M')^{\text{loc}}$ .*

*Proof.* To show that the first condition of Lemma 6.4 holds, we compute

$$1 - BA(X^{-1}) = \begin{bmatrix} 0 & -c(X^{-1}) & -q^2c(q^2X) \\ 0 & 1 - a(X^{-1}) & -q^4b(q^2X) \\ 0 & -b(X^{-1}) & 1 - a(q^2X) \end{bmatrix}.$$

It is straightforward to check that all entries in this matrix are divisible by  $1 - q^2X^2$ , which proves the first claim. The second claim follows from the fact that  $B'$  is the identity matrix.  $\square$

### 6.4 Three-variable Jones polynomials

In the previous section, we gave examples of skein modules of knot complements that extend to representations of  $\mathcal{H}_{q,t_1,t_2,1,1}$ . In this section, we use these modules to give example calculations of three-variable polynomials  $J_n(q, t_1, t_2) \in \mathbb{C}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$  that specialize to the colored Jones polynomials when  $t_1, t_2 = 1$ . More precisely, if  $Y_{t_1,t_2} = sT_0 \in D_q$ , we define

$$J_n(q, t_1, t_2) := \langle \emptyset, S_{n-1}(Y_{t_1,t_2} + Y_{t_1,t_2}^{-1}) \cdot \emptyset \rangle, \tag{6.2}$$

where  $\emptyset$  is the empty link and  $\langle -, - \rangle$  is the pairing from Corollary 5.3. We remark that the following equality is a direct consequence of Theorem 2.14 and the fact that  $Y_{1,1} + Y_{1,1}^{-1}$  is the longitude:

$$J_n(q) = J_n(q, t_1 = 1, t_2 = 1). \tag{6.3}$$

In other words, the polynomials  $J_n(q, t_1, t_2)$  specialize to the classical colored Jones polynomials of the knot  $K$ . We prove the following symmetry property of these three-variable polynomials, which extends the well-known symmetry for the classical colored Jones polynomials.

PROPOSITION 6.9. *Let  $\bar{K}$  be the mirror image of  $K$  and suppose that Conjecture 1 holds for  $K$ . Then*

$$J_n(K; q, t_1, t_2) = J_n(\bar{K}; q^{-1}, t_1^{-1}, t_2^{-1}).$$

*Proof.* The mirror map  $S^3 \rightarrow S^3$  induces a  $\mathbb{C}$ -linear isomorphism

$$K_q(S^3 \setminus K) \xrightarrow{\sim} K_{q^{-1}}(S^3 \setminus \bar{K}) \tag{6.4}$$

and we identify these two skein modules as vector spaces using this map. If  $\epsilon(K, q) : K_q(S^3 \setminus K) \rightarrow K_q(S^3)$  is the  $\mathbb{C}$ -linear map induced by the inclusion  $K \rightarrow S^3$ , then  $\epsilon(K, q) = \epsilon(\bar{K}, q^{-1})$  (under the identification (6.4)). Furthermore, the skein module  $K_{q^{-1}}(S^3 \setminus \bar{K})$  is the twist of the skein module  $K_q(S^3 \setminus K)$  by the isomorphism

$$\varphi : A_q \rtimes \mathbb{Z}_2 \rightarrow A_{q^{-1}} \rtimes \mathbb{Z}_2, \quad \varphi(X) = X, \quad \varphi(\hat{y}) = \hat{y}^{-1}, \quad \varphi(s) = s.$$

(Since this isomorphism sends  $s \mapsto s$ , it descends to an isomorphism of the spherical subalgebras.) This automorphism extends to  $\varphi : D_q \rightarrow D_{q^{-1}}$ . Let  $\Theta_{q,t_1,t_2} : \mathcal{H}_{q,t_1,t_2,1,1} \rightarrow D_q$  be the standard embedding given by Proposition 2.18. To prove the claim, we show that

$$\varphi(\Theta_{q,t_1,t_2}(Y_{t_1,t_2} + Y_{t_1,t_2}^{-1})) = \Theta_{q^{-1},t_1^{-1},t_2^{-1}}(Y_{t_1^{-1},t_2^{-1}} + Y_{t_1^{-1},t_2^{-1}}^{-1}). \tag{6.5}$$

Translating [NS04, Proposition 5.8] into our notation and specializing to  $t_3 = t_4 = 1$ , we have

$$\begin{aligned} \Theta_{q,t_1,t_2}(Y_{t_1,t_2} + Y_{t_1,t_2}^{-1}) &= A_{q,t_1,t_2}(X)(\hat{y} - 1) + A_{q,t_1,t_2}(X^{-1})(\hat{y}^{-1} - 1) + t_1 + t_1^{-1}, \\ A_{q,t_1,t_2}(X) &:= \frac{t_1^{-1}qX^{-1} - t_2 + t_2^{-1} - t_1q^{-1}X}{qX^{-1} - q^{-1}X}. \end{aligned} \tag{6.6}$$

We then have the equality

$$\varphi(A_{q,t_1,t_2}(X)) = A_{q^{-1},t_1^{-1},t_2^{-1}}(X^{-1}),$$

which proves (6.5) and completes the proof of the proposition. □

For the unknot, we compute a closed formula for  $J_n(q, t_1, t_2)$  in the following lemma, but for nontrivial knots this computation seems to be more difficult, so we include several example computations in the appendix.

**THEOREM 6.10.** *If  $K$  is the unknot, then*

$$J_n(K; q, t_1, t_2) = \frac{(t_1^{-1}q^2)^n - (t_1^{-1}q^2)^{-n}}{t_1^{-1}q^2 - (t_1^{-1}q^2)^{-1}}.$$

*Proof.* Using (6.6) and (4.3), it is straightforward to compute

$$(Y_{t_1,t_2} + Y_{t_1,t_2}^{-1}) \cdot \emptyset = -(t_1^{-1}q^2 + t_1q^{-2})\emptyset.$$

Then the claimed equality follows from the fact that  $\langle \emptyset, \emptyset \rangle = 1$  and from the identities for Chebyshev polynomials in Lemma 4.6. □

*Remark 6.11.* If  $q$  is specialized to  $q = 1$ , then the classical Jones polynomials are independent of the knot  $K$ . In all examples we have checked, this is also true for the polynomials  $J_n(K; q = 1, t_1, t_2)$  (when they are normalized as in (6.2)).

### 7. Five-parameter deformations

In this section, we discuss deformations of the nonsymmetric skein module  $M$  of a knot to a family of modules over the DAHA  $\mathcal{H}_{q,\underline{t}}$  for all parameters  $\underline{t} \in (\mathbb{C}^*)^4$ . Unfortunately, the deformations of  $M$  that we produce in this section do not seem to be canonical, and in particular depend on a choice of  $\mathbb{C}[X^{\pm 1}]$ -splitting of the module  $M$ . If  $M$  is the nonsymmetric skein module of the  $(2, 2p + 1)$  torus knot, there is a natural choice of such a splitting (see Remark 7.2), but for other knots this does not seem to be the case. For simplicity, in this section we will only discuss the trefoil.

We recall that  $\mathcal{H}_{q,\underline{t}}$  is the subalgebra of  $\text{End}_{\mathbb{C}}(\mathbb{C}[X^{\pm 1}])$  generated by  $X^{\pm 1}$  and the two operators

$$\begin{aligned} T_0 &= t_1 s \hat{y} - \frac{q^2 \bar{t}_1 X^2 + q \bar{t}_2 X}{1 - q^2 X^2} (1 - s \hat{y}), \\ T_1 &= t_3 s + \frac{\bar{t}_3 + \bar{t}_4 X}{1 - X^2} (1 - s). \end{aligned}$$

In §6, we showed that the operator  $T_0$  acts on the nonsymmetric skein module of several knot complements. However, the operator  $T_1$  does not even act on the nonsymmetric skein module  $V$  of the unknot because  $V$  is the sign representation of  $A_q \rtimes \mathbb{Z}_2$ . More precisely, we recall that  $V = \mathbb{C}[X^{\pm 1}]$  as a  $\mathbb{C}[X^{\pm 1}]$ -module, with the actions of  $\hat{y}$  and  $s$  determined by

$$\hat{y} \cdot 1 = -1, \quad s \cdot 1 = -1. \tag{7.1}$$

Then  $T_1 \cdot 1$  is a rational function in  $X$ , so, in the sign representation, the action of  $T_1$  on  $\mathbb{C}(X)$  does not preserve the subspace  $\mathbb{C}[X^{\pm 1}]$ .

To avoid this problem, we define a new operator  $T_1^- \in D_q$  that acts on  $V$  and show that the operators  $X, T_0$ , and  $T_1^-$  provide an action of  $\mathcal{H}_{q,\underline{t}}$  on the sign representation. In examples,



the nonsymmetric skein module splits over  $\mathbb{C}[X^{\pm 1}] \rtimes \mathbb{Z}_2$  as a sum of the standard and sign polynomial representations, and we can therefore use this splitting to define an action of  $\mathcal{H}_{q,t}$  on the nonsymmetric skein module. We describe this action for the trefoil and then use this action to define five-variable polynomials that specialize to the colored Jones polynomials of the trefoil.

**7.1 The Dunkl embedding for the sign representation**

Let  $V$  be the sign representation from (7.1) and let  $V^{\text{loc}}$  be the localization of  $V$  at all nonzero polynomials in  $X$ . Define the following operator:

$$T_1^- := t_3^{-1}s + \frac{t_3 - t_3^{-1} + (t_4 - t_4^{-1})X}{1 - X^2}(1 + s). \tag{7.2}$$

(Note that the sign in the term  $(1 + s)$  is not a typo: it is required for  $T_1^-$  to act on the sign representation  $V$ .) We recall from (2.5) that  $\mathcal{H}_{q,t}$  is generated by elements  $V_i$  and  $V_i^\vee$  (for  $i = 1, 2$ ) with relations (2.5).

LEMMA 7.1. *The operators  $T_0$  and  $T_1^-$  preserve the subspace  $V \subset V^{\text{loc}}$ . Furthermore, the operators  $X, T_0$ , and  $T_1^-$  generate a copy of  $\mathcal{H}_{q,t}$  inside of  $D_q$  via the map*

$$T_0 \mapsto T_0, \quad T_1 \mapsto T_1^-, \quad T_0^\vee \mapsto qT_0^{-1}X, \quad T_1^\vee \mapsto X^{-1}(T_1^-)^{-1}. \tag{7.3}$$

*Proof.* The operator  $s\hat{y}$  acts in the same way on the trivial and sign representations of  $D_q$ , and this implies that  $T_0$  preserves the subspace  $V \subset V^{\text{loc}}$ . Since the map  $D_q \rightarrow \text{End}_{\mathbb{C}}(\mathbb{C}(X))$  is injective for  $q$  not a root of unity, this also shows that  $T_0$  and  $T_0^\vee$  satisfy the first two relations in (2.5). Also, the final relation holds by definition.

To show that the third and fourth relations hold, we first note that as a  $\mathbb{C}[X + X^{-1}]$ -module, we have a splitting  $V = \mathbb{C}[X + X^{-1}]1 \oplus \mathbb{C}[X + X^{-1}]X$ . Furthermore, the operator  $T_1^-$  commutes with  $X + X^{-1} \in D_q$ , which shows that it suffices to check that the left-hand sides of the third and fourth relations annihilate the elements  $1, X \in V$ . These are straightforward computations, which we omit. □

**7.2 Five-parameter deformation for the trefoil**

We recall that the  $\mathbb{C}[X^{\pm 1}]$ -module structure of the nonsymmetric skein module  $M$  of the trefoil is

$$M \cong \mathbb{C}[X^{\pm 1}]u \oplus \mathbb{C}[X^{\pm 1}]v. \tag{7.4}$$

To define the  $A_q \rtimes \mathbb{Z}_2$ -module structure of  $M$ , we first define two operators  $S, P : M \rightarrow M$  as in equation (4.1). We can then write the action of the elements  $\hat{y}, s \in A_q \rtimes \mathbb{Z}_2$  in terms of matrices (with respect to the ordered basis  $(u, v)$  of  $M$ ):

$$s = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} S, \quad \hat{y} = \begin{bmatrix} -1 & q^2X^{-1} - q^6X^{-5} \\ 0 & q^6X^{-6} \end{bmatrix} P.$$

Remark 7.2. The element  $u \in M$  generates a submodule  $M' \subset M$ , and we write  $M''$  for the quotient, so that we have an exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.$$

If  $q$  is not a root of unity, then  $M'$  and  $M''$  are simple  $A_q \rtimes \mathbb{Z}_2$ -modules and, since this extension is nonsplit, it is easy to see that  $M'$  is the *unique* nontrivial submodule of  $M$  (see Lemma 4.10). The

module  $M''$  has a distinguished generator  $v''$  (the image of the empty link  $v$ ), so a  $\mathbb{C}[X^{\pm 1}]$ -splitting of this exact sequence is determined by the image of  $v''$  in  $M$ . The natural choice for this image is the empty link in  $M$ , which is  $v$ . In this sense, the choice of  $\mathbb{C}[X^{\pm 1}]$ -splitting given in (7.4) is natural. However, in general the analogue of the quotient  $M''$  has  $\mathbb{C}[X^{\pm 1}]$ -rank greater than 1, so a choice of  $\mathbb{C}[X^{\pm 1}]$ -splitting is not determined by a single element of  $M$ .

We now define the operator  $T_1^* : M \rightarrow M$  as follows:

$$T_1^* = \begin{bmatrix} T_1^- & 0 \\ 0 & T_1 \end{bmatrix}.$$

**THEOREM 7.3.** *The operators  $X$ ,  $T_0$ , and  $T_1^*$  provide an action of  $\mathcal{H}_{q,t}$  on  $M$  via the map (7.3).*

*Proof.* It was shown in §6.2 that the operator  $T_0$  preserves the subspace  $M \subset M^{\text{loc}}$ , and this implies that  $T_0$  and  $qT_0^{-1}X$  satisfy the first two relations of (2.5). The final relation holds by definition. Finally, since the splitting in (7.4) is a splitting of  $\mathbb{C}[X^{\pm 1}] \rtimes \mathbb{Z}_2$ -modules, the operator  $T_1^*$  preserves this splitting. Then the third and fourth relations of (2.5) follow from Lemma 7.1 and from the fact that  $T_1$  satisfies these relations in the standard polynomial representation.  $\square$

*Remark 7.4.* This action can be used to produce five-variable polynomials  $J_n(q, t_1, t_2, t_3, t_4)$  that restrict to the colored Jones polynomials  $J_n(q)$  using a similar formula to (6.2); however, the polynomials this produces are quite lengthy, so for the sake of brevity we omit them.

**ACKNOWLEDGEMENTS**

We would like to thank I. Cherednik for guidance with references and several helpful comments, S. Garoufalidis for suggesting the approach to the proof of Theorem 5.14, and R. Gelca for kindly allowing the use of his figures. We would also like to thank D. Bar-Natan, O. Chalykh, B. Cooper, P. Etingof, J. Kamnitzer, T. Lê, J. Marche, A. Marshall, G. Muller, A. Oblomkov, M. Pabiniak, D. Thurston, and B. Webster for enlightening conversations, and J. Stokman and P. Terwilliger for references. The second author is grateful to the users of the website MathOverflow, who have provided several helpful answers (see, e.g., [Ago10]).

**Appendix. Computer calculations of multi-variable Jones polynomials**

We now include computer computations of three-variable polynomials normalized as in (6.2). To shorten notation, we write  $t := t_1$  and  $v = t_2 - t_2^{-1}$ . (We remind the reader of the normalization conventions for the colored Jones polynomials; see Remark 2.15. In particular, the  $q$  of the KnotAtlas is our  $q^2$ .)

**A.1 The trefoil**

For the trefoil, we have the following polynomials:

$$\begin{aligned} J_2 &= vq - t^{-1}q^2 + (-t^{-1} + t)q^4 - tq^6 + vq^7 - vq^9 - t^{-1}q^{10} + (t^{-1} - t)q^{12} - vq^{15} + t^{-1}q^{18}, \\ J_3 &= v^2q^2 - t^{-1}vq^3 + (-1 + t^{-2} + t^2)q^4 + (-2t^{-1}v + tv)q^5 + (-1 + t^{-2})q^6 + (-t^{-1}v + tv)q^7 \\ &\quad + (t^{-2} + v^2)q^8 - t^{-1}vq^9 + (-1 + t^{-2})q^{10} - t^{-1}vq^{11} + t^{-2}q^{12} - tvq^{13} + (1 - t^2)q^{14} - tvq^{15} \\ &\quad + (1 - v^2)q^{16} + (t^{-1}v - tv)q^{17} + (1 - t^2 - v^2)q^{18} + (2t^{-1}v - tv)q^{19} + (2 - t^{-2})q^{20} \\ &\quad + (2t^{-1}v - tv)q^{21} + (-t^{-2} + t^2 - v^2)q^{22} + (t^{-1}v - tv)q^{23} + (2 - t^{-2} - t^2)q^{24} \\ &\quad + (t^{-1}v + tv)q^{25} + (-t^{-2} + t^2 - v^2)q^{26} + t^{-1}vq^{27} + t^{-1}vq^{29} + (1 - t^{-2} + v^2)q^{30} \\ &\quad + (t^{-1}v + tv)q^{31} + (-2 + t^2 - v^2)q^{32} - t^{-1}vq^{33} + (1 - t^{-2} + v^2)q^{34} + tvq^{35} + tvq^{37} \\ &\quad + (-1 + t^{-2})q^{38} - t^{-1}vq^{39} + (-t^{-2} + v^2)q^{40} + (-1 + t^{-2})q^{42} - t^{-1}vq^{43} - t^{-1}vq^{45} + t^{-2}q^{48}. \end{aligned}$$

**A.2 The (5, 2) torus knot**

For the (5, 2) torus knot, we have

$$J_2 = vq + (-t^{-1} + t)q^2 + (-t^{-1} + t)q^4 + vq^5 - tq^6 + vq^7 + (-t^{-1} + t)q^8 - vq^9 - t^{-1}q^{10} + vq^{11} + (t^{-1} - t)q^{12} - t^{-1}q^{14} - vq^{15} + (t^{-1} - t)q^{18} - vq^{21} + (t^{-1} - t)q^{24} - vq^{27} + t^{-1}q^{30}.$$

For the (5, 2) torus knot, the polynomial  $J_3$  is quite lengthy, so we omit it.

**A.3 The figure eight knot**

Since the figure eight knot is isotopic to its mirror image, Proposition 6.9 shows that  $J_n(q, t, v) = J_n(q^{-1}, t^{-1}, -v)$  (in the current notation for parameters). Explicitly, we have

$$\begin{aligned} J_2 &= -tq^{-10} - vq^{-7} + (t^{-1} - t)q^{-4} - vq^{-1} + vq + (-t^{-1} + t)q^4 + vq^7 - t^{-1}q^{10}, \\ J_3 &= t^2q^{-28} + tvq^{-25} + tvq^{-23} + (-1 + t^2)q^{-22} + (-t^2 + v^2)q^{-20} + tvq^{-19} + (-1 + t^2)q^{-18} \\ &\quad + -t^{-1}vq^{-17} + (-1 + t^2)q^{-16} - t^{-1}vq^{-15} + (1 - t^2 + v^2)q^{-14} + 2tvq^{-13} \\ &\quad + (-2 + t^{-2} + t^2 - v^2)q^{-12} - t^{-1}vq^{-11} + v^2q^{-10} - tvq^{-9} + (1 - t^2 + v^2)q^{-8} \\ &\quad + (-1 + t^{-2} - 2v^2)q^{-6} + (-t^{-1}v - tv)q^{-5} + (2 - t^{-2})q^{-4} + (2t^{-1}v - 2tv)q^{-3} \\ &\quad + (2 - 2t^2)q^{-2} + (2t^{-1}v - tv)q^{-1} + (1 - 2v^2) + (t^{-1}v - 2tv)q + (2 - 2t^{-2})q^2 \\ &\quad + (2t^{-1}v - 2tv)q^3 + (2 - t^2)q^4 + (t^{-1}v + tv)q^5 + (-1 + t^2 - 2v^2)q^6 + (1 - t^{-2} + v^2)q^8 \\ &\quad + t^{-1}vq^9 + v^2q^{10} + tvq^{11} + (-2 + t^{-2} + t^2 - v^2)q^{12} - 2t^{-1}vq^{13} + (1 - t^{-2} + v^2)q^{14} \\ &\quad + tvq^{15} + (-1 + t^{-2})q^{16} + tvq^{17} + (-1 + t^{-2})q^{18} - t^{-1}vq^{19} + (-t^{-2} + v^2)q^{20} \\ &\quad + (-1 + t^{-2})q^{22} - t^{-1}vq^{23} - t^{-1}vq^{25} + t^{-2}q^{28}. \end{aligned}$$

REFERENCES

Ago10 I. Agol, *Complete knot invariant?*, MathOverflow, URL: <http://mathoverflow.net/q/35687> (version: 2010-08-16).

AW85 R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985); MR 783216 (87a:05023).

Bar99 J. W. Barrett, *Skein spaces and spin structures*, Math. Proc. Cambridge Philos. Soc. **126** (1999), 267–275; MR 1670233 (99k:57006).

BC11 Y. Berest and O. Chalykh, *Quasi-invariants of complex reflection groups*, Composito Math. **147** (2011), 965–1002; MR 2801407.

BS12 Y. Berest and P. Samuelson, *Dunkl operators and quasi-invariants of complex reflection groups*, in *Mathematical aspects of quantization*, Contemporary Mathematics, vol. 583 (American Mathematical Society, Providence, RI, 2012), 1–23; MR 3013091.

BH95 G. W. Brumfiel and H. M. Hilden, *SL(2) representations of finitely presented groups*, Contemporary Mathematics, vol. 187 (American Mathematical Society, Providence, RI, 1995); MR 1339764 (96g:20004).

Bul97 D. Bullock, *Rings of SL<sub>2</sub>(C)-characters and the Kauffman bracket skein module*, Comment. Math. Helv. **72** (1997), 521–542; MR 1600138 (98k:57008).

BLF05 D. Bullock and W. Lo Faro, *The Kauffman bracket skein module of a twist knot exterior*, Algebr. Geom. Topol. **5** (2005), 107–118; (electronic); MR 2135547 (2006a:57012).

BP00 D. Bullock and J. H. Przytycki, *Multiplicative structure of Kauffman bracket skein module quantizations*, Proc. Amer. Math. Soc. **128** (2000), 923–931; MR 1625701 (2000e:57007).

BZ03 G. Burde and H. Zieschang, *Knots*, de Gruyter Studies in Mathematics, vol. 5, second edition (Walter de Gruyter, Berlin, 2003); MR 1959408 (2003m:57005).

- CKM14 S. Cautis, J. Kamnitzer and S. Morrison, *Webs and quantum skew Howe duality*, *Math. Ann.* **360** (2014), 351–390; [MR 3263166](#).
- CM15a L. Charles and J. Marché, *Knot state asymptotics I: AJ conjecture and Abelian representations*, *Publ. Math. Inst. Hautes Études Sci.* **121** (2015), 279–322; [MR 3349834](#).
- CM15b L. Charles and J. Marché, *Knot state asymptotics II: Witten conjecture and irreducible representations*, *Publ. Math. Inst. Hautes Études Sci.* **121** (2015), 323–361; [MR 3349835](#).
- CLPZ14 Q. Chen, K. Liu, P. Peng and S. Zhu, *Congruent skein relations for colored HOMFLY-PT invariants and colored Jones polynomials*, Preprint (2014), [arXiv:1402.3571](#).
- Che95 I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, *Ann. of Math. (2)* **141** (1995), 191–216; [MR 1314036](#) (96m:33010).
- Che05 I. Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, vol. 319 (Cambridge University Press, Cambridge, 2005); [MR 2133033](#) (2007e:32012).
- Che13 I. Cherednik, *Jones polynomials of torus knots via DAHA*, *Int. Math. Res. Not. IMRN* **2013** (2013), 5366–5425; [MR 3142259](#).
- CCG<sup>+</sup>94 D. Cooper, M. Culler, H. Gillet, D. D. Long and P. B. Shalen, *Plane curves associated to character varieties of 3-manifolds*, *Invent. Math.* **118** (1994), 47–84; [MR 1288467](#) (95g:57029).
- EOR07 P. Etingof, A. Oblomkov and E. Rains, *Generalized double affine Hecke algebras of rank 1 and quantized del Pezzo surfaces*, *Adv. Math.* **212** (2007), 749–796; [MR 2329319](#) (2008h:20006).
- Fox52 R. H. Fox, *On the complementary domains of a certain pair of inequivalent knots*, *Indag. Math.* **14** (1952), 37–40; [MR 0048024](#) (13,966c).
- FG00 C. Frohman and R. Gelca, *Skein modules and the noncommutative torus*, *Trans. Amer. Math. Soc.* **352** (2000), 4877–4888; [MR 1675190](#) (2001b:57014).
- Fuk05 S. Fukuhara, *Explicit formulae for two-bridge knot polynomials*, *J. Aust. Math. Soc.* **78** (2005), 149–166; [MR 2141874](#) (2006f:57013).
- GL05 S. Garoufalidis and T. T. Q. Lê, *The colored Jones function is q-holonomic*, *Geom. Topol.* **9** (2005), 1253–1293; (electronic); [MR 2174266](#) (2006j:57029).
- GL11 S. Garoufalidis and T. T. Q. Lê, *Asymptotics of the colored Jones function of a knot*, *Geom. Topol.* **15** (2011), 2135–2180; [MR 2860990](#).
- Gel02 R. Gelca, *Non-commutative trigonometry and the A-polynomial of the trefoil knot*, *Math. Proc. Cambridge Philos. Soc.* **133** (2002), 311–323; [MR 1912404](#) (2004c:57021).
- GS03 R. Gelca and J. Sain, *The noncommutative A-ideal of a  $(2, 2p + 1)$ -torus knot determines its Jones polynomial*, *J. Knot Theory Ramifications* **12** (2003), 187–201; [MR 1967240](#) (2004d:57015).
- GS04 R. Gelca and J. Sain, *The computation of the non-commutative generalization of the A-polynomial of the figure-eight knot*, *J. Knot Theory Ramifications* **13** (2004), 785–808; [MR 2088746](#) (2005f:57020).
- GL89 C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, *J. Amer. Math. Soc.* **2** (1989), 371–415; [MR 965210](#) (90a:57006a).
- Hab08 K. Habiro, *A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres*, *Invent. Math.* **171** (2008), 1–81; [MR 2358055](#) (2009b:57020).
- KM91 R. Kirby and P. Melvin, *The 3-manifold invariants of Witten and Reshetikhin–Turaev for  $sl(2, \mathbb{C})$* , *Invent. Math.* **105** (1991), 473–545; [MR 1117149](#) (92e:57011).
- Koo08 T. H. Koornwinder, *Zhedanov's algebra  $AW(3)$  and the double affine Hecke algebra in the rank one case. II. The spherical subalgebra*, *SIGMA Symmetry Integrability Geom. Methods Appl.* **4** (2008), Paper 052, 17; [MR 2425640](#) (2010e:33028).

- Lê06 T. T. Q. Lê, *The colored Jones polynomial and the A-polynomial of knots*, Adv. Math. **207** (2006), 782–804; [MR 2271986](#) (2007k:57021).
- LM85 A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58** (1985); [MR 818915](#) (87c:20021).
- Lus89 G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2** (1989), 599–635; [MR 991016](#) (90e:16049).
- Mac03 I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics, vol. 157 (Cambridge University Press, Cambridge, 2003); [MR 1976581](#) (2005b:33021).
- Min82 J. Minkus, *The branched cyclic coverings of 2 bridge knots and links*, Mem. Amer. Math. Soc. **35** (1982); [MR 643587](#) (83g:57004).
- NS04 M. Noumi and J. V. Stokman, *Askey–Wilson polynomials: an affine Hecke algebra approach*, in *Laredo lectures on orthogonal polynomials and special functions*, Advances in the Theory of Special Functions and Orthogonal Polynomials (Nova Science, Hauppauge, NY, 2004), 111–144; [MR 2085854](#) (2005h:42057).
- Obl04 A. Oblomkov, *Double affine Hecke algebras of rank 1 and affine cubic surfaces*, Int. Math. Res. Not. IMRN **2004** 877–912; [MR 2037756](#) (2005j:20005).
- Prz91 J. H. Przytycki, *Skein modules of 3-manifolds*, Bull. Pol. Acad. Sci. Math. **39** (1991), 91–100; [MR 1194712](#) (94g:57011).
- PS00 J. H. Przytycki and A. S. Sikora, *On skein algebras and  $Sl_2(\mathbb{C})$ -character varieties*, Topology **39** (2000), 115–148; [MR 1710996](#) (2000g:57026).
- RT90 N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), 1–26; [MR 1036112](#) (91c:57016).
- Ric79 R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Composito Math. **38** (1979), 311–327; [MR 535074](#) (80c:17009).
- Sah99 S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. of Math. (2) **150** (1999), 267–282; [MR 1715325](#) (2002b:33018).
- Sam12 P. Samuelson, *Kauffman bracket skein modules and the quantum torus*, PhD thesis, Cornell University (2012).
- Sam14 P. Samuelson, *A topological construction of Cherednik’s  $\mathfrak{sl}_2$  torus knot polynomials*, Preprint (2014), [arXiv:1408.0483](#).
- Sco83 P. Scott, *The geometries of 3-manifolds*, Bull. Lond. Math. Soc. **15** (1983), 401–487; [MR 705527](#) (84m:57009).
- Sik05 A. S. Sikora, *Skein theory for  $SU(n)$ -quantum invariants*, Algebr. Geom. Topol. **5** (2005), 865–897; (electronic); [MR 2171796](#) (2006j:57033).
- Sik14 A. S. Sikora, *Character varieties of abelian groups*, Math. Z. **277** (2014), 241–256; [MR 3205770](#).
- SW07 A. S. Sikora and B. W. Westbury, *Confluence theory for graphs*, Algebr. Geom. Topol. **7** (2007), 439–478; [MR 2308953](#) (2008f:57004).
- Sto03 J. V. Stokman, *Difference Fourier transforms for nonreduced root systems*, Selecta Math. (N.S.) **9** (2003), 409–494; [MR 2006574](#) (2004h:33040).
- Ter13 P. Terwilliger, *The universal Askey–Wilson algebra and DAHA of type  $(C_1^V, C_1)$* , SIGMA Symmetry Integrability Geom. Methods Appl. **9** (2013), Paper 047, 40; [MR 3116183](#).
- Tha01 M. Thaddeus, *Mirror symmetry, Langlands duality, and commuting elements of Lie groups*, Int. Math. Res. Not. IMRN **2001** (2001), 1169–1193, doi:[10.1155/S1073792801000551](#); [MR 1862614](#) (2002h:14019).
- Wal68 F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88, [MR 0224099](#) (36 #7146).

Yuri Berest berest@math.cornell.edu

Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA

Peter Samuelson psam@math.utoronto.ca

Department of Mathematics, University of Toronto, Toronto, ON, M4Y 1H5, Canada

*Current address:* Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA