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ON ZETA FUNCTIONS ASSOCIATED TO SYMMETRIC MATRICES III
AN EXPLICIT FORM OF L-FUNCTIONS

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Dedicated to Professor Hideo Shimizu on his 60th birthday

Abstract. In [I-S2], we gave an explicit form of zeta functions associated to the space of symmetric matrices. In this paper, the case of L-functions is treated. In the case of definite symmetric matrices, we show the rationality of special values of these L-functions.

Introduction

This is the third part of the series of our papers [I-S2] on zeta functions associated to the space of symmetric matrices. In the first part, we gave an explicit form of zeta functions, and in the second part, we discussed some analytic properties of them. The purpose of this paper is to give an explicit form of L-functions associated to that space.

For this space, two kinds of L-functions have been introduced by Sato [Sa2], Hashimoto, one of the author [Sai1], and by Arakawa [A]. The first ones are associated to Dirichlet characters, and were introduced as L-functions of the prehomogeneous vector space of symmetric matrices. The others are associated to a symmetric matrix with coefficients in a finite field and appeared in the calculation of the contribution of unipotent elements to the trace of some operators on the space of Siegel cusp forms.

There exists a close relation between these two kinds of L-functions. In fact, the second ones can be written by the first ones by means of the Gauss sums defined in Saito[Sai1]. Between these two kinds of L-functions, the first ones are easy to treat. For example, the analytic continuation and the functional equations of L-functions of the first kind were proved in [Sa2], [Sai1], [Sai2]. Furthermore our procedure of the computation of zeta functions.
functions in the first part can be easily applied to the case of L-functions of the first kind. The second kind of L-functions seem to have a rather complicated form. For simplicity, we assume $n \geq 3$ in this paper.

In §1, we give the definition of the two kinds of L-functions. Recalling the definition of the Gauss sums in [Sai1], we describe the relation between these two kinds of L-functions. We introduce one more matrix-valued Gauss sum and prove a result on it, which is a complement to the results in [Sai1].

In §2, we give an explicit form of L-functions assuming the result on orbital local series proved in §3. These results contain a generalization of [I-S1].

In the case of positive definite matrices, using these explicit forms, we prove the rationality of the values of the L-functions at non-positive integers.

In §3, we determine orbital local series for L-functions and complete the proof of theorems in §2.

§1. L-functions and Gauss sums

For a ring $R$, we denote by $S_n(R)$ the set of symmetric matrices of degree $n$ with coefficients in $R$. For a positive integer $n$, let $L_n$ (resp. $L_n^*$) be the lattice in $S_n(Q)$ consisting of integral (resp. half-integral) symmetric matrices of degree $n$, and $L_n^{(i)}$ (resp. $L_n^{*(i)}$) its subset consisting of elements with signature $(i, n-i)$. Then $SL_n(Z)$ acts on $L_n$ and $L_n^*$ by $g \cdot x = gx^tg$ for $g \in SL_n(Z)$ and $x \in L_n, L_n^*$. We define some functions on $L_n$ or $Z^*$, which we call characters in this paper. First, we consider characters defined modulo $p^v$ for an odd prime $p$ and then those modulo $2^v$. Lastly, we consider general ones. For a prime $p$ and a positive integer $\nu$, we set $R_{p,\nu} = Z/p^\nu Z$.

Let $p$ be an odd prime, and let $\varphi$ be a Dirichlet character with the conductor $f(\varphi) = p^\nu$ for a positive integer $\nu$. For $x \in L_n$ or $L_n^*$, set $\tilde{x} = x \mod p^\nu$ and define

$$\varphi^{(n)}(x) = \begin{cases} \varphi(\det \tilde{x}) & \text{if } \det \tilde{x} \in R_{p,\nu}^*, \\ 0 & \text{otherwise}. \end{cases}$$

Let $\chi_p$ and $\chi_0$ be the quadratic character and the trivial one modulo $p$ respectively. For $\psi = \chi_p$ or $\chi_0$ and and integer $r$ with $1 \leq r \leq n$, we define

$$\psi^{(r)}(x) = \begin{cases} \psi(\det \tilde{x}'') & \text{if } \text{rank}(\tilde{x}) = r, \\ 0 & \text{otherwise}, \end{cases}$$
where \( x' \) is an element of \( S_r(R_{p,1}) \) with \( g x' x = \begin{pmatrix} x' & 0 \\ 0 & 0 \end{pmatrix} \) for \( g \in GL_n(R_{p,1}) \) and \( \det x' \neq 0 \). The above definition is independent of the choice of \( x' \). For \( r = n \), the above two definitions are identical. For \( r = 0 \), we set

\[
\chi_p^{(0)}(x) = \chi_0^{(0)}(x) = \begin{cases} 
1 & \text{if } \text{rank}(x') = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

When it is necessary to indicate the prime \( p \), we write \( \chi_p^{(r)} \) instead of \( \chi_0^{(r)} \).

Let \( p = 2 \), and let \( \varphi \) and \( \chi_0 = \chi_{0,2} \) be as above. For \( x \in L_n \) and \( r \), \( 0 \leq r \leq n \), we define \( \varphi^{(n)} \) and \( \chi_0^{(r)} \) in the same way as in the case of \( p \) odd. For \( x \in L_n^* \), let \( Q_x \) be the quadratic form in \( t_1, t_2, \ldots, t_n \) associated to \( x \). Then \( Q_x \mod p \) is equivalent to one of

\[
t_1 t_2 + \cdots + t_r t_{r-1} + t_r^2,
\]

for \( r \) odd,

(1.1) \[ t_1 t_2 + \cdots + t_{r-1} t_r, \]

(1.2) \[ t_1 t_2 + \cdots + t_{r-3} t_{r-2} + t_{r-1} t_r + t_r^2, \]

for \( r \) even. We define for \( r \) even, \( 2 \leq r \leq n \), and \( x \in L_n^* \)

\[
\chi_p^{*,(r)}(x) = \begin{cases} 
1 & \text{if } Q_x \mod p \text{ is equivalent to (1.1)}, \\
-1 & \text{if } Q_x \mod p \text{ is equivalent to (1.2)}, \\
0 & \text{otherwise}.
\end{cases}
\]

For \( r = 0 \), we define \( \chi_p^{*,(0)}(x) = 1 \) if \( x \in 2L_n^* \), and \( \chi_p^{*,(0)}(x) = 0 \) otherwise.

We consider general characters. Let \( N_1, N_2, N_3 \) be three positive square-free integers coprime to each other. For an odd prime \( p \mid N_1 \), choose a character \( \varphi_p \) defined modulo a power of \( p \) with \( \varphi_p^2 \neq \chi_0 \). When \( 2 \mid N_1 \), we choose a non-trivial character \( \varphi_2 \) defined modulo a power of 2. For each \( p \mid N_2 \) and \( p \mid N_3 \), we choose integers \( r_p, 0 \leq r_p \leq n \). Let \( N_2 \) and \( N_3 \) be the product of primes \( p \) dividing \( N_2 \) such that \( r_p \) is odd or even respectively. We define \( N_2, N_3 \) similarly. For \( L_n \), we assume \( (2, N_2) = 1 \). For \( x \in L_n \), we define

\[
\psi(x) = \prod_{p \mid N_1} \varphi_p^{(n)}(x) \prod_{p \mid N_2} \chi_p^{(r_p)}(x) \prod_{p \mid N_3} \chi_{0,p}^{(r_p)}(x).
\]
For $L^*_n$, we assume $(2, N_1 N_2^* N_3) = 1$, and we define a character $\psi$ by (1.3) replacing $\chi_2^{(r_p)}$ by $\chi_2^{(r_p)}$ when $2 \mid N_2^*$. It is easy to see $\psi$ is invariant under the action of $SL_n(\mathbb{Z})$.

For $\psi$, and $L = L_n, \ L^*_n$, we define

$$
\zeta_i(s, L, \psi) = c_n \sum_{x \in L^{(i)}/SL_n(\mathbb{Z})} \psi(x) \mu(x) |\det x|^{-s},
$$

where

$$
c_n = \frac{2 \prod_{k=1}^n \Gamma(k/2)}{\pi^{n(n+1)/4}},
$$

and $\mu(x)$ is the volume attached to $x$, the definition of which is given in §1 of the part I. This series converges absolutely for $Re(s) > \frac{n+1}{2}$ unless $n = 2$ and $i = 1$.

We give the definition of another kind of $L$-functions introduced by Arakawa. Let $p$ be an odd prime. For $a \in R$ and a positive integer $m$, we set $e_m(a) = \exp(2\pi a \sqrt{-1}/m)$. For $S \in S_n(R_{p,1})$ and $x \in L^*_n$, set

$$
\tau_S^{(n)}(x) = \sum_{y \sim S} e_p(\text{tr}(xy)),
$$

where $y$ is extended over all $y \in S_n(R_{p,1})$ which are equivalent to $S$. Here we understand $e_p(\text{tr}(z)) = e_p(\text{tr}(\bar{z}))$ for $z \in S_n(R_{p,1})$ with $\bar{z} (\in S_n(\mathbb{Z}))$ mod $p = z$. Then Arakawa’s $L$-function is defined by

$$
\zeta_i(s, L, S) = c_n \sum_{x \in L^{(i)}/SL_n(\mathbb{Z})} \tau_S^{(n)}(x) \mu(x) |\det x|^{-s},
$$

for $L = L_n, \ L^*_n$. This series converges absolutely also for $Re(s) > \frac{n+1}{2}$ unless $n = 2$ and $i = 1$.

To describe the relation between these $L$-functions, we recall the Gauss sums introduced in [Sai1]. For $\eta = \chi_p$, or $\chi_0$, and $x \in S_n(R_{p,1})$ of rank $r$, we define

$$
\eta(x) = \eta(\det x'),
$$

where $x'$ is an element of $S_r(R_{p,1})$ such that $^tgxg = \begin{pmatrix} x' & 0 \\ 0 & 0 \end{pmatrix}$ with $g \in GL_n(R_{p,1})$ and $\det x' \neq 0$. Then $\eta(x)$ is well-defined. If $r = 0$, we set $\eta(x) = 1$. We define

$$
W^n_r(x, \eta) = \sum_y \eta(y) e_p(\text{tr}(xy)),
$$
where \( y \) runs through all elements of \( S_n(R_{p,1}) \) of rank \( r \). For \( x \in L_n \) or \( L^*_n \), we set \( W^n_r(x, \eta) = W^n_r(\tilde{x}, \eta) \) with \( \tilde{x} \equiv x \mod p \). Then for \( S \in S_n(R_{p,1}) \) of rank \( r \), we have

\[
\tau_S^n(x) = \frac{1}{2}(W^n_r(x, \chi_0) + \chi_p(S)W^n_r(x, \chi_p)).
\]

For two integers \( r, t \) such that \( 0 \leq r, t \leq n \) and \( r \equiv t \mod 2 \), we define \( W^n(r, t) \) as follows. When \( r = t = 1 \mod 2 \), we set

\[
W^n(r, t) = W^n(x, \chi_p)\chi_p(x),
\]

where \( x \) is an element of \( S_n(R_{p,1}) \) of rank \( t \). Then by Cor. 1.2 of [Sai1], this is independent of the choice of \( x \) (denoted by \( W^n_o(i, j) \) with \( r = 2i - 1, t = 2j - 1 \) in [Sai1]). When both of \( r \) and \( t \) are even, we set

\[
W^n(r, t) = W^n_r(x, \chi_p)
\]

for \( x \in S_n(R_{p,1}) \) with rank \( x = t \). This is also independent of the choice of \( x \) by Cor. 1.14 of [Sai1] (The proof of Cor. 1.14 there is incomplete in the case where \( n \) is even and rank \( x = t = n \). But this case can be deduced easily from Prop. 1.12 of [Sai1]). Let \( G(\chi_p) \) be the usual Gauss sum for \( \chi_p \). In these notations, we can prove

**Proposition 1.1.** Let \( p \) be an odd prime and let \( x \in S_n(R_{p,1}) \).

(1) If \( r \) is odd, then

\[
W^n_r(x, \chi_p) = \sum_{j=0}^{[n-1]/2} W^n(r, 2j + 1)\chi_p^{(2j+1)}(x),
\]

\[
W^n_r(x, \chi_0) = G(\chi_p)^r \sum_{j=1}^{[n/2]} W^{n-1}(r, 2j - 1)\chi_p^{(2j)}(x)
\]

\[
- G(\chi_p)^{r-1} \sum_{j=1}^{[(n+1)/2]} W^{n-1}(r - 1, 2j - 2)(\chi_0^{(2j)}(x) + chi_0^{(2j-1)}(x))
\]

\[
+ W^n_r(O_n, \chi_0)\chi_0^{(0)}(x).
\]

(2) Let \( r \geq 2 \) be even, and let \( \tilde{x} \) be an element of \( S_{n+1}(R_{p,1}) \) such that \( \chi_p(\tilde{x}) = \chi_p(x) \) and rank \( \tilde{x} \) = rank \( x \) + 1. Then one has

\[
W^n_r(x, \chi_p) = \sum_{j=0}^{[n/2]} W^n(r, 2j)(\chi_0^{(2j)}(x) + \chi_0^{(2j+1)}(x)),
\]

\[
W^n_r(x, \chi_0) = G(\chi_p)^{-r-1}W^{n+1}(\tilde{x}, \chi_p) - W^n_{r+1}(x, \chi_0).
\]
Here we understand that $\chi_p^{(m)}(x) = 0$ for $m > n$ when $n$ is even, and $O_n$ is the zero matrix of degree $n$.

**Proof.** These assertions follow easily from Cor. 1.2, Cor. 1.14, Prop. 1.13, Prop. 1.11 and Prop. 1.12 of [Sai1].

This shows that the L-functions $\zeta_i(s, L, S)$ can be written as linear combinations of $\zeta_i(s, L, \chi)$. For example, if the rank $r$ of $S$ is odd, then by (1) of Prop. 1.1. and (1.4) we have

$$
\zeta_i(s, L, S) = \frac{1}{2} \left( \chi_p(S) \sum_{j=0}^{[n-1]/2} W^n(r, 2j + 1) \zeta_i(s, L, \chi_p^{(2j+1)}) \right.
$$

$$
+ G(\chi_p)^r \sum_{j=1}^{[n/2]} W^{n-1}(r, 2j - 1) \zeta_i(s, L, \chi_p^{(2j)})
$$

$$
- G(\chi_p)^{r-1} \sum_{j=1}^{[(n+1)/2]} W^{n-1}(r - 1, 2j - 2)(\zeta_i(s, L, \chi_0^{(2j)}) + \zeta_i(s, L, \chi_0^{(2j-1)}))
$$

$$
- W^n_r(O_n, \chi_0) \zeta_i(s, L, \chi_0^{(0)}))
.$$}

We can prove a similar formula for $S$ of rank $r$ even by (2) of Prop. 1.1. Hence the rationality of special values of $\zeta_i(s, L, S)$ follows from that of $\zeta_i(s, L, \psi)$.

Here we insert a result on Gauss sums, which is a complement to Th. 1.15 and Cor. 1.17 of [Sai] (in Th. 1.15, $W^n_o(\chi_p^2)$ should be read $W^n_o(\chi_p)$). There, for $u, v$ such that $0 \leq u, v \leq [n/2]$, we define

$$
W^n_{e}(u, v) = \begin{cases} 
W^n_{2u}(x, \chi_0)\chi_p(x) & \text{if } n \text{ is even and } n = 2u, \\
(W^n_{2u+1}(x, \chi_0) \\
+ W^n_{2u}(x, \chi_0))\chi_p(x) & \text{otherwise},
\end{cases}
$$

with $x \in S_n(R_{p,1})$ of rank $x = 2v$, which is independent of the choice of $x$, and the Gauss sum

$$
W^n_{e}(\chi_p) = (W^n_e(i - 1, j - 1)).
$$

We define one more matrix-valued Gauss sum

$$
U^n_e(\chi_p) = (W^n(2i - 2, 2j - 2)),
$$

where the $(i, j)$ component of $U^n_e(\chi_p)$ is $W^n(2i - 2, 2j - 2)$. 


Theorem 1.2. The notation being as above, one has

\[ U_c^n(\chi_0)W_c^n(\chi_p) = p^{n(n+1)/2}E_{[n/2]+1}, \]

where \( E_{[n/2]+1} \) is the unit matrix of degree \([n/2] + 1\).

Proof. We give a proof only for the case \( n \) odd, since the other case can be treated in the same way. Then the \((i,j)\) component of the product of matrices on the left hand side is equal to

\[ \sum_{k=1}^{[n/2]+1} W_{2i-2}^n(x, \chi_p)(W_{2k-2}^n(y, \chi_0) + W_{2k-1}^n(y, \chi_0))\chi_p(y), \]

where \( x, y \in S_n(R_p) \) of rank \( 2k - 2, 2j - 2 \) respectively. Using the fact that (cf. Prop. 1.13 of [Sail])

\[ W_{2i-2}^n(x, \chi_p) = W_{2i-2}^n(x', \chi_p) \]

for \( x, x' \in S_n(R_p) \) of rank \( 2k - 1, 2k - 2 \) respectively, we see (1.5) is equal to

\[ \sum_{z \in S_n(R_p)} \chi_p(y)e_p(\text{tr}(yz))W_{2i-2}^n(z, \chi_p) \]

\[ \quad = \sum_w \chi_p(y)\chi_p(w)\sum_{z \in S_n(Z/pZ)} e_p(\text{tr}((y + w)z)) \]

\[ \quad = p^{n(n+1)/2}\delta_{ij}, \]

where \( w \) runs through all elements of \( S_n(R_p) \) of rank \( 2i - 2 \). Here we used the fact that the rank of \( y \) is even. This completes the proof.

\[ \text{§2. Explicit form of L-functions} \]

In this section, we give an explicit form of L-functions assuming the results in §3 and discuss the rationality of the values of L-functions at non-positive integers. As for the calculation of the L-functions, we follow the procedure of [I-S2], and only give an outline.

Let \( N_1, N_2, N_3 \) and \( \psi \) be as in §1. We set \( \psi_p = \varphi_p^{(n)}, \chi_p^{(r_p)}, \text{or} \chi_{0,p}^{(r_p)} \) according to whether \( p \) divides \( N_1, N_2, \text{or} N_3 \), and extend \( \psi_p \) to \( S_n(R_{p,\nu}) \) for a large \( \nu \) and to \( S_n(Z_p) \) naturally. For \( p \) prime to \( N_1N_2N_3 \), let \( \psi_p \) be the characteristic functions of \( S_n(R_{p,\nu}) \) or \( S_n(Z_p) \). For \( p = 2 \), let \( S_n(Z_p) \)
be the subset of \( S_n(Z_p) \) consisting of elements \((x_{ij})\) such that \( x_{ii} \equiv 0 \mod p \) for all \( i \), and let \( S_n(R_{p,\nu})_e \) be the similar subset of \( S_n(R_{p,\nu}) \).

If \( 2 \mid N_2^e \), we set \( \psi_p^*(x) = \bar{\chi}_2^*(r_p)(x) = \chi_2^*(r_p)(y) \) for \( x \in S_n(R_{p,\nu})_e \) or \( S_n(Z_p)_e \) taking \( y \in L_n^* \) such that \( 2y \equiv x \mod p^\nu \). When \( (2, N_2^e) = 1 \), let \( \psi_p^* \) be the characteristic function of \( S_n(R_{p,\nu}) \) or \( S_n(Z_p) \). In the following, we assume \( r_p \geq 1 \), since the case of \( r_p = 0 \) can be easily reduced to the case where \( p \nmid N_2 N_3 \).

For \( i, 0 \leq i \leq n \), let
\[
\delta = (-1)^{n-i}, \quad \epsilon = (-1)^{(n-i)(n-i+1)/2},
\]
and set
\[
\xi_i(s, L_n, \psi) = \sum_{d=1}^{\infty} a_i(d)|d|^{-s}, \quad \xi_i(s, L_n^*, \psi) = \sum_{d=1}^{\infty} a_i^*(d)2^{ns}|d|^{-s}.
\]

The first step is to express \( a_i(d) \) and \( a_i^*(d) \) by local data. For this, we introduce some notations. Let \( \iota_p \) and \( \varepsilon_p \) be the constant function with value 1 and the Hasse invariant on \( S_n(Z_p) \) or \( S_n(R_{p,\nu}) \) respectively, and for \( a, b \in \mathbb{Q}_p^* \), let \( (a, b)_p \) be the Hilbert symbol of \( a \) and \( b \). For \( \omega_p = \iota_p \) or \( \varepsilon_p \) and \( d \in Z_p, d \neq 0 \), we define
\[
\lambda_p(\psi_p, d, \omega_p) = \lim_{\nu \to \infty} \lambda_{p,\nu}(\psi_p, d, \omega_p) \quad (p = 2),
\]
\[
(\lambda_p^*(\psi_p^*, d, \omega_p) = \lim_{\nu \to \infty} \lambda_{p,\nu}^*(\psi_p^*, d, \omega_p) \quad (p = 2),
\]
where
\[
\lambda_{p,\nu}(\psi_p, d, \omega_p) = 2^{-\delta_2 r_p v(d)+(n(n-1)/2)\nu}|SL_n(R_{p,\nu})|^{-1} \sum_{x \in S_n(R_{p,\nu}, d)} \psi_p(x)\omega_p(x)
\]
\[
(\lambda_{p,\nu}^*(\psi_p^*, d, \omega_p) = 2^{-\delta_2 r_p v(d)+(n(n-1)/2)\nu}|SL_n(R_{p,\nu})|^{-1} \times \sum_{x \in S_n(R_{p,\nu}, d)_e} \psi_p^*(x)\omega_p(x) \quad (p = 2).
\]

Here \( v \) is the additive valuation of \( Z_p \) such that \( v(p) = 1 \), and
\[
S_n(R_{p,\nu}, d) = \{ x \in S_n(R_{p,\nu}) \mid \det x \equiv d \mod p^\nu \}
\]
\[
(S_n(R_{p,\nu}, d)_e = S_n(R_{p,\nu}, d) \cap S_n(R_{p,\nu})_e \quad (p = 2).
\]
For \( \omega = \iota \) or \( \epsilon \), and \( d \in \mathbb{Z}, d \neq 0 \), we define
\[
\lambda_f(\psi, d, \omega) = \prod_p \lambda_p(\psi_p, d, \omega_p)
\]
\[
(\lambda_f^*(\psi, d, \omega) = \lambda_2(\psi_p^*, d, \omega_p) \prod_{p \neq 2} \lambda_p(\psi_p, d, \omega_p) \text{ for } p = 2).
\]

Then by Siegel’s formula and the invariance of \( \psi \) in a genus, in the same way as Prop. 2.2 of [I-S], we obtain
\[
a_i(d) = c_n(\lambda_f(\psi, d, \iota) + \epsilon \lambda_f(\psi, d, \epsilon))|d|^{(n+1)/2},
\]
\[
a_i^*(d) = c_n \prod_{p \mid N_1} \varphi_p(2^{-n}) \prod_{p \mid N_2} \chi_p(2^{-n})(\lambda_f^*(\psi, d, \iota) + \epsilon \lambda_f^*(\psi, d, \epsilon))|d|^{(n+1)/2}.
\]

As in the case of zeta-functions, our L-functions depends only on \( \delta \) and \( \epsilon \), and we set
\[
\xi(s, L, \psi, \delta, \epsilon) = \xi_i(s, L, \psi).
\]

To sum up the above quantities, we introduce another local data
\[
\lambda_p(\psi_p, d, \omega_p, \{n_i\}, \{d_i\}), \quad \lambda^*_p(\psi_p^*, d, \omega_p, \{n_i\}, \{d_i\}),
\]
and some power series. Let \( n = n_1 + n_2 + \cdots + n_m \) be a partition of \( n \) into \( m \) positive integers. We denote this by \( \{n_i\} \) and call \( m \) the length of \( \{n_i\} \). A partition is called even if all \( n_i \) are even and odd otherwise. A sequence \( t_1, t_2, \cdots, t_m \) of integers of the same length \( m \) is called a sequence associated to \( \{n_i\} \) if it satisfies \( t_1 < t_2 < \cdots < t_m \). For \( \{n_i\}, \{t_i\} \) as above, let \( S_n(R_{p, \nu}, d, \{n_i\}, \{t_i\}) \) be the subset of \( S_n(R_{p, \nu}, d) \) consisting of elements equivalent to
\[
\begin{pmatrix}
p^{t_1}x_1 & 0 & \cdots & 0 \\
0 & p^{t_2}x_2 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & p^{t_m}x_m
\end{pmatrix}
\]
with respect to \( GL_n(R_{p, \nu}) \), for \( x_i \in S_{n_i}(R_{p, \nu}, R_{p, \nu}^X) \), and let
\[
S_n(R_{p, \nu}, d, \{n_i\}, \{t_i\}) = S_n(R_{p, \nu}, d, \{n_i\}, \{t_i\}) \cap S_n(R_{p, \nu}).
\]

The matrix of the form (2.1) will be denoted by \( (\oplus p^{t_i}x_i) \). Similarly as above, for \( \{n_i\} \) and \( \{t_i\} \), we define
\[
\lambda_p(\psi, d, \omega, \{n_i\}, \{t_i\}) = \lim_{\nu \to \infty} \lambda_{p, \nu}(\psi, d, \omega, \{n_i\}, \{t_i\}),
\]
\[
(\lambda_p^*(\psi_p^*, d, \omega, \{n_i\}, \{t_i\}) = \lim_{\nu \to \infty} \lambda_{p, \nu}^*(\psi_p^*, d, \omega, \{n_i\}, \{t_i\}) \text{ for } p = 2).
\]
where

\[
\lambda_{p,\nu}(\psi_p, d, \omega_p, \{n_i\}, \{t_i\}) = 2^{-\delta_2, p} v(d)^{(n(n-1)/2)} \sum_{\tau \in S_n(R(d), \{n_i\}, \{t_i\})} \psi_p(x)\omega_p(x)
\]

and

\[
\lambda_{p,\nu}^*(\psi_p^*, d, \omega_p, \{n_i\}, \{t_i\}) = 2^{-\delta_2, p} v(d)^{(n(n-1)/2)} \sum_{\tau \in S_n^*(R(d), \{n_i\}, \{t_i\})} \psi_p^*(x)\omega_p(x)
\]

for \( p = 2 \).

Let \( t = \sum_{i=1}^m n_i t_i \). Then we have

\[
\lambda_p(\psi_p, d, \omega_p) = \sum_{\{n_i\}, \ t = v(d)} \lambda(\psi_p, d, \omega_p, \{n_i\}, \{t_i\})
\]

\[
(\lambda_p^*(\psi_p^*, d, \omega_p) = \sum_{\{n_i\}, \ t = v(d)} \lambda_p^*(\psi_p^*, d, \omega_p, \{n_i\}, \{t_i\}) \text{ for } p = 2).
\]

Here \( \{n_i\} \) and \( \{t_i\} \) run through all partitions of \( n \) and all sequences associated to them satisfying \( v(d) = t \). By these constants, we define orbital local series with characters as follows. For \( \psi_p, \psi_p^*, \omega_p \) as above, we set

\[
\lambda_p(\psi_p, d, \omega_p, \{n_i\}, \{t_i\}) = \sum_{\{n_i\}, \ t = v(d)} \lambda(\psi_p, d, \omega_p, \{n_i\}, \{t_i\})
\]

\[
(\lambda_p^*(\psi_p^*, d, \omega_p, \{n_i\}, \{t_i\}) = \sum_{\{n_i\}, \ t = v(d)} \lambda_p^*(\psi_p^*, d, \omega_p, \{n_i\}, \{t_i\}) \text{ for } p = 2)
\]

\[
\times \begin{cases} 
((-1)^{(n(n+1)/2)}, p)^t & \text{if } n \text{ is odd and } \omega_p = \varepsilon_p, \\
1 & \text{otherwise},
\end{cases}
\]

and for \( d_0 \in \mathbb{Z}^\times_p \), define

\[
Z_n(u, \psi_p, \omega_p, d_0) = \sum_{\{n_i\}} \lambda_p^*(\psi_p, d_0 p^t, \omega_p, \{n_i\}, \{t_i\}) p^{(n(n+1)/2)} u^t
\]

\[
(\lambda_p^*(\psi_p^*, d_0 p^t, \omega_p, \{n_i\}, \{t_i\}) p^{(n(n+1)/2)} u^t \text{ for } p = 2),
\]

where \( \{n_i\} \) runs through all partitions of \( n \) and \( \{t_i\} \) runs through all sequences associated to \( \{n_i\} \) satisfying \( 0 \leq t_1 \). In the case \( n \) is even, define

\[
Z_{n,0}(u, \psi_p, t_p, d_0) = \frac{1}{2} (Z_n(u, \psi_p, t_p, d_0) - Z_n(-u, \psi_p, t_p, d_0)),
\]

\[
Z_{n,e}(u, \psi_p, t_p, d_0) = \frac{1}{2} (Z_n(u, \psi_p, t_p, d_0) + Z_n(-u, \psi_p, t_p, d_0)),
\]
and define $Z_{n,o}^*(u, \psi^*_p, \iota_p, d_0)$ and $Z_{n,e}(u, \psi^*_p, \iota_p, d_0)$ similarly. We denote the series associated to the characteristic functions of $S_n(Z_p)$, or $S_n(Z_p)_e$ simply by

$$Z_n(u, \omega_p, d_0), \quad Z_n(u, \omega_p, d_0), \quad Z_{n,o}^*(u, \omega_p, d_0),$$

and so on. These are calculated in §5 of [I-S2]. The other series will be calculated in §3.

We treat the cases of $n$ odd and $n$ even separately. First let $n$ be odd. To state our result, we introduce some notations. For $\psi$, we define Dirichlet characters $\psi$ and $\psi$ by

$$\Phi = \prod_{p \mid N_1} \chi_{p^r}, \quad \tilde{\psi} = \prod_{p \mid N_1} \varphi_p \prod_{p \mid N_2} \chi_p.$$

For $p \mid N_2N_3$, set

$$A_p(u, \psi_p, \iota_p) = \frac{Z_n(u, \psi_p, \iota_p, 1)/Z_n(u, \iota_p, 1)}{(1 - p^{(n-1)/2}u)^{-1}} \quad \text{if } p \mid N_2^e, \quad \text{otherwise},$$

$$A_p(u, \psi_p, \epsilon_p) = \frac{Z_n(u, \psi_p, \epsilon_p, 1)/Z_n(u, \epsilon_p, 1)}{(1 - u)^{-1}} \quad \text{if } p \mid N_2^o, \quad \text{otherwise}.$$

If $p = 2 \mid N_2^e$, we set

$$A_p^*(u, \psi_p^*, \omega_p) = \frac{Z_{n}^*(u, \psi_p^*, \omega_p, 1)/Z_{n}^*(u, \omega_p, 1)}{A^*(s, \psi, \omega)}$$

For $\omega = \iota$, $\epsilon$, and $L_n$ define

$$A(s, \psi, \omega) = \prod_{p \mid N_2N_3} A_p\left(\prod_{q \mid N_1} \varphi_q(p) \prod_{q \mid N_2^e, q \neq p} \chi_q(p) p^{-s}, \psi_p, \omega_p\right)$$

and for $L_n^*$, define $A^*(s, \psi, \omega) = A(s, \psi, \omega)$ if $2 \nmid N_2^e$ and if $2 \mid N_2^e$, define $A^*(s, \psi, \omega)$ replacing $A_2(u, \psi_p^*, \omega_p)$ by $A_2^*(u, \psi_p^*, \omega_p)$ in the above definition of $A(s, \psi, \omega)$. In these notations, we can prove

**Theorem 2.1.** Let $n$ be an odd integer $\geq 3$, and assume $r_p \geq 1$ for
\( p \mid N_2N_3 \). Let \( A(s, \psi, \omega) \), \( A^*(s, \psi, \omega) \), and \( \tilde{\psi}, \tilde{\psi} \) be as above. Then one has

\[
\xi(s, L_n, \psi, \delta, \epsilon) = \left\{ \begin{array}{l}
\prod_{i=1}^{[n/2]} B_{2i} \prod_{i=1}^{[n/2]} L(s - 2i, \bar{\psi}) L(2s - (2i - 1), \bar{\psi}^2) \\
\times \left( 2^{(n-1)/2} A(s, \psi, \bar{\psi}) L(s - \frac{n-1}{2}, \bar{\psi}) \prod_{i=1}^{[n/2]} L(2s - (2i - 1), \bar{\psi}^2) \right) \\
+ \epsilon \delta^{(n+1)/2} (-1)^{(n^2-1)/8} A(s, \psi, \bar{\psi}) L(s, \bar{\psi}) \prod_{i=1}^{[n/2]} L(2s - 2i, \bar{\psi}^2) \right) \right.
\]

\[
\xi(s, L_n^*, \psi, \delta, \epsilon) = \left\{ \begin{array}{l}
\prod_{i=1}^{[n/2]} B_{2i} \prod_{p \mid N_1} \varphi_p(2^{-n}) \prod_{p \mid N_2} \chi_p(2^{-\sigma_p}) \\
\times \left( A^*(s, \psi, \bar{\psi}) L(s - \frac{n-1}{2}, \bar{\psi}) \prod_{i=1}^{[n/2]} L(2s - (2i - 1), \bar{\psi}^2) \right) \\
+ \epsilon \delta^{(n+1)/2} (-1)^{(n^2-1)/8} A^*(s, \psi, \bar{\psi}) L(s, \bar{\psi}) \prod_{i=1}^{[n/2]} L(2s - 2i, \bar{\psi}^2) \right) \right.
\]

Proof. We give a proof only for \( L_n \). We note \( (d, N_1) = 1 \) if \( \lambda_f(\psi, d, \omega) \neq 0 \). For \( \delta d > 0 \), let

\[
d = \delta \prod_p p^{r_p} = p^{r_p} d_0, \quad d_0 \in Z_p^\times.
\]

By the results in §3, \( Z_n(u, \psi, p, t_p, d_0) \) is independent of \( d_0 \) if \( p \nmid N_1N_2 \), and we see for \( \omega = \bar{\psi} \)

\[
\lambda_f(\psi, d, \bar{\psi}) = \prod_{p \mid N_1} 2^{-\delta_2p} \varphi_p(d) (p^{-2})^{[n/2]} \prod_{p \mid N_2} \chi_p(d_0, p) \prod_{(p, N_1) = 1} \lambda_p(p, t_p, \bar{\psi}) \\
= \tilde{\psi}(\delta) \prod_{p \mid N_1} 2^{-\delta_2p} (p^{-2})^{[n/2]} \\
\times \prod_{(p, N_1) = 1} \left( \prod_{q \mid N_1} \varphi_q(p) \prod_{q \mid N_2, q \neq p} \chi_q(p) \right)^{t_p} \lambda_p(p, t_p, \bar{\psi}).
\]

From this we see

\[
\sum_{\delta d > 0} \lambda_f(\psi, d, \bar{\psi}) d^{(n+1)/2-s}
\]
= \tilde{\psi}(\delta) \prod_{p \mid N_1} 2^{-\delta_2, p} (p^{-2})_{[n/2]}^{-1} \\
\times \prod_{(p,N_1)=\mathbb{Z}_p} \left( \sum_{q \mid N_1} \chi_q(p) \prod_{q \mid N_2} \chi_q^*(p) \right)^{t_p} \lambda_p(\psi_p, \psi_p^*, \epsilon_p) \lambda_p^{(n+1)}(n-p, \epsilon_p) \\
= \tilde{\psi}(\delta) \prod_{p \mid N_1} 2^{-\delta_2, p} (p^{-2})_{[n/2]}^{-1} \\
\times \prod_{(p,N_1)=1} Z_n \left( \prod_{q \mid N_1} \chi_q(p) \prod_{q \mid N_2} \chi_q^*(p) \right)^{n-p, \psi_p, \epsilon_p, 1).}

From this we easily obtain our formula. The case of \omega = \epsilon can be treated in the same way, and will be omitted.

From this theorem, we can deduce the following result on the rationality of the values at non-positive integers of L-functions.

**Corollary 2.2.** Let Q(\psi) be the field generated by the values of \psi over Q. Then for a positive integer m, the values \xi(1-m, L_n, \psi, \delta, \epsilon) and \xi(1-m, L^*_n, \psi, \delta, \epsilon) are contained in Q(\psi).

**Proof.** Since

\[
\prod_{i=1}^{[n/2]} L(2(1-m) - 2i, \psi^2) = 0,
\]

we have

\[
\xi(1-m, L_n, \psi, \delta, \epsilon) = \frac{\prod_{i=1}^{[n/2]} B_{2i}}{2^{n-1}(n-1)!} \tilde{\psi}(\delta) A(1-m, \psi, \epsilon) 2^{(n-1)/2} \\
\times L(1-m - \frac{n-1}{2}, \tilde{\psi}) \prod_{i=1}^{[n/2]} L(1-2(m+i-1), \tilde{\psi}^2).
\]

Our assertion for \(L_n\) easily follows from this. The case of \(L^*_n\) is similar.

Now we turn to the case of \(n\) even. We introduce more notations. For a quadratic field \(K\), we denote by \(d_K\) the discriminant of \(K\) and for \(K = Q \oplus Q\), we set \(d_K = 1\). For a quadratic field \(K\), we denote by \(\chi_K\) the Dirichlet character corresponding to \(K\), and for \(K = Q \oplus Q\) by \(\chi_K\) the trivial character.
To describe the \( \psi \)-part of the L-function, we define two Dirichlet series \( D(s, \psi, \delta) \) and \( D^*(s, \psi, \delta) \). If \( N_2^\varepsilon \neq 1 \), we set \( D(s, \psi, \delta) = D^*(s, \psi, \delta) = 0 \).

For \( K \) as above and an odd prime \( p \mid N_2N_3 \), we set

\[
B_p(u, \psi_p, \iota_p, K) = \begin{cases} 
Z_{n,o}(u, \psi_p, \iota_p, d_K/p)/Z_{n,o}(u, \iota_p, d_K/p) & \text{if } p \mid N_2N_3, p \mid d_K, \\
Z_{n,e}(u, \psi_p, \iota_p, d_K)/Z_{n,e}(u, \iota_p, d_K) & \text{if } p \mid N_2N_3, p \not\mid d_K,
\end{cases}
\]

for \( p = 2 \mid N_3 \)

\[
B_p(u, \psi_p, \iota_p, K) = \begin{cases} 
Z_{n,o}(u, \psi_p, \iota_p, d_K/p^3)/Z_{n,o}(u, \iota_p, d_K/p^3) & \text{if } p^3 \mid d_K, \\
Z_{n,e}(u, \psi_p, \iota_p, d_K/p^2)/Z_{n,e}(u, \iota_p, d_K/p^2) & \text{if } p^2 \mid d_K, \\
Z_{n,e}(u, \psi_p, \iota_p, d_K)/Z_{n,e}(u, \iota_p, d_K) & \text{if } p \not\mid d_K,
\end{cases}
\]

and for \( K \) with \( (d_K, 2) = 1 \)

\[
\tilde{B}_2(u, K) = \left\{ \begin{array}{ll}
(1 - (1 - u^2)(1 - 2^{n-1}u^2)(1 - \chi_K(2)2^{n/2-1}u^2)-1) & \text{if } 2 \not\mid N_1, \\
(1 + \chi_K(2)2^{-n/2}-1)(1 + 2^{-n} + 2\chi_K(2)2^{-n-n/2}) & \text{if } 2 \mid N_1.
\end{array} \right.
\]

If \( 2 \mid N_2^\varepsilon \) for \( L = L^*_n \), we define \( B^*_2(u, \psi^*_2, \iota_2, K) \) in the same way as above taking \( Z_{n,o}(u, \psi^*_2, \iota_2, d_0), Z_{n,e}(u, \psi^*_2, \iota_2, d_0) \) instead of \( Z_{n,o}(u, \psi_2, \iota_2, d_0),Z_{n,e}(u, \psi_2, \iota_2, d_0) \).

Using these functions, we define

\[
B(s, \psi, \iota, K) = \prod_{p \mid N_2N_3} B_p(\hat{\psi}(p)p^{-s}, \psi_p, \iota_p, K) \times \left\{ \begin{array}{ll}
\hat{\psi}(d_K)\tilde{B}_2(\hat{\psi}(2)^{-2s}, K) & \text{if } 2 \not\mid d_K, \\
\hat{\psi}(d_K/4) & \text{if } 2 \mid d_K,
\end{array} \right.
\]

\[
B^*(s, \psi, \iota, K) = \hat{\psi}(d_K) \left\{ \begin{array}{ll}
\prod_{p \mid N_2N_3} B_p(\hat{\psi}(p)p^{-s}, \psi_p, \iota_p, K) & \text{if } 2 \not\mid N_2^\varepsilon, \\
B_2^*(\hat{\psi}(2)^{-s}, \psi^*_2, \iota_2, K) \times \prod_{p \mid N_2N_3/2} B_p(\hat{\psi}(p)p^{-s}, \psi_p, \iota_p, K) & \text{if } 2 \mid N_2^\varepsilon.
\end{array} \right.
\]

and the Dirichlet series by

\[
D^*(s, \psi, \delta) = (-1)^{[n/4]} \sum_{d_K \mid N_1} 2(2\pi)^{-n/2}(\frac{n}{2} - 1)!|d_K|^{(n-1)/2}
\]

\[
\times B^*(s, \psi, \iota, K)\hat{\psi}(d_K)L(\frac{n}{2}, \chi_K) \frac{L(2s, \hat{\psi}(2))L(2s - n + 1, \hat{\psi}(2))}{L(2s - (n/2 - 1), \hat{\psi}(2)\chi_K)} |d_K|^{-s},
\]
\[ D(s, \psi, \delta) = (-1)^{[n/4]}2^{2s} \sum_{\substack{d_K > 0 \atop \delta d_K > 0 \atop (d_K, N_1) = 1}} 2(2\pi)^{-n/2}(\frac{n}{2} - 1)!|d_K|^{(n-1)/2} \times \text{B}(s, \psi, \nu, K) L\left(\frac{n}{2}, \chi_K\right) \frac{L(2s, \hat{\psi}^2) L(2s - n + 1, \psi^2)}{L(2s - (n/2 - 1), \hat{\psi}^2 \chi_K)} |d_K|^{-s}. \]

Here \( K \) runs through quadratic fields or \( Q \oplus Q \) such that \((-1)^{n/2}d_K > 0\), and \( d'_K = d_K/(d_K, 4) \). In the above definition, we understand \( \hat{\psi}^2 \chi_K \) as a character modulo \( N_1 d_K \).

Next we introduce notations necessary to describe the \( \varepsilon \)-part of the \( L \)-function. We set

\[ \kappa(n, \delta; \psi) = \begin{cases} 1 & \text{if } (-1)^{n/2}dN_2^2 \equiv 1 \mod 4, \\ 0 & \text{otherwise,} \end{cases} \]

and denote by \( H \) the quadratic field or \( Q \oplus Q \) such that \( d_H = (-1)^{n/2}dN_2^2 \) when \( \kappa(n, \delta; \psi) = 1 \). For a prime \( p \mid N_2N_3 \), we set

\[ B_p(u, \psi_p, \varepsilon) = Z_n(u, \psi_p, \varepsilon_p, d_H/p)/Z_n(u, \varepsilon_p, d_H/p) \times \begin{cases} (1 - ((-1)^{n/2}d_H/p, p)p^{-n/2})^{-1} & \text{if } p \mid N_2^c, \\ 1 & \text{otherwise}, \end{cases} \]

and when \( 2 \mid N_2^c \) for \( L_n^* \), for \( p = 2 \), we set

\[ B_p^*(u, \psi_p, \varepsilon_p) = Z_n^*(u, \psi_p^*, \varepsilon_p, d_H)/Z_n^*(u, \varepsilon_p, d_H). \]

Finally we define

\[ B(s, \psi, \varepsilon) = \prod_{p \mid N_2N_3} B_p(\hat{\psi}(p)p^{-s}, \psi_p, \varepsilon_p), \]

\[ B^*(s, \psi, \varepsilon) = \begin{cases} \prod_{p \mid N_2N_3} B_p(\hat{\psi}(p)p^{-s}, \psi_p, \varepsilon_p) & \text{if } 2 \nmid N_2^c, \\ B_2^*(\hat{\psi}(p)p^{-s}, \psi_p, \varepsilon_p) \times \prod_{p \mid N_2N_3/2} B_p(\hat{\psi}(p)p^{-s}, \psi_p, \varepsilon_p) & \text{if } 2 \mid N_2^c. \end{cases} \]

In these notation, we can prove

**Theorem 2.3.** Let \( n \) be even \( \geq 4 \), and assume \( r_p \geq 1 \) for \( p \mid N_2N_3 \).
Let the notation be as above. Then one has

\[
\xi(s, L_n, \psi, \delta, \epsilon) = \frac{\prod_{i=1}^{n/2-1} B_{2i}}{2^{n-1} \binom{n-2}{2}} \psi((-1)^{n/2}) \left((-1)^{[n/4]} D(s, \psi, \delta) \prod_{i=1}^{[n/2]-1} L(2s - 2i, \psi^2) \right. \\
\left. + \epsilon \kappa(n, \delta, \psi) (-1)^{n(n+2)/8} \psi(d_H) \frac{2^{(n+2)/2} B_{n/2, \psi}}{n} \times B(s, \psi, \epsilon) \prod_{i=1}^{n/2} L(2s - (2i - 1), \psi^2) \right),
\]

\[
\xi(s, L_n^*, \psi, \delta, \epsilon) = 2^{ns} \frac{\prod_{i=1}^{n/2-1} B_{2i}}{2^{n-1} \binom{n-2}{2}} \psi((-1)^{n/2}) \left(\prod_{p \mid N_1} \varphi_p(2^{-n}) \prod_{p \mid N_2^o} \chi_p(2^{-r_p}) \right. \\
\left. \times \left((-1)^{[n/4]} D^*(s, \psi, \delta) \prod_{i=1}^{[n/2]-1} L(2s - 2i, \psi^2) \right. \\
\left. + \epsilon \kappa(n, \delta, \psi) (-1)^{n(n+2)/8} \psi(d_H) \frac{2B_{n/2, \psi}}{n} \times B^*(s, \psi, \epsilon) \prod_{i=1}^{n/2} L(2s - (2i - 1), \psi^2) \right),
\]

where

\[
B'_{n/2, \psi} = 2\left(\frac{n}{2}\right)! \left(2\pi\right)^{-n/2} L\left(\frac{n}{2}, \chi_H\right).
\]

**Proof.** We treat the case of \(L_n^*.\) In this case, \((N_1 N_2^o N_3, 2) = 1.\) As in the case of \(n\) odd, the L-function is, up to the factor

\[
2^{ns} \prod_{p \mid N_1} \varphi_p(2^{-n}) \prod_{p \mid N_2^o} \chi_p(2^{-r_p}),
\]

the sum of the following two series

\[
(2.1) \quad c_n \sum_{d=1}^{\infty} \lambda^*_d(d, \psi, \epsilon) |d|^{(n+1)/2-s},
\]

\[
(2.2) \quad c_n \sum_{d=1}^{\infty} \lambda^*_d(d, \psi, \epsilon) |d|^{(n+1)/2-s}.
\]
We note \((d, N_1) = 1\) if \(\lambda^*_N(d, \psi, \omega) \neq 0\).

First we calculate the series \((2.1)\). Let \(d = \det 2x\) for \(x \in L^*_n\). Then there exists \(K\) and a positive integer \(f\) such that \((-1)^{n/2}d = d_K f^2\). Let \(d_0, p\) be as in the proof of Th. 2.1. Then by the results in §3, we have

\[
\lambda^*_N(\psi, d, \varepsilon) = \prod_{p | N_1} \varphi_p(d)(p^{-2})^{-1}(1 + ((-1)^{n/2}d, p))p^{-n/2})
\]

\[
\times \lambda^*_N(\psi^*, 2^{d_0, 2}, \varepsilon) \prod_{(p, 2N_1) = 1} \lambda_p(\psi_p, p^{d_0, p}, \varepsilon_p).
\]

If \(N_2^p \neq 1\), this vanishes by \((2)\) of Th. 3.1. Hence we assume \(N_2 = 1\). Let \(L^{(i)}_n(K)\) be the subset of \(L^*_n\) consisting of all the elements \(x\) such that \((-1)^{n/2} \det 2x = d_K f^2\) for a positive integer \(f\), and set

\[
\zeta^{(i)}_K(s, \psi, \varepsilon) = \left(\prod_{p | N_1} \varphi_p(2^{-n})\right)^{-1} 2^{-ns} c_n \sum_{L^{(i)}_n(K) / SL_n(Z)} \psi(x) \mu(x) |det x|^{-s}.
\]

Then the series \((2.1)\) is the sum of these series over \(K\) such that \((d_K, N_1) = 1\) and \((-1)^{n/2}d_K > 0\). Since \(d_K\) is fixed, we see as in the case of \(n\) odd

\[
\zeta^{(i)}_K(s, \psi, \varepsilon) = c_n \zeta((-1)^{n/2}d_K)|d_K|^{(n+1)/2-s} \prod_{p | N_1} (p^{-2})^{-1}(1 - \chi_K(p)p^{-n/2})^{-1}
\]

\[
\times \prod_{(p, N_1) = 1, p \neq 2, p | d_K} Z_n,o(\psi(p)p^{-s}, \psi_p, \varepsilon_p, d_K/p)
\]

\[
\times \prod_{(p, N_1d_K) = 1, p \neq 2} Z_n,o(\psi(2)^{-s}, \psi^*, \varepsilon_2, d_K/2^2)
\]

\[
\times \begin{cases} 
Z_n,o(\psi(2)^{-s}, \psi^*, \varepsilon_2, d_K/2^2) & \text{if } 8 | d_K, \\
Z_n,o(\psi(2)^{-s}, \psi^*, \varepsilon_2, d_K) & \text{if } 4 \parallel d_K, \\
Z_n,o(\psi(2)^{-s}, \psi^*, \varepsilon_2, d_K) & \text{if } 2 \nmid d_K.
\end{cases}
\]

Then we see

\[
\zeta^{(i)}_K(s, \psi, \varepsilon) = 2^{-1} c_n \zeta((-1)^{n/2}d_K)|d_K|^{(n+1)/2-s} L(\frac{n}{2}, \chi_K)
\]

\[
\times B^*_p(s, \psi, \varepsilon, K) \frac{L(2s, \psi^2)L(2s - n + 1, \psi^2)}{L(2s - n/2 + 1, \psi^2 \chi_K)} \prod_{i=1}^{n/2-1} L(2s - 2i, \psi^2).
\]

From this we obtain our result for \(\varepsilon\).
To compute the series associated to \( \varepsilon \), first we note \( d \) contributing to (2.2) is of the form \( \delta N_2^2 f^2 \) for a positive integer \( f \) by the results in §5 of [I-S2] and Th. 3.1 and Th. 3.2. At \( p = 2 \), \( d_{0,2}/\delta N_2^2 \in Q_{\sum}^2 \). Taking account of the factor \( 2^{-1}(1 + ((-1)^{n/2}d_{0,2} - 1)_{12}) \) of \( Z^*_n(u, \ell_2, d_{0,2}) \) (cf. Th. 5.3 of [I-S2]), we see (2.2) vanishes unless \((-1)^{n/2}\delta N_2^2 \equiv 1 \pmod{4} \). Let \( H \) be the quadratic field or \( Q \oplus Q \) such that \( d_J = (-1)^{n/2}d_{0,2} \). Then we see under this condition that (2.2) is equal to

\[
\begin{align*}
&c_n \psi((-1)^{n/2}d_H) \\
&\times \left( \prod_{p|N_1} \left( (p^{-2})_{n/2}^{-1}(1 - \chi_H(p)p^{-n/2})^{-1} \right) \right) Z^*_n(\psi(2)2^{-s}, \psi_2, \varepsilon_2, d_H) \\
&\times \prod_{(p, 2N_1, N_2^2) = 1} Z_n(\psi(p)p^{-s}, \psi_2, \varepsilon_2, d_H) \prod_{p|N_2^2} Z_n(\psi(p)p^{-s}, \psi_2, \varepsilon_2, d_H/p),
\end{align*}
\]

and hence is equal to

\[
\frac{|\prod_{i=1}^{n/2-1} B_{2i}|}{2^{n-1}(n-2)!} (-1)^{n(n+2)/8}\psi((-1)^{n/2}d_H)\chi_H(2)
\]

\[
\times 2(2\pi)^{-n(n/2 - 1)!} L(\frac{n}{2}, \chi_H) B^*(s, \psi, \varepsilon) \prod_{i=1}^{n/2} L(2s - (2i - 1), \psi^2).
\]

This completes the proof. The case of \( L_n \) can be treated in the same way and will be omitted.

We give a special case of the above result as a corollary, which is a generalization of Th. 1 of [I-S1].

**Corollary 2.4.** Let \( L = L_n^* \), and \( \psi = \chi_p^{(r_p)} \) for an odd prime \( p \) and an odd integer \( r_p \). Then one has

\[
\xi(s, L_n^*, \psi, \delta, \varepsilon) = \frac{2^{n_s}|\prod_{i=1}^{n/2-1} B_{2i}|}{2^{n-1}(n-2)!} (-1)^{n(n+2)/8}
\]

\[
= \begin{cases} 
B^*(s, \psi, \varepsilon)\chi_p(2^{-r_p})\chi_H(2) \\
\times 2(2\pi)^{-n(n/2 - 1)!} L(\frac{n}{2}, \chi_H) \prod_{i=1}^{n/2} \zeta(2s - (2i - 1)) \\
0 & \text{if } (-1)^{n/2}\delta p \equiv 1 \pmod{4}, \\
& \text{otherwise}.
\end{cases}
\]

In the above corollary, \( B^*(s, \psi, \varepsilon) \) can be given explicitly by

\[
C_{r_p}(p^{-s})((-1)^{(n-r_p+1)/2}, p)p\left(p^{r_p-2s}, p^2\right)^{-1}((n-r_p+1)/2].
\]
in the notation of §3.

Next we discuss the rationality of the values of the L-functions at non-positive integers. For a Dirichlet character \( \varphi \) and a Dirichlet series \( A(s) = \sum_{n=1}^{\infty} a_n n^{-s} \), we set

\[
(A | R_{\varphi})(s) = \sum_{n=1}^{\infty} a_n \varphi(n) n^{-s}.
\]

For the trivial character \( \chi_{0,p} \) modulo \( p \), we set

\[
(A | R_{\chi_{0,p}})(s) = \sum_{(n,p) = 1} a_n n^{-s},
\]

and

\[
(A | U_2)(s) = \sum_{8|n} a_n n^{-s}.
\]

Lastly we set

\[
(A | I)(s) = A(s).
\]

Let \( D^*(s, \delta) \) be the Dirichlet series introduced in §1 of [I-S2], which is \( D(s, \psi, \delta) \) for the trivial \( \psi \) and is associated to Eisenstein series of weight \( (n + 1)/2 \). For abbreviation, we set \( D^*(s) = D^*(s, \delta) \). We show our series \( D(s, \psi, \delta) \) and \( D^*(s, \psi, \delta) \) can be written as a linear combination of Dirichlet series of the form

\[
(2.3) \quad \Pi R_{\varphi} \prod_{p \in S} R_{\psi, \delta} \prod_{p \in T} R_{\chi_{0,p}}(s), \quad B(s)(A | R_{\psi} \prod_{p \in S} R_{\chi_{p}} \prod_{p \in T} R_{\chi_{0,p}} U_2)(s)
\]

for \( A(s) = D^*(s) \), where \( B(s) \) is a rational function in \( p^{-s} \) for \( p | N \). First we note

\[
(D^* | R_{\psi})(s)
\]

\[
= (-1)^{\lfloor n/4 \rfloor} \sum_{d_K|N, d_K > 0} 2(2\pi)^{-n}(n/2 - 1)! |d_K|^{(n-1)/2} \psi(d_K)L(\frac{n}{2}, \chi_K)
\]

\[
\times \frac{L(2s, \hat{\psi}^2) L(2s - n + 1, \hat{\psi}^2) d_K |\delta|^{-s}}{L(2s - (n/2 - 1), \hat{\psi}^2 \chi_K)}
\]

\[
= (-1)^{\lfloor n/4 \rfloor} \sum_{d_K|N, d_K > 0, (d_K, N_1) = 1} (\eta_K | R_{\psi})(s),
\]
where
\[ \eta_K(s) = 2(2\pi)^{-n}(n/2 - 1)!|d_K|^{(n-1)/2} \]
\[ \times L\left(\frac{n}{2}, \chi_K\right) \frac{L(2s, \psi^2)L(2s - n + 1, \psi^2)}{L(2s - (n/2 - 1), \psi^2 \chi_K)} |d_K|^{-s}. \]

Let \( p \) be an odd prime with \((p, N_1) = 1\). Then we see
\[
(D^* | R_{\psi^p} R_{\chi_0, p})(s) = (-1)^{[n/4]} \sum_{(d_K, p, N_1) = 1, \bigl(-1\bigr)^{n/2}d_K > 0} \frac{(1 - \psi^2(p)p^{-2s})(1 - \psi^2(p)p^{n-1-2s})}{(1 - \psi^2(p)\chi_K(p)p^{n/2-2s})} (\eta_K | R_{\psi^p})(s).
\]
Hence for \( \varepsilon = \pm 1 \) we have
\[
(D^* | R_{\psi^p} R_{\chi_0, p} (\varepsilon R_{\chi_p} + I)/2)(s) = (-1)^{[n/4]} \frac{(1 - \psi^2(p)p^{-2s})(1 - \psi^2(p)p^{n-1-2s})}{(1 - \psi^2(p)\chi_K(p)p^{n/2-1-2s})} \times \sum_{(d_K, N_1 p) = 1, \chi_p(d_K) = \varepsilon} (\eta_K | R_{\psi^p})(s),
\]
and
\[
\sum_{(-1)^{n/2}d_K > 0, (d_K, N_1 p) = 1, \chi_p(d_K) = \varepsilon} (\eta_K | R_{\psi^p})(s)
= (-1)^{[n/4]} \frac{(1 - \psi^2(p)\chi_K(p)p^{n/2-1-2s})}{(1 - \psi^2(p)p^{-2s})(1 - \psi^2(p)p^{n-1-2s})} \times (D^* | R_{\psi^p} R_{\chi_0, p} (\varepsilon R_{\chi_p} + I)/2)(s).
\]
Subtracting the above series for \( \varepsilon = \pm 1 \) from \((D^* | R_{\psi^p})(s)\), we obtain an expression for
\[
\sum_{p | d_K} (\eta_K | R_{\psi^p})(s),
\]
by the series of type (2.3).

Now let \( p = 2 \) and let \( \chi_2 \) be the character modulo 8 such that \( \chi_2(m) = (-1)^{(m^2-1)/8} \) for an odd integer \( m \). Then by the above procedure we obtain an expression of
\[
\sum_{(-1)^{n/2}d_K > 0, (N_1 p, d_K) = 1, \chi_2(d_K) = \varepsilon} (\eta_K | R_{\psi^p})(s)
\]
for \( \varepsilon = \pm 1 \) and
\[
\sum_{-1}^{1/2 \delta d_K > 0, \ p | d_K} (\eta_K | R_{\varepsilon})(s)
\]
as a linear combination of Dirichlet series of the form (2.3). Now we see
\[
\sum_{-1}^{1/2 \delta d_K > 0, \ p | d_K} (\eta_K | R_{\varepsilon}(I - U_2))(s)
\]
\[
= \sum_{-1}^{1/2 \delta d_K > 0, \ p^2 \| d_K} (1 - \psi^2(p)p^{-2s})(1 - \psi^2(p)p^{n-1-2s})(\eta_K | R_{\varepsilon})(s).
\]
This shows that
\[
\sum_{-1}^{1/2 \delta d_K > 0, \ p^2 \| d_K} (\eta_K | R_{\varepsilon})(s), \ \sum_{-1}^{1/2 \delta d_K > 0, \ p^3 \| d_K} (\eta_K | R_{\varepsilon})(s)
\]
can be written as linear combinations of Dirichlet series of type (2.3). Since the rational functions \( B_p(u, \psi_p, \zeta, K) \) and \( B^*_p(u, \psi_p, \zeta, K) \) depend only on \( \chi_K(p) \) for an odd prime \( p \) (whether \( (p, d_K) = 1, \ \chi_K(2) = \pm 1, \ p^2 \| d_K, \) or \( p^3 \ | d_K \) for \( p = 2 \)), combining the above results for odd primes and for \( p = 2 \), our assertion can be easily verified. If \( \Delta(s) \) is a Dirichlet series associated to a holomorphic modular form of weight \( (n + 1)/2 \), by a result of [S], the series of type (2.3) for \( \Delta(s) = 1 \) are also Dirichlet series associated to holomorphic modular forms of weight \( (n + 1)/2 \).

Therefore in the case of \( \delta = 1 \), for \( \Delta(s) = D^*(s) \) the Dirichlet series of type (2.3) for \( \Delta(s) = 1 \) are holomorphic at non-positive integers, as in the case of zeta functions. By the results in §5 of [I-S2], Th. 3.1 and Th. 3.2, we can check that in the expression of \( D(d, \psi, \delta) \), or \( D^*(s, \psi, \delta) \) as a linear combination of functions of the form (2.3), the denominators of \( \Delta(s) \)’s do not vanish at non-positive integers. Hence we obtain
\[
\zeta(1 - m, L, \psi, 1, \epsilon)
\]
\[
= \epsilon \kappa(n, \delta, \psi) \frac{\prod_{i=1}^{n/2-1} B_{2i}}{2^{n-1}(n-2)!} \psi((-1)^{n/2})2(2\pi)^{-n}(n/2 - 1)! L(n/2, \chi H)
\]
\[
\times \prod_{i=1}^{n/2-1} L(1 - 2(m + i - 1), \psi^2)
\]
\[
\times \begin{cases} 2^{n/2} B(1 - m, \psi, \epsilon) & \text{if } L = L_n, \\
2^{n(1-m)} \prod_{p | N_1} \varphi_p(2^{-n}) \times \prod_{p | N_2} \chi_p(2^{-r_p}) B^*(1 - m, \psi, \epsilon) & \text{if } L = L_n^*. \end{cases}
\]
From this we obtain the following result.

**Proposition 2.5.** Let $n$ be an even integer $\geq 4$, and let $Q(\psi)$ be as in Cor. 2.2. Assume $\delta = 1$. Then one has

$$\xi(1-m, L, \psi, 1, \epsilon) \in Q(\psi)$$

for a positive integer $m$ and $L = L_n, L_n^*$. 

As a special case, we can prove

**Corollary 2.6.** Let $n$ be as in Prop. 2.5. Let $L = L_n^*$ and assume $N = N_1 = p$ for an odd prime $p$, or $N = N_2 = p$ for an odd prime $p$ and $r_p = n$. Let $\varphi$ be a character modulo $p$ such that $\varphi^2 \neq \chi_0, \chi$ in the first case and $\varphi \chi$ in the second case. Let $\psi$ be the character defined by these data as in (1.3). Then one has

$$\xi(1-m, L_n^*, \psi, 1, \epsilon)$$

$$= 2^{n(1-m)} \frac{\prod_{i=1}^{n/2-1} B_{2i}}{2^{n-1}(n-2)!} \epsilon \varphi(2^{-n}) \kappa(n, 1, \epsilon)(-1)^n n(n+2)/8$$

$$\times \psi((-1)^{n/2} 2(2\pi)^{-n}(\frac{n}{2} - 1)! \zeta(\frac{n}{2})$$

$$\times \prod_{i=1}^{n/2-1} L(1 - 2(m + i - 1), \varphi^2),$$

for a positive integer $m$. Here we understand $L(s, \varphi^2) = \zeta(s)$ in the second case.

This is a generalization of Th. 2 of [A]. The case of $n = 2$ can be treated by the functional equation in [Sai2].

§3. **Orbital local series**

In this section, we determine the orbital local series for L-fuctions and composites our calculation. Throughout this section, we fix a prime $p$ and sometimes we abbreviate the suffix $p$, for example, $R_{p, \nu} = R_{\nu}$, $\varphi(n) = \varphi_p(n)$.

For non-negative integers $m$, $n$, and $d \in \mathbb{Z}_p$, and indeterminates $U$, $q$, we set

$$\beta(n, d) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ ((-1)^{n/2} d, p) & \text{if } n \text{ is even,} \end{cases}$$
\( (U, q)_m = \begin{cases} 
\prod_{i=1}^{m} (1 - q^{i-1}) U & \text{if } m \geq 1, \\
1 & \text{if } m = 0,
\end{cases} \)

\( (p^{-2})_m = (p^{-2}, p^{-2})_m = \begin{cases} 
\prod_{i=1}^{m} (1 - p^{-2i}) & \text{if } m \geq 1, \\
1 & \text{if } m = 0.
\end{cases} \)

For an integer \( r, 0 \leq r \leq n \) we define a polynomial \( C_r(u) \) in \( u \) by

\[
C_r(u) = (p^{-2})_{[r/2]}^{-1} (p^{-2})_{[(n-r)/2]}^{-1} p^{r(n-r)/2} u^{n-r}.
\]

We recall some formal power series introduced in [I-S2]. For a partition \( \{n_i\} \) of length \( m \) and a sequence \( \{t_i\} \) associated to it, we set

\[
Q(\{n_i\}, \{t_i\}) = -\sum_{i=1}^{m} \frac{n_i(n_i + 1)}{2} t_i - \sum_{j < i} n_i n_j t_j,
\]

\[
\tilde{Q}(\{n_i\}, \{t_i\}) = Q(\{n_i\}, \{t_i\}) + \frac{n + 1}{2} \sum_{i=1}^{m} n_i t_i.
\]

We define

\[
X_n(u, \varepsilon) = \sum_{\{n_i\}} \sum_{1 \leq t_1} \left( \prod_{i=1}^{m} (p^{-2})_{[n_i/2]}^{-1} \right) p^{\tilde{Q}(\{n_i\}, \{t_i\})} u^{\sum_{i=1}^{m} n_i t_i},
\]

\[
Y_n(u, \varepsilon) = \sum_{\{n_i\}, \text{even}} \sum_{1 \leq t_1} \left( \prod_{i=1}^{m} (p^{-2})_{[n_i/2]}^{-1} \right) p^{\tilde{Q}(\{n_i\}, \{t_i\})} u^{\sum_{i=1}^{m} n_i t_i},
\]

\[
X_n(u, \varepsilon) = \sum_{\{n_i\}, \text{even}} \sum_{1 \leq t_1, t_1 \equiv 0 \text{mod} 2} \left( \prod_{i=1}^{m} (p^{-2})_{[n_i/2]}^{-1} \right) p^{\tilde{Q}(\{n_i\}, \{t_i\})} \times \left( \prod_{n_i \text{ even}, t_i \text{ odd}} p^{-n_i/2} \right) u^{\sum_{i=1}^{m} n_i t_i},
\]

\[
Y_n(u, \varepsilon) = \sum_{\{n_i\}, \text{even}} \sum_{1 \leq t_1, t_1 \equiv 0 \text{mod} 2} \left( \prod_{i=1}^{m} (p^{-2})_{[n_i/2]}^{-1} \right) p^{\tilde{Q}(\{n_i\}, \{t_i\})} \times \left( \prod_{n_i \text{ even}, t_i \text{ even}} p^{-n_i/2} \right) u^{\sum_{i=1}^{m} n_i t_i}.
\]

Here \( \{n_i\} \) runs through all partitions of \( n \), even ones in \( Y(u, \varepsilon) \) and \( \{t_i\} \) runs through all sequences associated to \( \{n_i\} \) with \( t_1 \geq 1 \), satisfying \( t_i \equiv 0 \text{ mod } 2 \) for \( n_i \text{ odd} \) in the latter two cases.
By Prop. 5.6 and Prop. 5.9 of [I-S2], we have explicitly

\[
X_n(u, \iota) = (p^{-2})^{-1}[n/2]u^n(1 - p^{(n-1)/2}u)^{-1} \times \begin{cases} 
(pu^2, p^2)_{[n/2]}^{-1} & \text{if } n \text{ is odd}, \\
(1 - p^{-(n+1)/2}u)(u^2, p^2)_{[n/2]}^{-1} & \text{if } n \text{ is even}, 
\end{cases}
\]

\[
Y_n(u, \iota) = (p^{-2})^{-1}[n/2]u^n(u^2, p^2)_{n/2}^{-1},
\]

\[
X_n(u, \varepsilon) = (p^{-2})^{-1}[n/2] \begin{cases} 
\varepsilon^{n+1/2}(u^2, p^2)_{n/2}^{-1} & \text{if } n \text{ is odd}, \\
\varepsilon^{-n/2}u^n(pu^2, p^2)_{n/2}^{-1} & \text{if } n \text{ is even}, 
\end{cases}
\]

\[
Y_n(u, \varepsilon) = (p^{-2})^{-1}[n/2] \begin{cases} 
\varepsilon^n(u^2, p^2)_{n/2}^{-1} & \text{if } n \text{ is odd}, \\
\varepsilon^n(pu^2, p^2)_{n/2}^{-1} & \text{if } n \text{ is even}.
\end{cases}
\]

First we treat the case of odd primes.

**Theorem 3.1.** Let \( p \) be an odd prime, and let \( d_0 \in \mathbb{Z}_p^* \).

1. Let \( \psi = \varphi^{(n)} \). Then, one has for \( \omega = \iota \) or \( \varepsilon \)

\[
Z_n(u, \varphi^{(n)}, \omega, d_0) = (p^{-2})^{-1}[n/2]\varphi(d_0)(1 + \beta(n, d_0)p^{-n/2}).
\]

2. Let \( \psi = \chi_p^{(r)} \).
   (a) Let \( \omega = \iota \). If \( n \) is odd, then

\[
Z_n(u, \chi_p^{(r)}, \iota, d_0) = C_r(u)
\times \begin{cases} 
((-1)^{n-r/2}d_0, p)p^{-(n-r)/2}(p^r u^2, p^2)_{[(n-r)/2]}^{-1} & \text{if } r \text{ is odd}, \\
((-1)^{r/2}, p)p^{-r/2} \\
\times (1 - p^{(n-1)/2}u)^{-1}(p^{r+1} u^2, p^2)_{[(n-r)/2]}^{-1} & \text{if } r \text{ is even}.
\end{cases}
\]

Let \( n \) be even. If \( r \) is odd, then

\[
Z_{n,e}(u, \chi_p^{(r)}, \iota, d_0) = Z_{n,e}(u, \chi_p^{(r)}, \iota, d_0) = 0,
\]

and if \( r \) is even, then

\[
Z_{n,o}(u, \chi_p^{(r)}, \iota, d_0) = C_r(u)((-1)^{r/2}, p)(1 - p^{n-1} u^2)^{-1}(p^{r+1} u^2, p^2)_{(n-r)/2}^{-1}
\times (1 - p^{-r}) + (1 - p^{-(n-1)/2})p^{(n+r-1)/2}u,
\]
Z_{n, e}(u, \chi_p^{(r)}, \iota, d_0) = C_r(u)((-1)^{r/2}, p)(1 - p^{n-1}u^2)^{-1}(p^r u^2, p^2)^{(n-r)/2}
\times p^{r/2}(-(1 - p^{-r})
+ (1 + ((-1)^{n/2}d_0, p)p^{-n/2})(1 - ((-1)^{n/2}d_0, p)p^{n/2-1}u^2)).

(b) Let \omega = \varepsilon. Then one has

Z_n(u, \chi_p^{(r)}, \varepsilon, d_0) = C_r(u)
\times \begin{cases}
((-1)^{(n-r)/2}d_0, p)(p^{r+1}u^2, p^2)^{(n-r)/2} & \text{if } n \text{ is odd and } r \text{ is odd,} \\
((-1)^{r/2}, p)(1 + u)(p^r u^2, p^2)^{(n-r+1)/2} & \text{if } n \text{ is odd and } r \text{ is even,} \\
((-1)^{(n-r+1)/2}, p)(p^r u^2, p^2)^{(n-r+1)/2} & \text{if } n \text{ is even and } r \text{ is odd,} \\
((-1)^{r/2}, p)(((-1)^n/2d_0, p) + p^{-n/2})
\times (p^{r+1}u^2, p^2)^{(n-r)/2} & \text{if } n \text{ is even and } r \text{ is even.}
\end{cases}

(3) Let \psi = \chi_0^{(r)}.

(a) Let \omega = \iota. If n is odd, then

Z_n(u, \chi_0^{(r)}, \iota, d_0) = C_r(u)(1 - p^{(n-1)/2}u)^{-1}
\times \begin{cases}
(1 - p^{-(n-2r+1)/2}u)(p^r u^2, p^2)^{(n-r)/2} & \text{if } r \text{ is odd,} \\
(p^{r+1}u^2, p^2)^{(n-r)/2} & \text{if } r \text{ is even,}
\end{cases}

If n is even, then

Z_{n, o}(u, \chi_0^{(r)}, \iota, d_0) = C_r(u)(1 - p^{n-1}u^2)^{-1}
\times \begin{cases}
(p^{r+1}u^2, p^2)^{(n-r)/2} & \text{if } r \text{ is odd,} \\
(1 - p^{-(n-r)})p^{(n-1)/2}u(p^r u^2, p^2)^{(n-r)/2} & \text{if } r \text{ is even,}
\end{cases}

Z_{n, e}(u, \chi_0^{(r)}, \iota, d_0) = C_r(u)(1 - p^{n-1}u^2)^{-1}
\times \begin{cases}
p^{(n-1)/2}u(p^{r+1}u^2, p^2)^{(n-r)/2} & \text{if } r \text{ is odd,} \\
((1 + ((-1)^{n/2}d_0, p)p^{-n/2})
\times (1 - ((-1)^{n/2}d_0, p)p^{n/2-1}u^2)
\times (p^{r+1} - p^{r-1})u^2)(p^r u^2, p^2)^{(n-r)/2} & \text{if } r \text{ is even.}
\end{cases}
(b) Let $\omega = \varepsilon$. Then one has

$$Z_n(u, \chi_{\omega}^{(r)}, \varepsilon, d_0) = C_r(u)$$

$$\times \begin{cases} p^{-(n-r)/2}(p^{r+1}u, p^2) -1 & \text{if } n \text{ is odd and } r \text{ is odd,} \\ (p^{r/2} + p^{r/2}u)(p^ru^2, p^2) -1/2 \text{[(n-r+1)/2]} & \text{if } n \text{ is odd and } r \text{ is even,} \\ p^{r/2}u(p^ru^2, p^2) -1/2 \text{[(n-r+1)/2]} & \text{if } n \text{ is even and } r \text{ is odd,} \\ (p^{-(n-r)/2} + ((-1)^{n/2}d_0, p)p^{-r/2}) \\ \times (p^{r+1}u^2, p^2) -1/2 \text{[(n-r)/2]} & \text{if } n \text{ is even and } r \text{ is even.} \end{cases}$$

Proof. These formulas can be proved in the same way as in the case of zeta functions, and we give a proof only for formulas in (2). Let $d = d_0p^t$ with $d_0 \in Z_p^\times$. If $\chi_p^{(r)}(x) \neq 0$ for $x \in S_n(R_v, d_0p^t, \{n_i\}, \{t_i\})$, then $n_1 = r$ and $t_1 = 0$. In the following we assume this condition. Let $x = (\oplus p^ix_i)$.

Then $\chi_p^{(r)}(x) = \chi_p(det x_i) = (det x_1, p)$.

Let $\omega = \varepsilon$. Then in the same way as in §3 of the part I, we see

$$\lambda(\chi_p^{(r)} , d, \{n_i\}, \{t_i\}) = p^{Q(\{n_i\}, \{t_i\})}(\prod_{i=1}^m p^{-2})^{-1/2}\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\})$$

where

$$\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}) = (p-1)^{(m-1)} \sum_{d_1d_2\cdots d_m \equiv \delta_0 \mod p} (d_1, p) \prod_{i=1}^m (1 + \beta(n_i, d_i)p^{-n_i/2}).$$

The summation is extended over $d_1, d_2, \cdots, d_m \in R_1$ such that $d_1d_2\cdots d_m \equiv \delta_0 \mod p$. For $i, 1 \leq i \leq m-1$, let $n_i = n_{i+1}$. Then $\{n_i'\}$ gives a partition of $n - r$.

If $n$ is odd, then we see easily

$$\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}) = \begin{cases} 0 & \text{if } r \text{ is odd, and } \{n_i'\} \text{ is odd,} \\ ((-1)^{(n-r)/2}d_0, p)p^{-(n-r)/2} & \text{if } r \text{ is odd, and } \{n_i'\} \text{ is even,} \\ ((-1)^{r/2}, p)p^{-r/2} & \text{if } r \text{ is even.} \end{cases}$$

Let $n$ be even. If $r$ is odd, $\{n_i'\}$ is odd and as above $\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}) = 0$. If $r$ is even, then

$$\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}) = \begin{cases} ((-1)^{r/2}, p)p^{-r/2} & \text{if } \{n_i'\} \text{ is odd,} \\ ((-1)^{r/2}, p)p^{-r/2} & \text{if } \{n_i'\} \text{ is even,} \\ +((-1)^{(n-r)/2}d_0, p)p^{-(n-r)/2} & \text{if } \{n_i'\} \text{ is even.} \end{cases}$$
By the same calculation as in the proof of Lemma 5.7 of [I-S2], we see

\[ Z_n(u, \chi_p^{(r)}, \iota, d_0) = (p^{-2})_{[r/2]}^{-1} \left\{ \begin{array}{ll}
(1 - 1)^{(n-r)/2}d_0, p)^{-1/2}Y_{n-r}(p^{r/2}u, \iota) & \text{if } r \text{ is odd}, \\
((-1)^{r/2}, p)^{-1/2}X_{n-r}(p^{r/2}u, \iota) & \text{if } r \text{ is even},
\end{array} \right. \]

if \( n \) is odd and

\[ Z_n(u, \chi_p^{(r)}, \iota, d_0) = (p^{-2})_{[r/2]}^{-1} \left( \begin{array}{c}
((-1)^{r/2}, p)^{-1/2}X_{n-r}(p^{r/2}u, \iota) \\
+ ((-1)^{(n-r)/2}d_0, p)^{-1/2}Y_{n-r}(p^{r/2}u, \iota)
\end{array} \right) \]

if \( n \) and \( r \) are even. This proves (a).

Next assume \( \omega = \varepsilon \). In this case, as above we have

\[ \lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = p^{Q(\{n_i\}, \{t_i\})} \left( \prod_{i=1}^{m}(p^{-2})_{[n_i/2]}^{-1} \right) \]

\[ \times \varepsilon(d_0, \{n_i\}, \{t_i\}) \Lambda(\chi_p, \varepsilon, \{n_i\}, \{t_i\}), \]

where

\[ \Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = (p - 1)^{-(m-1)} \sum_{d_1d_2 \cdots d_m \equiv d_0 \mod p} (d_1, p) \prod_{i=1}^{m} (1 + \beta(n_i, d_i) p^{-n_i/2})(d_i, p)^{t_i}, \]

and

\[ \varepsilon(d_0, \{n_i\}, \{t_i\}) = (p^t, d_0) \prod_{i<j} (p^{n_i}, p^{n_j})^{t_it_j} \prod_{i=1}^{m} ((-1)^{n_i(n_i+1)/2}, p)^{t_i}. \]

Let \( n \) be odd. If \( r \) is odd, by substituting

\[ d_1 = d_0 d_2^{-1} \cdots d_m^{-1} \]

we see easily \( \Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) \) vanishes unless \( t_i \equiv 1 \mod 2 \) for \( n_i \) odd, \( i > 1 \). If this condition is satisfied, then

\[ \Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = (d_0, p) \prod_{n_i \text{ even}, t_i \text{ even}} (1 - 1)^{n_i/2}, p)^{-n_i/2}. \]
We see $t \equiv n - r \equiv 0 \mod 2$ and

$$\prod_{i<j}(p^{n_i}, p^{n_j})^{t_it_j} \prod_{i=1}^{m} ((-1)^{n_i(n_i+1)/2}, p)^{t_i} \prod_{n_i \text{ even}, t_i \text{ even}} ((-1)^{n_i/2}, p)$$

$$= ((-1)^{r(r+1)/2}, p) \prod_{1<j}(p^r, p^{n_j})^{t_j} \prod_{i<j} (p^{n_i}, p^{n_j}) \prod_{i=1}^{m} ((-1)^{n_i(n_i+1)/2}, p)$$

$$= ((-1)^{r(r+1)/2}, p)((-1)^{n(n+1)/2}, p)$$

$$= ((-1)^{(n-r)/2}, p).$$

Hence we have

$$\varepsilon(d_0, \{n_i\}, \{t_i\})\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = ((-1)^{(n-r)/2}d_0, p) \prod_{n_i \text{ even}, t_i \text{ even}} p^{-n_i/2}.$$

Since $t \equiv 0 \mod 2$ in this case, in the same way as in Lemma 5.8 of the part I by the above formula we see

$$Z_n(u, \chi_p^{(r)}, \varepsilon, d_0) = ((-1)^{(n-r)/2}d_0, p)(p^{-2})_{[r/2]} Y_{n-r}(p^{r/2}u, \varepsilon).$$

If $r$ is even, we see easily $\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})$ vanishes unless there exists $t_0$ such that $t_i \equiv t_0 \mod 2$ for $n_i$ odd. Assume this condition. Then we have

$$\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = (d_0, p)^{t_0} \prod_{n_i \text{ even}, t_i \not\equiv t_0 \mod 2, 2 \leq i} ((-1)^{n_i/2}, p)p^{-n_i/2}$$

$$\times \begin{cases} ((-1)^{r/2}, p)p^{-r/2} & \text{if } t_1(=0) \equiv t_0 \mod 2, \\ 1 & \text{if } t_1(=0) \not\equiv t_0 \mod 2. \end{cases}$$

Hence by a similar calculation, we obtain

$$\varepsilon(d_0, \{n_i\}, \{t_i\})\Lambda(\chi_p, \varepsilon, \{n_i\}, \{t_i\}) = ((-1)^{(n+1)/2}, p)^{t_0} ((-1)^{r/2}, p)$$

$$\times \prod_{n \text{ even}, t_i \not\equiv t_0 \mod 2, 2 \leq i} p^{-n_i/2} \times \begin{cases} p^{-r/2} & \text{if } t_0 \equiv 0 \mod 2, \\ 1 & \text{if } t_0 \not\equiv 0 \mod 2. \end{cases}$$

Since $t \equiv t_0 \mod 2$, from this we see

$$Z_n(u, \chi_p^{(r)}, \varepsilon, d_0) = (p^{-2})_{[r/2]}((-1)^{r/2}, p)(p^{-r/2}X_{n-r}(p^{r/2}u, \varepsilon) + Y_{n-r}(p^{r/2}u, \varepsilon)).$$
Let $n$ be even. If $r$ is odd, then $\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})$ vanishes unless $t_i \equiv 1 \mod 2$ for $n_i$ odd, $i > 1$. If this condition is satisfied, then
\[
\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = (d_0, p) \prod_{n_i \text{ even}, t_i \text{ even}} ((-1)^{n_i/2}, p)p^{-n_i/2}.
\]
In this case we see $t \equiv n - r \equiv 1 \mod 2$ and
\[
\prod_{i < j} (p^{n_i}, p^{n_j})^{t_i t_j} \prod_{i=1}^{m} ((-1)^{n_i(n_i+1)/2}, p)^{t_i} \prod_{n_i \text{ even}, t_i \text{ even}} ((-1)^{n_i/2}, p)
\]
\[
= ((-1)^{r(r+1)/2}, p) \prod_{1 < j} (p^{r}, p^{n_j})^{t_j} \prod_{i=1}^{m} ((-1)^{n_i(n_i+1)/2}, p)
\]
\[
= ((-1)^{(n-r+1)/2}, p).
\]
Hence we have
\[
Z_n(u, \chi_p^{(r)}, \varepsilon, d_0) = ((-1)^{(n-r+1)/2}, p)(p^{-2})^{-1}Y_{n-r}(p^{r/2}u, \varepsilon)
\]
If $r$ is even, then we see easily $\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})$ vanishes unless there exists $t_0$ such that $t_i \equiv t_0 \mod 2$ for $n_i$ odd. If this is satisfied, in the same way as above we have
\[
\varepsilon(d_0, \{n_i\}, \{t_i\}) \Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})
\]
\[
= ((-1)^{n/2}d_0, p)^{t_0}((-1)^{r/2}, p) \prod_{n_i \text{ even}, t_i \equiv 0 \mod 2, 2 \leq i} p^{-n_i/2}
\]
\[
\times \begin{cases} 
  p^{-r/2} & \text{if } t_0 \equiv 0 \mod 2, \\
  1 & \text{if } t_0 \not\equiv 0 \mod 2,
\end{cases}
\]
for $\{n_i\}$ odd, and
\[
\varepsilon(d_0, \{n_i\}, \{t_i\}) \Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})
\]
\[
= ((-1)^{(n-r)/2}d_0, p) \prod_{n_i \text{ even}, t_i \equiv 1 \mod 2, 2 \leq i} p^{-n_i/2}
\]
\[
+ ((-1)^{r/2}, p)p^{-r/2} \prod_{n_i \text{ even}, t_i \equiv 0 \mod 2, 2 \leq i} p^{-n_i/2},
\]
for $\{n_i\}$ even. This shows
\[
Z_n(u, \chi_p^{(r)}, \varepsilon, d_0) = (p^{-2})^{-1}((-1)^{r/2}, p)
\]
\[
\times \left( ((-1)^{n/2}d_0, p)Y_{n-r}(p^{r/2}u, \varepsilon) + p^{-r/2}X_{n-r}(p^{r/2}u, \varepsilon) \right).
\]
This completes the proof.
For $p = 2$, in the same way as above, we can prove

**Theorem 3.2.** Let $p = 2$, and let $d_0 \in \mathbb{Z}^\times$.

1. Let $\psi = \varphi^{(n)}$ for a non-trivial character $\varphi$. Then one has

\[
Z_n(u, \varphi^{(n)}, \iota, d_0) = 2^{-1}(p^{-2})^{-1/2} \varphi(d_0)
\times \begin{cases}
1 & \text{if } n \text{ is odd}, \\
1 + ((-1)^{n/2}d_0, -1)p^{-n} \\
& + ((-1)^{n/2}d_0, p) + ((-1)^{n/2}d_0, -p))p^{-n-n/2} & \text{if } n \text{ is even},
\end{cases}
\]

and

\[
Z_n(u, \varphi^{(n)}, \varepsilon, d_0) = p^{-[(n+1)/2]}\varphi(d_0)(p^{-2})^{-1/2}
\times \begin{cases}
(-1)^{(n^2-1)/8}(d_0, -1)^{(n+1)/2)} & \text{if } n \text{ is odd}, \\
(-1)^{(n+2)/8}2^{-1}(1 + ((-1)^{n/2}d_0, -1)) \\
& \times (1 + ((-1)^{n/2}d_0, -1)p^{-n/2}) \text{ if } n \text{ is even}.
\end{cases}
\]

2. Let $\psi = \tilde{\chi}_p^{(r)}$ with $r$ even.

(a) Let $\omega = \iota$. Then one has

\[
Z_{n,o}(u, \tilde{\chi}_p^{(r)}, \iota, d_0) = 2^{-1}p^{-3r/2}C_r(u)(1 - p^{(n-1)/2}u)^{-1}(p^{r+1}u^2, p^2)^{-1/2}u \times (1 - p^{-r})(p^{-r}u^2, p^2)^{-1/2}
\]

if $n$ is odd, and if $n$ is even

\[
Z_{n,o}(u, \tilde{\chi}_p^{(r)}, \iota, d_0) = 2^{-1}C_r(u)p^{-r/2}(-1 - p^{-r})
\]

\[
+ (1 - p^{-r})p^{(n-1)/2}u \times (1 - p^{n-1}u^2)^{-1}(p^{r}u^2, p^2)^{-1/2}u
\]

Z_{n,e}(u, \tilde{\chi}_p^{(r)}, \iota, d_0) = 2^{-1}C_r(u)(1 - p^{n-1}u^2)^{-1}p^{-r/2}
\]

\[
\times \left(p^{-r}(1 - p^{r-1}u^2) + (1 - p^{n-1}u^2)(p^{-n}((-1)^{n/2}d_0, -1)
\]

\[
+ p^{-(3n/2-r)}((-1)^{n/2}d_0, 2) + ((-1)^{n/2}d_0, -2))
\]

\[
\times (p^{r}u^2, p^2)^{-1/2}u.
\]

(b) Let $\omega = \varepsilon$.

If $n$ is odd, then

\[
Z_{n,n}(u, \tilde{\chi}_p^{(r)}, \varepsilon, d_0) = p^{-(n+r+1)/2}(-1)^{(n^2-1)/8}((-1)^{(n+1)/2}, d_0)C_r(u)
\]

\[
\times (p^{-r/2} + p^{r/2}u)(p^{r}u^2, u^2)^{-1/2}u
\]

\[
\times (p^{r}u^2, p^2)^{-1/2}u.
\]
If \( n \) is even, then
\[
Z_n^*(u, \chi_p^{(r)}, \varepsilon, d_0) = p^{-(n+r)/2}2^{-1}(1 + ((-1)^{n/2}d_0, -1))(-1)^{n(n+2)/8}C_r(u)
\]
\[
\times ((-1)^{n/2}d_0, p)p^{-r/2}(p^{r+1}u^2, p^2)^{-1}_{[(n-r)/2]}
\]

(3) Let \( \psi = \chi_0^{(r)} \).
(a) Let \( \omega = \iota \).

If \( n \) is odd, then
\[
Z_n(u, \chi_0^{(r)}, \iota, d_0) = 2^{-1}C_r(u)(1 - p^{(n-1)/2}u)^{-1}
\]
\[
\times \begin{cases} 
(1 - p^{-(n-2r+1)/2}u)(p^{r}u^2, p^2)^{1/[(n-r)/2]} & \text{if } r \text{ is odd,} \\
(p^{r+1}u^2, p^2)^{1/[(n-r)/2]} & \text{if } r \text{ is even.}
\end{cases}
\]

If \( n \) is even, then
\[
Z_{n,0}(u, \chi_0^{(r)}, \iota, d_0) = 2^{-1}C_r(u)(1 - p^{-n}u^2)^{-1}
\]
\[
\times \begin{cases} 
(p^{r+1}u^2, p^2)^{1/[(n-r)/2]} & \text{if } r \text{ is odd,} \\
(1 - p^{-(n-r)/2})p^{(n-1)/2}u(p^{r}u^2, p^2)^{1/[(n-r)/2]} & \text{if } r \text{ is even,}
\end{cases}
\]

and
\[
Z_{n,e}(u, \chi_0^{(r)}, \iota, d_0) = 2^{-1}C_r(u)(1 - p^{-n}u^2)^{-1}
\]
\[
\times \begin{cases} 
p^{-(n-1)/2}u(p^{r+1}u^2, p^2)^{1/[(n-r)/2]} & \text{if } r \text{ is odd,} \\
(1 - p^{-(n-r)/2})(((-1)^{n/2}d_0, -1)p^{-n}
+ (((-1)^{n/2}d_0, 2) + ((-1)^{n/2}d_0, -2))p^{-n-n/2}) & \text{if } r \text{ is even.}
\end{cases}
\]

(b) Let \( \omega = \varepsilon \).

If \( n \) is odd, then
\[
Z_n(u, \chi_0^{(r)}, \varepsilon, d_0) = p^{-(n+1)/2}((-1)^{(n^2-1)/8}(d_0, -1)^{(n+1)/2})C_r(u)
\]
\[
\times \begin{cases} 
p^{-r/2}(p^{r+1}u^2, p^2)^{1/[(n-r+1)/2]} & \text{if } r \text{ is odd,} \\
p^{-r/2}u(p^{r}u^2, p^2)^{1/[(n-r+1)/2]} & \text{if } r \text{ is even.}
\end{cases}
\]

If \( n \) is even, then
\[
Z_n(u, \chi_0^{(r)}, \varepsilon, d_0) = (-1)^{n(n+2)/8}p^{-n/2}2^{-1}(1 + ((-1)^{n/2}d_0, -1))C_r(u)
\]
\[
\times \begin{cases} 
p^{r/2}u(p^{r}u^2, p^2)^{1/[(n-r+1)/2]} & \text{if } r \text{ is odd,} \\
(p^{-r/2}u(p^{r}u^2, p^2)^{1/[(n-r+1)/2]} & \text{if } r \text{ is even.}
\end{cases}
\]
Proof. We give a proof for (2), since the other case can be treated in the same way, and give a proof for the case \( r \geq 1 \). The case \( r = 0 \) can be treated in a similar way. Let \( x \in S_n^*(R_\nu, d, \{n_i\}, \{t_i\}) \) and \( 2y \equiv x \mod p^r \) for \( y \in L_n^* \). Let \( x \) be equivalent to \( (\oplus p^i x_i) \), with \( x_i \in S_{n_i}(R_\nu) \). We see \( Q_y \) is equivalent to (1.1) or (1.2) if and only if \( n_1 = r \), \( t_1 = 0 \) and 
\[ \quad \det x_1, 2 = ((-1)^r/2 d_1, 2) = 1 \quad \text{or} \quad -1 \quad \text{respectively}, \] and \( \chi_p^*(r)(x) = ((-1)^r/2 d_1, 2) \). Hence we have as in the proof of Prop. 3.6 of the part I
\[
\Lambda^*(\chi_p^*(r), d_0, \{n_i\}, \{t_i\}) = p^Q(\{n_i\}, \{t_i\}) (\prod_{i=1}^m (p^{-1})_{[n_i/2]}^r) \Lambda^*(\chi_p^*(r), \nu, \{n_i\}, \{t_i\}),
\]
where
\[
\Lambda^*(\chi_p^*(r), \nu, \{n_i\}, \{t_i\}) = 2^{-1} p^{-2(m-1)} 
\times \sum_{d_1d_2\cdots d_m \equiv d_0 \mod p^3} ((-1)^{r/2} d_1, 2) ((-1)^{r/2} d_1, -1) + 1 + ((-1)^{r/2} d_1, 2) p^{-r/2} p^{-r} 
\times \prod_{n_i \text{ even}, 2 \leq i} (1 + ((-1)^{n_i/2} d_i, -1) p^{-n_i} + ((-1)^{n_i/2} d_i, 2) p^{-n_i - n_i/2} + ((-1)^{n_i/2} d_i, -2) p^{-n_i - n_i/2}),
\]
Here the summation is extended over all \( d_i \in R_3 \) such that
\[ d_1d_2 \cdots d_m \equiv d_0 \mod 2^3. \]
If \( n \) is odd, then we see
\[
\Lambda^*(\chi_p^*(r), \nu, \{n_i\}, \{t_i\}) = 2^{-1} p^{-3r/2}
\]
This shows that
\[
Z_n^*(u, \chi_p^*(r), \nu, d_0) = 2^{-1} p^{-3r/2} (p^{-2})_{r/2}^{-1} X_{n-r}(p^{r/2} u, \nu).
\]
If \( n \) is even, then we see
\[
\Lambda^*(\chi_p^*(r), \nu, \{n_i\}, \{t_i\})
= \frac{1}{2} \begin{cases} 
    p^{-3r/2} & \text{if } \{n_i\} \text{ is odd}, \\
    (p^{-3r/2} + p^{-3r/2} ((-1)^{n/2} d_0, -1) p^{-(n-r)} + p^{-r} ((-1)^{n/2} d_0, 2) + ((-1)^{n/2} d_0, -2)) p^{-3(n-r)/2} & \text{if } \{n_i\} \text{ is even}.
\end{cases}
\]
This shows that
\[
Z_n^\ast(u, \hat{X}_p^\ast(r), \epsilon, d_0)
= 2^{-1}(p^{-2r})^{-1/2} \left( p^{-3r/2} X_{n-r}(p^{r/2} u, \nu) + \left( ((-1)^{n/2} d_0, -1) p^{-n-r/2} \right. \right.
+ \left. \left( ((-1)^{n/2} d_0, 2) + ((-1)^{n/2} d_0, -2) p^{-3(n/2-r/2)} Y_{n-r}(p^{r/2} u, \nu) \right. \right.
\]
\[
\left. \left. \right. \left. \right. \right) \right).
\]

Let \( \omega = \epsilon \). Then by means of the remark after Prop. 3.9 of the part I we have
\[
\Lambda^\ast(\hat{X}_p^\ast(r), d, \epsilon, \{n_i\}, \{t_i\})
= p^{Q(\{n_i\}, \{t_i\})} \left( \prod_{i=1}^m (p^{-2})^{-1} \right) \Lambda^\ast(\hat{X}_p^\ast(r), \epsilon, \{n_i\}, \{t_i\}),
\]
where
\[
\Lambda^\ast(\hat{X}_p^\ast(r), \epsilon, \{n_i\}, \{t_i\})
= 2^{-1}p^{-2(m-1)} \sum_{d_1d_2\cdots d_m \equiv d_0 \text{mod } p^3} \left( p^{r} d_0 \right) \prod_{i=1}^m ((-1)^{n_i(n_i+1)/2} d_i, p)^{t_i}
\times \prod_{i<j} (d_i, d_j) \times \prod_{i<j} (p^{n_i}, p^{n_j})^{t_i t_j}
\times \left( p^{-r/2} (1 + ((-1)^{r/2} d_1, p) p^{-r/2}) \epsilon_r(d_1) \right) \times \prod_{2 \leq i} \epsilon'_n(d_i).
\]

Here \( \epsilon_{n_i}(d_i) \) is as in the proof of Lemma 3.8 of the part I, and
\[
\epsilon'_n(d_i) = (1 + ((-1)^{n_i/2} d_i, p) p^{-n_i/2}) \epsilon_{n_i}(d_i).
\]

If \( \{n_i\} \) is odd, \( \Lambda^\ast(\hat{X}_p^\ast(r), \epsilon, \{n_i\}, \{t_i\}) \) vanishes unless there exists \( t_0 \) such that \( t_i \equiv t_0 \text{ mod } 2 \) for \( n_i \) odd. We note this implies \( t \equiv t_0 \text{ mod } 2 \) if \( n \) is odd. Under this condition, by the same calculation as in the proof of Prop. 3.8 of the part I we obtain
\[
\Lambda^\ast(\hat{X}_p^\ast(r), \epsilon, \{n_i\}, \{t_i\})
= p^{-[(n+1)/2]} ((-1)^{(n+1)/2} d_0, p)^{t_0} p^{-r/2}
\prod_{n_i \text{ even}, \ t_i \not\equiv t_0 \text{ mod } 2} p^{-n_i/2}
\times \begin{cases} 
(-1)^{(n-1)/2} (d_0, (-1)^{(n+1)/2}) & \text{if } n \text{ is odd}, \\
(-1)^{(n+2)/2} 2^{-1} (1 + ((-1)^{n/2} d_0, -1)) & \text{if } n \text{ is even}.
\end{cases}
\]
If \( \{n_i\} \) is even, then

\[
\Lambda^*(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = p^{-(n+r)/2}2^{-1}(1 + ((-1)^{n/2}d_0, -1))(-1)^{n(n+2)/8}
\times \left( ((-1)^{n/2}d_0, p) \prod_{t_i \text{ even}} p^{-n_i/2} + \prod_{t_i \text{ odd}} p^{-n_i/2} \right).
\]

This shows that \( Z_n^*(u, \chi_p^{(r)}, \varepsilon, d_0) \) is equal to

\[
= p^{-(n+r+1)/2}(-1)^{(n^2-1)/8}(d_0, -1)^{(n+1)/2}(p^{-2})^{-1}_{[r/2]}
\times (X_{n-r}(p^{r/2}u, \varepsilon) + p^{-r/2}Y_{n-r}(p^{r/2}u, \varepsilon)),
\]

if \( n \) is odd, and is equal to

\[
2^{-1}(1 + ((-1)^{n/2}d_0, -1))p^{-(n+r)/2}(-1)^{n(n+2)/8}(p^{-2})^{-1}_{r/2}
\times ((-1)^{n/2}d_0, p)p^{-r/2}Y_{n-r}(p^{r/2}u, \varepsilon) + X_{n-r}(p^{r/2}u, \varepsilon),
\]

if \( n \) is even. This completes the proof.

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