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ON THE ALTITUDE OF NODES IN RANDOM TREES

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1. Introduction. Let T_n denote a tree with n nodes that is rooted at node r. (For definitions not given here see [**4**] or [**10**].) The *altitude* of a node u in T_n is the distance $\alpha = \alpha(u, T_n)$ between r and u in T_n. The *width* of T_n at altitude k is the number $W_k = W_k(T_n)$ of nodes at altitude k in T_n, where $k = 0, 1, \cdots, n-1$. Our main object here is to determine the expected values $\mu(n)$ and $\mu(n, k)$ of α and W_k over all trees T_n in certain families $\mathscr{F}$ of rooted trees.

We consider families $\mathscr{F}$ whose generating function $y(x)$ satisfies a relation of the type $y(x) = \Gamma\{y(x)\}$ with a suitable operator Γ. In § 2 we establish formal relations for the generating functions of $\mu(n)$ and $\mu(n, k)$. After some preliminaries in § 3, we apply these results in § 4 to *simply generated* families, that is, to families $\mathscr{F}$ whose generating function $y(x)$ satisfies a functional relation of the type $y = x\theta(y)$ where $\theta(y)$ is a power series in y. We show that under certain conditions

$$\mu(n) \sim (\pi n/2A)^{1/2} \quad \text{and} \quad \mu(n, k) \sim Ake^{-Ak^2/2n} \quad \text{if } k = O(n^{1/2}),$$

where A is a constant that depends upon $\mathscr{F}$. § 3 and 4 are devoted to the technical details of establishing our main general asymptotic results (Theorems 4.3 and 4.5). In § 5 we apply these relations to obtain specific results for three particular, simply generated families: labelled trees, plane trees and binary trees. In §§ 6 and 7 we consider two non-simply generated families of trees, rooted unlabelled trees and recursive trees. A relation of the type $\mu(n) \sim cn^{1/2}$ holds for the rooted unlabelled trees but $\mu(n) \sim \log n$ for the recursive trees.

2. General results. Let y_n denote the number of trees T_n in any given family $\mathscr{F}$ of rooted trees. (More precisely, if there are weights associated with the trees in $\mathscr{F}$, then y_n denotes the sum of the weights of the trees T_n in $\mathscr{F}$; this will be discussed further in § 3.) We assume $y_1 = 1$ and that any non-trivial tree T_n in $\mathscr{F}$ may be constructed by joining the roots of a collection of smaller trees $B_1, \ldots, B_j$ in $\mathscr{F}$ to a new node r that serves as the root of the tree thus formed; the subtrees $B_1, \ldots, B_j$ are called the *branches* of T_n. We further assume that this construction, subject to whatever conditions are implicit in the definition of $\mathscr{F}$, gives rise to a recurrence relation for y_n in terms of $y_1, \ldots, y_{n-1}$ for $n \geqq 2$. Finally, we assume that there exists an operator $\gamma\{g_1(x), g_2(x), \cdots\}$ defined for any (possibly infinite) sequence of power series

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$g_1(x), g_2(x), \cdots$ such that if

$$y = y(x) = \sum_1^\infty y_n x^n,$$

then the recurrence relation for y_n can be expressed in terms of generating functions as

$$y(x) = \gamma\{y(x), y(x^2), \cdots\}.$$

For notational convenience we shall write this simply as

$$(2.1) \quad y(x) = \Gamma\{y(x)\}.$$

More generally, if $h(x, z)$ is any power series in two variables x and z, we let

$$\Gamma\{h(x, z)\} = \gamma\{h(x, z), h(x^2, z^2), \cdots\}$$

where $h(x, z)$, $h(x^2, z^2), \cdots$ are interpreted as functions of x for fixed values of z.

Let

$$p_{nmk} = \Pr\{W_k(T_n) = m\}$$

where the probability is taken over all trees T_n in $\mathscr{F}$. If

$$P_k(x, z) = \sum_{n,m} p_{nmk} y_n x^n z^m,$$

then clearly $P_k(x, 1) = y(x)$. It is easy to see that p_{nm0} equals one or zero according as $m = 1$ or $m \neq 1$ and that if T_n has branches $B_1, \ldots, B_j$ then $W_{k+1}(T_n) = W_k(B_1) + \cdots + W_k(B_j)$ for $k \geqq 1$. These observations and assumption (2.1) imply the following result.

THEOREM 2.1. *$P_0(x, z) = zy(x)$ and $P_{k+1}(x, z) = \Gamma\{P_k(x, z)\}$ for $k \geqq 0$.*

Let

$$F_k(x) = \sum_{n=1}^\infty \mu(n, k) y_n x^n \quad \text{and} \quad S_k(x) = \sum_{n=1}^\infty \mu_2(n, k) y_n x^n$$

where $\mu(n, k)$ and $\mu_2(n, k)$ denote the first and second factorial moments of $W_k(T_n)$ over all trees T_n in $\mathscr{F}$. The following relations are obvious.

THEOREM 2.2. *$F_k(x) = (\partial/\partial z\, P_k(x, z))_{z=1}$ and $S_k(x) = (\partial^2/\partial z^2\, P_k(x, z))_{z=1}$.*

Let

$$M(x) = \sum_{n=1}^\infty \mu(n) n y_n x^n \quad \text{and} \quad V(x) = \sum_{n=1}^\infty \mu_2(n) n y_n x^n$$

where $\mu(n)$ and $\mu_2(x)$ denote the first and second factorial moments of $\alpha = \alpha(u, T_n)$ over all nodes u of all trees T_n in $\mathscr{F}$. It is not difficult to see that $\Pr\{\alpha = k\} = \mu(n, k)/n$. This implies the following results.

THEOREM 2.3. *$M(x) = \sum_{k=1}^\infty kF_k(x)$ and $V(x) = \sum k(k-1)F_k(x)$.*

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3. Enumerating simply generated families of trees. We say a family $\mathscr{F}$ of rooted trees is a *simply generated* family if relation (2.1) takes the form

$$(3.1) \quad y = x\theta(y)$$

where

$$\theta(y) = 1 + c_1 y + c_2 y^2 + \cdots$$

is a power series in $y = y(x)$ with non-negative coefficient c_i. (Labelled trees, plane trees, and binary trees are probably the most familiar examples of simply generated families.) The coefficients y_n have a straightforward interpretation in terms of the c_i's.

The *out-degree* of a node in a rooted tree is the number of edges incident with the node that lead away from the root; thus the out-degree is one less than the total number of edges incident with the node except for the root node itself in which case the two are equal. If there are $D_i(T_n)$ nodes of out-degree i in the tree T_n then the *weight* $w(T_n)$ of T_n is defined by the formula

$$(3.2) \quad w(T_n) = \prod_i c_i^{D_i(T_n)}$$

where we adopt the convention that $c_0 = 1$. It is not difficult to see that assumption (3.1) implies that

$$y_n = \sum w(T_n)$$

where the sum is over all the rooted plane trees T_n. There are only a finite number of rooted plane trees T_n and each different family $\mathscr{F}$ may be regarded as just a different assignment of weights to these trees. If, for example, $c_i = 0$ for some i then trees with any nodes of out-degree i are assigned zero weight, i.e., they are in effect not counted in the family at all. The choice of the c_i's may reflect, for example, whether different orderings of the branches are taken into account in distinguishing between trees in $\mathscr{F}$. It should be pointed out that although the class of simply generated families of trees—in which the weights are determined by a relation of type (3.2)—includes some of the most common families of trees, it certainly does not include all families. Two non-simply generated families will be considered later.

Various authors have, in effect, applied a result formulated by Darboux [**3**, p. 20] to enumerate asymptotically various families of trees. (See, for example, [**14**; **12**; **4**, p. 503; **5**; and **1**].) In any particular case it is necessary to establish that the generating function of the family satisfies certain hypotheses. The following result gives conditions sufficient to ensure that Darboux's theorem can in fact be applied.

THEOREM 3.1. *Suppose $\theta(t) = 1 + c_1 t + c_2 t^2 + \cdots$ is a regular function of t when $|t| < R \leqq +\infty$ and let*

$$y = y(x) = x + y_2 x^2 + y_3 x^3 + \cdots$$

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denote the solution of $y(x) = x\theta(y(x))$ *in the neighbourhood of* $x = 0$. *If*

(i) $c_1 > 0$ *and* $c_j > 0$ *for some* $j \geqq 2$,
(ii) $c_i \geqq 0$ *for* $i \geqq 2$, *and*
(iii) $\tau\theta'(\tau) = \theta(\tau)$ *for some* τ, *where* $0 < \tau < R$,

then $y(x)$ *is regular in the disk* $|x| \leqq \rho = \tau/\theta(\tau)$ *except at* $x = \rho$; *furthermore,* $y(x)$ *has an expansion in the neighbourhood of* ρ *of the form*

$$(3.3) \quad y(x) = \tau - b(\rho - x)^{1/2} - b_2(\rho - x) \cdots,$$

where $b = \rho^{-1}(2\tau/\theta''(\tau))^{1/2}$, *and*

$$(3.4) \quad y_n \sim \frac{b}{2\sqrt{\pi}} \rho^{-n+1/2} n^{-3/2}, \quad \text{as } n \to \infty.$$

Proof. Let $f(t) = t\theta'(t) - \theta(t)$. Then $f(0) = -1$ and $f'(t) = t\theta''(t) > 0$ for $0 < t < R$ by (i) and (ii). Therefore, $f(t)$ is a strictly increasing function for $0 \leqq t < R$ and it follows from (iii) that $t\theta'(t) - \theta(t) < 0$ for $0 \leqq t < \tau$.

Consider the functional relation $F(x, y) \equiv y - x\theta(y) = 0$. Then $F_y = 1 - x\theta'(y)$, and the observations in the last paragraph imply that $F_y \neq 0$ when $|x| < \rho = \tau/\theta(\tau)$ and $|y| < \tau$. Since $F_y(\rho, \tau) = 0$, it follows from the implicit function theorem that $y = y(x)$ is regular for $|x| < \rho$, that $y(\rho) = \tau$, and that $x = \rho$ is a singularity of $y(x)$. Moreover, if $|x| = \rho$ but $x \neq \rho$ then $|y(x)| < y(\rho) = \tau$ since $y_1 = 1$ and $y_2 = c_1 > 0$; hence, $|\theta'(y(x))| < \theta'(\tau) = \rho^{-1}$, by (i) and (ii). Thus $|x\theta'(y(x))| < 1$ if $|x| = \rho$ but $x \neq \rho$ and, consequently, $F_y(x, y(x)) \neq 0$ if $|x| \leqq \rho$ except when $x = \rho$. Since $F_x \neq 0$, $F_y = 0$, and $F_{yy} \neq 0$ at (ρ, τ), it follows that $y(x)$ is regular for $|x| \leqq \rho$ except at $x = \rho$; hence, y has the expansion (3.3) around $x = \rho$. It follows from (3.3) and (3.1) that

$$\tfrac{1}{2}b^2 = \lim_{x \to \rho^-} (\tau - y(x)) \cdot y'(x) = \lim_{x \to \rho^-} \frac{y(\tau - y(x))}{x(1 - x\theta')}.$$

If we use L'Hôpital's rule to evaluate the limit, we find that $\frac{1}{2}b^2 = \tau(\rho^2\theta''(\tau))^{-1}$, as required. Darboux's theorem [**3**; p. 20] states that if y has the expansion (3.3), where ρ is the unique singularity of y on its circle of convergence, then conclusion (3.4) holds.

We remark that condition (i) can be replaced by the condition that $c_i > 0$ and $c_j > 0$ for some i and j such that g.c.d. $(i, j) = 1$. Condition (iii) is clearly satisfied if $\lim_{t \to R^-} (t\theta'(t) - \theta(t)) > 0$.

We also observe, for later use, that it follows from Taylor's expansion of $\theta'(y)$ about $y = \tau$ and equation (3.3) that

$$(3.5) \quad x\theta' = 1 - b\rho\theta''(\tau)(\rho - x)^{1/2} + \cdots$$

in the neighbourhood of ρ. It is easy to see that θ' is a regular function of x when $|x| \leqq \rho$ except when $x = \rho$. The same is true for θ'' and θ'''.

We state without proof some straightforward asymptotic results that we shall use later. (See [**9**, § 4] for a proof of one of the results; the other proofs are similar.)

LEMMA 3.1. *Let*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad B(x) = \sum_{n=0}^{\infty} b_n x^n, \quad A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n$$

and suppose there exist constants α, β, a, b, *and* ρ, *where* a, b, *and* $\rho > 0$, *such that*

$$a_n \sim a\rho^{-n} n^{-\alpha} \quad \text{and} \quad b_n \sim b\rho^{-n} n^{-\beta}, \quad \text{as } n \to \infty.$$

(i) *If* $\alpha = 0$ *and* $\beta = \frac{1}{2}$, *then* $c_n \sim 2ab\rho^{-n} n^{1/2}$.
(ii) *If* $\alpha = \beta = \frac{1}{2}$, *then* $c_n \sim \pi ab\rho^{-n}$.
(iii) *If* $\alpha = \beta = \frac{3}{2}$, *then* $c_n \sim A(\rho)b_n + B(\rho)a_n$.
(iv) *If* $\alpha = \frac{3}{2}$ *and* $\beta = \frac{1}{2}$ *and* $A(\rho) \neq 0$, *then* $c_n \sim A(\rho)b_n$. (*This same conclusion also holds if the assumption about* a_n *is replaced by the assumption that* $a_n = O(\rho^{-n} n^{-3/2})$ *or the assumption that* $A(x)$ *is regular when* $|x| < \rho + \epsilon$ *for some* $\epsilon > 0$.)

4. Results for simply generated families of trees. In this section we specialize the results in § 2 to simply generated families of trees whose generating functions y satisfy a relation of the type $y = x\theta(y)$. When deriving results of an asymptotic nature, we shall assume $\theta(y)$ satisfies the hypothesis of Theorem 3.1 so that $y_n \sim c\rho^{-n} n^{-3/2}$ as $n \to \infty$, where $c = \frac{1}{2}b(\rho/\pi)^{1/2}$. For notational convenience we shall let $C_n\{f(x)\}$ denote the coefficient of x^n in any power series $f(x)$.

THEOREM 4.1. *If* $y = x\theta(y)$ *then* $F_k(x) = y(x\theta')^k$ *and*

$$S_k(x) = y^2(x\theta'')\{(x\theta')^{k-1} + (x\theta')^k + \cdots + (x\theta')^{2k-2}\} \quad \text{for } k \geqq 0.$$

Proof. If $\Gamma\{y\} = x\theta(y)$, then it follows from Theorem 2.1 that

$$\frac{\partial}{\partial z} P_{k+1} = \frac{\partial}{\partial z} \Gamma\{P_k\} = x\theta'(P_k) \frac{\partial}{\partial z} P_k.$$

Hence,

$$F_{k+1}(x) = x\theta' \cdot F_k(x)$$

by Theorem 2.2. The required formula for $F_k(x)$ now follows immediately by induction since $F_0(x) = y$. The formula for $S_k(x)$ may be established in a similar way, starting with the fact that $S_0(x) = 0$.

THEOREM 4.2. *If* $y = x\theta(y)$, *then* $\mu(n, k) \sim Ak + 1$ *and*

$$\mu_2(n, k) \sim 3A^2 \binom{k}{2} + k(2A + \tau^2 \rho\theta'''(\tau))$$

for fixed values of k *as* $n \to \infty$, *where* $A = \tau\rho\theta''(\tau)$.

Proof. Theorem 4.1 implies that

$$F_k' = y'(x\theta')^k + ky(x\theta')^{k-1}\{y' \cdot x\theta'' + \theta'\}.$$

It follows from (3.5) and Lemma 3.1 (iii) that

$$C_n\{ky \cdot \theta'(x\theta')^{k-1}\} = O(y_n).$$

When we apply Lemma 3.1 (iv) to the remaining terms and use the fact that $\rho\theta'(\tau) = 1$, we find that

$$n\mu(n, k)y_n = C_n\{F_k'\} \sim ny_n\{1 + k\tau\rho\theta''(\tau)\} = ny_n\{1 + kA\}$$

for fixed values of k as $n \to \infty$, as required. The proof of the corresponding result for $\mu_2(n, k)$ is similar and we shall omit the details.

If $\sigma^2(n, k)$ denotes the variance of $W_k(T_n)$ over all trees T_n in $\mathscr{F}$, then it follows from Theorem 4.2 that

$$\sigma^2(n, k) \sim \tfrac{1}{2}A^2k(k - 3) + k(A + \tau^2\rho\theta'''(\tau))$$

for fixed values of k as $n \to \infty$.

We now give a series of lemmas that will enable us to determine the asymptotic behaviour of $\mu(n, k)$ when $k = O(n^{1/2})$ as $k \to \infty$.

LEMMA 4.1. *If $y = x\theta(y)$, then*

$$C_n\{y(x\theta')^k\} = \frac{1}{2\pi i}\int (y\theta')^k\theta^{n-1-k}(\theta - y\theta')y^{-n}dy,$$

where the integration is around the circle $y = \tau e^{is}$ with $-\pi < s \leqq \pi$.

Proof. Cauchy's theorem implies that

$$C_n\{y(x\theta')^k\} = \frac{1}{2\pi i}\int y(x\theta')^k x^{-n-1}dx,$$

where the integration is along a small circle around $x = 0$. If we let $x = y/\theta(y)$ we obtain the required integrand; the new path of integration is a simple closed curve in the y-plane which, since the integrand is a regular function for $|y| \leqq \tau$, may be taken to be the circle $|y| = \tau$.

LEMMA 4.2. *If $y = \tau e^{is}$ and $A = \tau\rho\theta''(\tau)$, then*

$$(y\theta')^k\theta^{n-1-k}(\theta - y\theta')y^{-n} = \rho^{-n}e^{-\frac{1}{2}Ans^2+Akis} \cdot (-Ais + O(s^2 + k|s|^3 + ns^4))$$

provided that $ks^2 \to 0$ and $ns^3 \to 0$ as $k, n \to \infty$.

Proof. The functions $\theta(y)$ and $\theta'(y)$ are regular in the neighbourhood of $y = \tau$. If we apply relations such as $1 + Bs + O(s^2) = e^{Bs+O(s^2)}$ and $1 + Cs^2 + O(s^3) = e^{Cs^2+O(s^3)}$ to their Taylor expansions and recall that $\rho\theta'(\tau) = 1$

we find, after some simplification, that

$$\theta = \rho^{-1}\tau \cdot \exp\left(is - \tfrac{1}{2}As^2 + O(s^3)\right),$$
$$y\theta' = \rho^{-1}\tau \cdot \exp\left((A+1)is + O(s^2)\right), \quad \text{and}$$
$$\theta - y\theta' = \rho^{-1}\tau(-Ais + O(s^2)),$$

as $s \to 0$. These relations imply the required result.

LEMMA 4.3. *Let*

$$I_1 = \frac{1}{2\pi i}\int (y\theta')^k \theta^{n-1-k}(\theta - y\theta')y^{-n}dy$$

where the integration is over the arc $y = \tau e^{is}$ *with* $|s| \leqq \epsilon_n = n^{-1/2} \log n$. *If* $k = O(n^{1/2})$ *as* $k, n \to \infty$, *then*

$$I_1 = \tau\rho^{-n}(A/2\pi)^{1/2}kn^{-3/2}e^{-Ak^2/2n} + O(\rho^{-n}n^{-3/2}).$$

Proof. We use the estimate for the integrand given in Lemma 4.2 and observe that $dy = i\tau(1 + O(s))ds$. This implies that

$$I_1 = (A\tau/2\pi i)\rho^{-n}\int_{-\epsilon_n}^{+\epsilon_n} s\, e^{-Ans^2/2+Akis} \cdot (1 + O(|s| + ks^2 + n|s|^3))\, ds.$$

It is not difficult to see that the contribution of the O-terms to the integral is $O(n^{-3/2})$. To evaluate the main term of the integral we change variables and let $s = tn^{-1/2} + ikn^{-1}$. We find that

$$I_1 = (A\tau/2\pi i)\rho^{-n}e^{-Ak^2/2n}\int e^{-At^2/2}(tn^{-1} + ikn^{-3/2})dt + O(\rho^{-n}n^{-3/2}),$$

where the integration is over the line segment from $-ikn^{-1/2} - \log n$ to $-ikn^{-1/2} + \log n$. If we appeal to Cauchy's theorem and the fact that $k = O(n^{1/2})$, it is not difficult to show that we may replace this path of integration by the interval $[-\log n, \log n]$. When we let $n \to \infty$ we obtain the relation

$$I_1 = (A\tau/2\pi)k\rho^{-n}n^{-3/2}e^{-Ak^2/2n}\int_{-\infty}^{+\infty} e^{-At^2/2}dt + O(\rho^{-n}n^{-3/2}).$$

This implies the required result.

LEMMA 4.4. *Let*

$$I_2 = \frac{1}{2\pi i}\int (y\theta')^k \theta^{n-1-k}(\theta - y\theta')y^{-n}dy$$

where the integration is over the arc $y = \tau e^{is}$ *with* $\epsilon_n = n^{-1/2} \log n \leqq |s| \leqq \pi$. *If* $k = O(n^{1/2})$ *as* $k, n \to \infty$, *then* $I_2 = O(\rho^{-n}n^{-3/2})$.

Proof. Let $d_1, d_2, \cdots$ denote suitable positive constants and recall that

$1 - \cos s \geqq 2s^2/\pi^2$ when $-\pi < s \leqq \pi$. If $y = \tau e^{is}$, then

$$|1 + c_1 y|^2 = (1 + c_1\tau)^2 - 2c_1\tau(1 - \cos s)$$
$$< (1 + c_1\tau)^2 - d_1 s^2 < (1 + c_1\tau - d_2 s^2)^2.$$

Consequently, if $y = \tau e^{is}$ and $|s| \geqq \epsilon_n$, then

$$|\theta(y)| \leqq |1 + c_1 y| + \theta(\tau) - 1 - c_1\tau < \theta(\tau) - d_2\epsilon_n^2 = \tau\rho^{-1}(1 - d_3\epsilon_n^2).$$

Since $|y\theta'(y)| \leqq \tau\rho^{-1}$ when $|y| = \tau$, it follows, therefore, that

$$|I_2| = O(\rho^{-n})(1 - d_3\epsilon_n^2)^{n-1-k}.$$

This implies the required result.

THEOREM 4.3. *If* $y = x\theta(y)$ *and* $k = O(n^{1/2})$ *as* $k, n \to \infty$, *then*

$$\mu(n, k) \sim Ake^{-Ak^2/2n},$$

where $A = \tau\rho\theta''(\tau)$.

Proof. Theorem 4.1 and Lemmas 4.1, 4.3, and 4.4 imply that

$$\mu(n, k)y_n = C_n\{y(x\theta')^k\} = I_1 + I_2$$
$$= \tau(A/2\pi)^{1/2}k\rho^{-n}n^{-3/2}e^{-Ak^2/2n} + O(\rho^{-n}n^{-3/2}).$$

The required result now follows from (3.4) and the definitions of A and b.

It can also be shown, by a similar argument, that

$$\mu_2(n, k) = An(e^{-Ak^2/2n} - e^{-2Ak^2/n}) + O(k)$$

when $k = O(n^{1/2})$ as $k, n \to \infty$. This implies that

$$\sigma^2(n, k) = An(e^{-Ak^2/2n} - e^{-2Ak^2/n}) - A^2k^2e^{-Ak^2/n} + O(k)$$

when $k = O(n^{1/2})$ as $k, n \to \infty$. In particular, if $k \sim \lambda n^{1/2}$ as $n \to \infty$, then

$$\sigma^2(n, k) \sim An(e^{-A\lambda^2/2} - A\lambda^2 e^{-A\lambda^2} - e^{-2A\lambda^2}).$$

THEOREM 4.4. *If* $y = x\theta(y)$, *then*

$$M(x) = xy'(xy' \cdot y^{-1} - 1) \quad \text{and} \quad V(x) = 2M(x)(xy' \cdot y^{-1} - 1).$$

Proof. Theorems 2.3 and 4.1 imply that

$$M = \sum_{k=1}^{\infty} kF_k = y\sum_{k=1}^{\infty} k(x\theta')^k = yx\theta'(1 - x\theta')^{-2} = xy'(xy' \cdot y^{-1} - 1),$$

since $x\theta' = 1 - y/xy'$. The formula for $V(x)$ can be derived in a similar way.

THEOREM 4.5. *If* $y = x\theta(y)$, *then*

$$\mu(n) \sim c\pi\tau^{-1}n^{1/2} \quad \text{and} \quad \mu_2(n) \sim 4\pi c^2\tau^{-2}n$$

as $n \to \infty$, *where* $c = \frac{1}{2}b(\rho/\pi)^{1/2} = \tau(2\pi A)^{-1/2}$.

Proof. It was shown in [**9**] that $C_n\{xy' \cdot y^{-1}\} \sim \tau^{-1}ny_n$. Hence,

$$\mu(n)ny_n = C_n\{xy' \cdot y^{-1} - 1)\} \sim \tau^{-1}\pi c^2\rho^{-n} \sim \tau^{-1}c\pi n^{3/2}y_n$$

as $n \to \infty$, by Lemma 3.1(ii). This implies the required relation for $\mu(n)$. The proof of the corresponding result for $\mu_2(n)$ is similar except that it requires the use of Lemma 3.1(i).

If $\sigma^2(n)$ denotes the variance of the altitude $\alpha = \alpha(u, T_n)$ over all nodes u of all trees T_n in $\mathscr{F}$, then it follows from Theorem 4.5 that

$$\sigma^2(n) \sim \pi c^2\tau^{-2}(4 - \pi)n, \quad \text{as } n \to \infty.$$

The limiting distribution of the altitude α can be readily deduced from Theorem 4.3.

THEOREM 4.6. *For each fixed constant t,*

$$\Pr\{\alpha \leqq tn^{1/2}\} = 1 - e^{-t^2/2} + O(n^{-1/2}) \quad \textit{as } n \to \infty.$$

We remark that the constant A that appears in Theorem 4.3 may assume any positive value for suitable familes $\mathscr{F}$. For example, when

$$y = x(1 + \beta y + \tfrac{1}{4}\gamma^2y^2),$$

where β and γ are positive constants, we find that $A = \gamma(\beta + \gamma)^{-1}$; this takes on all values in the interval $(0, 1)$ as β varies throughout the interval $(0, \infty)$. When $y = x(1 - \beta y)^{-\gamma}$ we find that $A = 1 + \gamma^{-1}$ and this takes on all values in the interval $(1, \infty)$ as γ varies throughout the interval $(1, \infty)$. When $y = xe^y$ we find that $A = 1$.

The graphs of the functions

$$u = u(v) = \pm\tfrac{1}{2}\mu(n, vn^{1/2})/n^{1/2} = \pm\tfrac{1}{2}Ave^{-Av^2/2}$$

are give in Figure 1 for a few values of A. These give some indication of how the (two-dimensional) profile of a random tree depends on A.

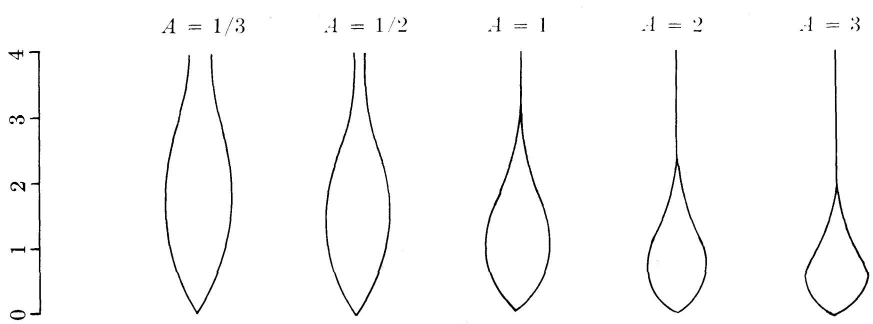

FIGURE 1. The width of trees.

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5. Explicit formulae for some families of simply generated trees. We now apply Theorems 4.1 and 4.4 to obtain explicit formulae for $\mu(n, k)$, $\mu_2(n, k)$, $\mu(n)$, and $\mu_2(n)$ for three particular families of simply generated trees, namely, labelled trees, plane trees, and binary trees. We leave it as an exercise for the reader to verify that the asymptotic behaviour of these variables agrees with the results of Theorems 4.2, 4.3, and 4.5, where applicable.

Let $\mathscr{F}$ denote the set of rooted trees with labelled nodes; if the tree T_n is in $\mathscr{F}$ then its nodes are labelled $1, 2, \cdots, n$ and two such trees are considered the same if and only if their roots have the same label and nodes i and j are joined in one tree if and only if they are joined in the other for $1 \leqq i < j \leqq n$. The non-isomorphic rooted trees T_4 and the number of ways of labelling their nodes are illustrated in Figure 2.

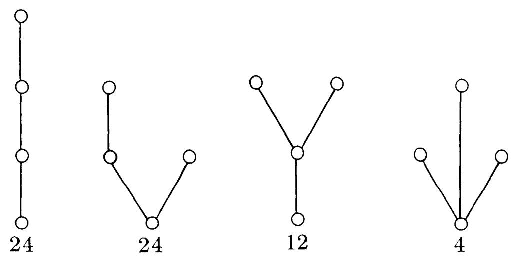

FIGURE 2. Labelled trees.

Let y denote the generating function for the family $\mathscr{F}$ of rooted labelled trees; it is well-known (see, e.g. [**14**] or [**10**]) that

$$(5.1) \quad y = xe^y = \sum_{n=1}^{\infty} n^{n-1} \frac{x^n}{n!}.$$

Hence,

$$F_k(x) = y(x\theta')^k = y^{k+1} = (k+1) \sum_{n=k+1}^{\infty} (n)_{k+1} n^{n-k-2} \frac{x^n}{n!},$$

by Theorem 4.1, equation (5.1), and Lagrange's inversion formula. Therefore,

$$(5.2) \quad \mu(n, k) = (k+1)(n)_{k+1}/n^{k+1}$$

and, similarly, we find that

$$\mu_2(n, k) = \sum_{j=k+2}^{2k+1} j(n)_j/n^j.$$

Theorem 4.4 and equation (5.1) imply that

$$M(x) = y^2(1-y)^{-2} = \sum_{j=1}^{\infty} jy^{j+1}$$

from which it follows that

$$(5.3)\quad \mu(n) = n^{-1} \sum_{j=2}^{n} j(j-1)(n)_j/n^j.$$

Similarly, we find that

$$(5.4)\quad \mu_2(n) = n^{-1} \sum_{j=3}^{n} j(j-1)(j-2)(n)_j/n^j.$$

We leave it as an exercise for the reader to show that these formulae can be rewritten as

$$\mu(n) = \sum_{j=2}^{n} (n)_j/n^j$$

and

$$\mu_2(n) = 2(n+1) - 4\sum_{j=1}^{n} (n)_j/n^j.$$

Meir and Moon [**7**] derived results equivalent to (5.2), (5.3), and (5.4) by a somewhat different argument. We also remark that Stepanov [**15**] has determined the asymptotic behaviour of the higher moments of the variable $W_k(T_n) \cdot n^{-1/2}$ when $k \sim cn^{1/2}$ for some positive constant c for rooted labelled trees.

Let $\mathscr{F}$ denote the set of rooted plane trees. The nodes of such trees are not labelled, although the root is distinguishable from the remaining nodes, and two such trees are considered the same if and only if they have the same ordered set of branches with respect to the root. The different rooted plane trees with four nodes are illustrated in Figure 3.

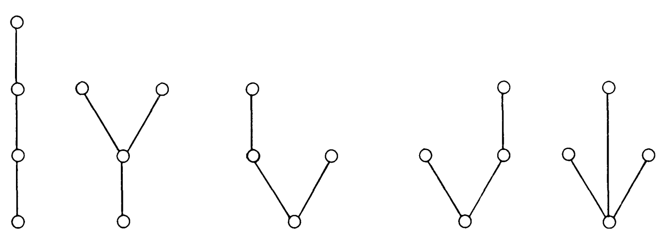

FIGURE 3. Plane trees.

Let y denote the generating function for the family $\mathscr{F}$ of rooted plane trees. Then (see, e.g., [**14**, p. 197] or [**4**, p. 67])

$$(5.5)\quad y = x(1 + y + y^2 + \cdots) = x(1-y)^{-1} = \tfrac{1}{2}(1 - (1-4x)^{1/2}) = \sum_{n=1}^{\infty} \binom{2n-2}{n-1} \frac{x^n}{n}.$$

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Hence,

$$F_k(x) = y(x\theta')^k = yx^k(1-y)^{-2k}$$
$$= x^{-k}y^{2k+1} = (2k+1)\sum_{n=k+1}^{\infty}\binom{2n-1}{n+k-1}\frac{x^n}{n+k},$$

by Theorem 4.1, equation (5.5), and Lagrange's inversion formula. Therefore,

$$\mu(n, k) = (2k+1)(n-1)_k/(n+k)_k$$

and, similarly, we find that

$$\mu_2(n, k) = 2\sum_{j=k+1}^{2k}(2j+1)(n-1)_j/(n+j)_j.$$

Theorem 4.4 and equation (5.5) imply that

$$M(x) = xy(1-2y)^{-2} = xy(1-4x)^{-4}$$
$$= \tfrac{1}{2}x\{(1-4x)^{-1} - (1-4x)^{-1/2}\}$$

from which it follows that

$$\mu(n) = \tfrac{1}{2}\left\{4^{n-1}\Big/\binom{2n-2}{n-1} - 1\right\}. \tag{5.6}$$

Similarly, we find that

$$\mu_2(n) = n - 4^{n-1}\Big/\binom{2n-2}{n-1}.$$

Formula (5.6) was given earlier by Vološin [**16**].

Let $\mathscr{F}$ denote the family of rooted binary trees. This family is the subset of rooted plane trees consisting of those trees in which the out-degree of every node is either two or zero. The different rooted binary trees with seven nodes are illustrated in Figure 4.

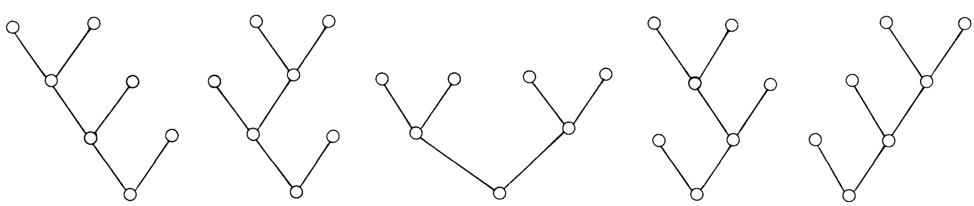

FIGURE 4. Binary trees.

Let y denote the generating function of the family $\mathscr{F}$ of rooted binary trees. Then (see, e.g., [**2**])

$$y = x(1+y^2) = \frac{1}{2x}(1-(1-4x^2)^{1/2}) = \sum_{m=0}^{\infty}\binom{2m}{m}\frac{x^{2m+1}}{m+1}. \tag{5.7}$$

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Hence,

$$F_k(x) = y(x\theta')^k = y(2xy)^k$$
$$= (k+1)2^k \sum_{m=k}^{\infty} \binom{2m+1-k}{m+1} \frac{x^{2m+1}}{2m+1-k},$$

by Theorem 4.1, equation (5.7), and Lagrange's inversion formula. Therefore,

$$\mu(2m+1, k) = (k+1)2^k(m)_k/(2m)_k$$

and, similarly, we find that

$$\mu_2(2m+1, k) = \sum_{j=k}^{2k-1} (j+1)2^j(m)_j/(2m)_j.$$

Theorem 4.4 and equation (5.7) imply that

$$M(x) = 2xy^2(1-2xy)^{-2} = 2(y-x)(1-4x^2)^{-1}$$
$$= x^{-1}\{(1-2x^2)(1-4x^2)^{-1} - (1-4x^2)^{-1/2}\}$$

from which it follows that

$$(5.8) \quad \mu(2m+1) = 2^{2m+1} \Big/ \binom{2m+1}{m} - 2.$$

Similarly, we find that

$$\mu_2(2m+1) = 4(m+3) - 6 \cdot 2^{2m+1} \Big/ \binom{2m+1}{m}.$$

It can be shown that (5.8) is equivalent to a result derived by Knuth [**6**, p. 590].

6. Rooted unlabelled trees. Let $\mathscr{F}$ denote the family of (non-isomorphic) rooted unlabelled trees. The rooted unlabelled trees with four nodes are illustrated in Figure 5. Let y denote the generating function for the family $\mathscr{F}$ of

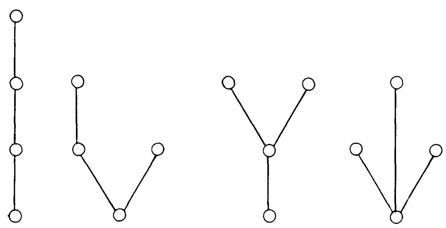

FIGURE 5. Rooted unlabelled trees.

rooted unlabelled trees. Then (see, e.g., [**14**, p. 149] or [**4**, p. 52])

$$(6.1) \quad y = x \exp \sum_{j=1}^{\infty} y(x^j)/j.$$

Thus the rooted unlabelled trees are not a simply generated family, but their generating function does satisfy relation (2.1) with

$$\Gamma\{y(x)\} = x \exp \sum_{j=1}^{\infty} y(x^j)/j.$$

Otter [**12**] (see also [**4**, pp. 209–213]) showed that $y(x)$ has an expansion of the type

(6.2) $\quad y(x) = 1 - b(\rho - x)^{1/2} - b_2(\rho - x) \cdots$

in the neighbourhood of $\rho = 0.3383 \cdots$, where $b = 2.6811 \cdots$. From this he deduced that

(6.3) $\quad y_n \sim c\rho^{-n}n^{-3/2}$

as $n \to \infty$, where $c = \frac{1}{2}b(\rho/\pi)^{1/2} = 0.4399 \cdots$.

We remark, for later use, that it follows from (6.1) that

(6.4) $\quad xy'(x)(1 - y(x)) = y(x) \cdot Q(x)$

where

(6.5) $\quad Q(x) = 1 + \sum_{k=2}^{\infty} y'(x^k)x^k.$

Relations (6.2) and (6.4) imply (see [**4**, p. 213]) that

(6.6) $\quad Q(\rho) = \frac{1}{2}b^2\rho = 1.21600 \cdots$.

THEOREM 6.1. *If $\mathscr{F}$ denotes the family of rooted unlabelled trees, then $F_0(x) = y$ and*

$$F_{k+1}(x) = y \sum_{j=1}^{\infty} F_k(x^j) \quad \textit{for } k = 0, 1, \cdots.$$

Proof. The result is obvious when $k = 0$. Theorem 2.1 and relation (6.1) imply that

$$\frac{\partial}{\partial z} P_{k+1} = \frac{\partial}{\partial z}\left\{x \exp \sum_{j=1}^{\infty} P_k(x^j, z^j)/j\right\} = P_{k+1} \cdot \sum_{j=1}^{\infty} \frac{\partial}{\partial z}\{P_k(x^j, z^j)\}z^{j-1}.$$

The required result now follows from Theorem 2.2 upon letting $z = 1$.

COROLLARY 6.1. $\mu(n, 1) \to 2 + y(\rho^2) + y(\rho^3) + \cdots$ *as* $n \to \infty$.

Proof. Theorem 6.1 implies

$$F_1(x) = \sum_{n=1}^{\infty} \mu(n, 1)y_n x^n = y(x)\{y(x) + y(x^2) + \cdots\}.$$

Now, $y(\rho) = 1$ and the functions $y(x^2), y(x^3), \cdots$ are regular when $|x| < \rho + \epsilon$ for some positive ϵ, by (6.2). The required result now follows from Lemma (3.1), parts (iii) and (iv).

Notice that $\mu(n, 1)$ is the expected degree of the root of a rooted unlabelled tree with n nodes and that this equals the expected number of trees in a random forest of rooted unlabelled trees with $n - 1$ nodes altogether. Palmer and Schwenk [**13**] have shown how to enumerate the forests with a given number of nodes consisting of a given number of trees from certain families. In particular, they obtained, in effect, the result in Corollary 6.1 for forests of rooted unlabelled trees. The relation in Corollary 6.1 can be rewritten as

$$\mu(n, 1) \rightarrow 2 + \sum_{j=1}^{\infty} y_j \rho^{2j}/(1 - \rho^j)$$

and they obtained the estimate $2.1918 \cdots$ for the limiting value on the right hand side. More generally, it can be shown that

$$\mu(n, k + 1) \rightarrow \mu(n, k) + F_k(\rho) + F_k(\rho^2) + \cdots$$

for $k \geqq 0$.

Relations for $S_{k+1}(x)$ can also be derived from Theorem 2.2 and relation (6.1). In particular, if $\sigma^2(n, 1)$ denotes the variance of the number of nodes of altitude one in a random rooted unlabelled tree T_n, then it can be shown that

$$\sigma^2(n, 1) \rightarrow \sum_{j=1}^{\infty} jy(\rho^j) = 1.4741 \cdots$$

as $n \rightarrow \infty$.

THEOREM 6.2. *If $\mathscr{F}$ denotes the family of rooted unlabelled trees, then*

$$M(x) = y \sum_{j=1}^{\infty} M(x^j) + xy' - y. \tag{6.7}$$

Proof. It follows from Theorem 2.3 and 6.1 that

$$\begin{aligned} M(x) &= \sum_{k=0}^{\infty} (k + 1)F_{k+1}(x) = y \sum_{k=0}^{\infty} k \sum_{j=1}^{\infty} F_k(x^j) + \sum_{k=1}^{\infty} F_k(x) \\ &= y \sum_{j=1}^{\infty} M(x^j) + xy' - y. \end{aligned}$$

COROLLARY 6.2. $\mu(n) \sim (1/2c)n^{1/2} = (1.1365 \cdots)n^{1/2}$ *as* $n \rightarrow \infty$.

Proof. Formula (6.7) can be rewritten as

$$M(x) = xy'\left\{xy' + \sum_{j=2}^{\infty} M(x^j) + Q(x) - 1\right\} \Big/ Q(x),$$

in view of (6.4) and (6.5). The functions $\sum_{j=2}^{\infty} M(x^j)$ and $(Q(x))^{-1}$ are regular when $|x| \leqq \rho + \epsilon$ for some positive ϵ. It follows, therefore, that

$$\begin{aligned} n\mu(n)y_n = C_n\{M(x)\} &= C_n\{(xy')^2/Q(x)\} + O(ny_n) \\ &= \tfrac{1}{4}b^2\rho^{1-n}/Q(\rho) + O(ny_n) = \tfrac{1}{2}\rho^{-n} + O(ny_n) \end{aligned}$$

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from (6.3), parts (iv) and (iii) of Lemma 3.1, and (6.6). This and (6.3) imply the required result.

If $\sigma^2(n)$ denotes the variance of the altitude $\alpha = \alpha(u, T_n)$ over all nodes u of all rooted unlabelled trees T_n, then it can be shown that

$$\sigma^2(n) \sim \frac{4-\pi}{2Q(\rho)} n = (.3529 \cdots)n \quad \text{as } n \to \infty.$$

7. Recursive trees. A tree T_n with n labelled nodes, rooted at node 1, is a *recursive* tree if $n = 1$ or if T_n can be constructed by successively joining the j-th node to one of the first $j - 1$ nodes for $2 \leqq j \leqq n$. The branches of T_n with respect to the root node 1 are themselves recursive trees, or rather they would be if the nodes in each branch were relabelled accoıding to the size of the original labels. The recursive trees with four nodes are illustrated in Figure 6. Let y denote the generating function of the family $\mathscr{F}$ of recursive trees.

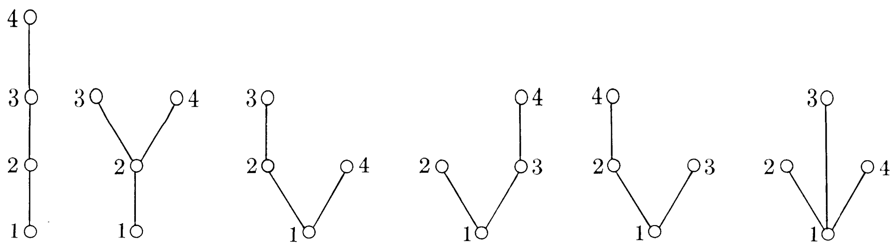

FIGURE 6. Recursive trees.

Then (see, e.g., [**8**])

(7.1) $y' = e^y$

so

$$y = -\ln(1-x) = \sum_{n=1}^{\infty} (n-1)! \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Thus the recursive trees are not a simply generated family, but their generating function does satisfy relation (2.1) with

(7.2) $\Gamma\{y(x)\} = \int_0^x e^{y(t)} dt.$

THEOREM 7.1. *If $\mathscr{F}$ denotes the family of recursive trees, then*

$$F_k(x) = \frac{y^{k+1}}{(k+1)!} \quad \textit{for } k = 0, 1, \cdots.$$

Proof. Theorem 2.1 and relation (7.3) imply that

$$\frac{\partial}{\partial z} P_{k+1} = \frac{\partial}{\partial z} \int_0^x e^{P_k(t,z)} dt = \int_0^x e^{P_k(t,z)} \frac{\partial}{\partial z} (P_k(t, z))dt.$$

Hence,

$$F_{k+1}(x) = \int_0^x e^{y(t)} F_k(t)dt = \int_0^x y'(t)F_k(t)dt,$$

by Theorem 2.2 and relation (7.1). The required result now follows by induction on k since $F_0(x) = y$.

If $1 \leqq k \leqq n$, let

$$f(n, k) = \sum \frac{1}{a_1 a_2 \cdots a_k}$$

where the sum is over all positive integers a_i such that $1 \leqq a_i \leqq n - 1$ and $a_i \neq a_j$ for $1 \leqq i, j \leqq k$; we adopt the convention that $f(n, 0) = 1$ and $f(n, k) = 0$ if $n \leqq k$.

COROLLARY 7.1. $\mu(n, k) = f(n, k)/k!$ *for* $0 \leqq k \leqq n - 1$.

Proof. The numbers $f(n, k)$ clearly satisfy the relation

$$f(n + 1, k + 1) = f(n, k + 1) + (k + 1)f(n, k)/n.$$

The numbers $\mu(n, k)$ satisfy the relation

$$\mu(n + 1, k + 1) = \mu(n, k + 1) + \mu(n, k)/n;$$

this follows from the fact that $F_{k+1}' = y' \cdot F_k$, or $(1 - x)F_{k+1}' = F_k$. The required result now follows readily by induction. (The result can also be established by a purely combinatorial argument; see [**11**].)

COROLLARY 7.2. $\mu(n, 1) = \log n + \gamma + O(n^{-1})$ *and*

$$\mu(n, k) = \frac{1}{k!}\{(\log n)^k + k\gamma(\log n)^{k-1} + O(k^2(\log n)^{k-2})\}$$

if $k \geqq 2$ *and* $k = o(\log n)$ *as* $n \to \infty$, *where* $\gamma = 0.5772\cdots$ *denotes Euler's constant.*

Proof. We may suppose that $k \geqq 2$. Let $g(n, k)$ be defined as $f(n, k)$ was except that now the restriction $a_i \neq a_j$ is dropped. Then

$$g(n, k) = \left(\sum_{j=1}^{n-1} j^{-1}\right)^k = (\log n + \gamma + O(n^{-1}))^k. \tag{7.3}$$

The terms in $g(n, k)$ consist of those in $f(n, k)$ plus those in which at least two factors are equal. Hence,

$$0 \leqq g(n, k) - f(n, k) \leqq \binom{k}{2} g(n, k - 2) \sum_{j=1}^{n-1} j^{-2} \tag{7.4}$$

$$< k^2(\log n + 1)^{k-2} = O(k^2(\log n)^{k-2}).$$

The required result now follows from (7.3), (7.4), and Corollary 7.2.

Similarly, it can be shown that

$$S_k(x) = \sum_{j=1}^{k} \binom{2j}{j} \frac{y^{j+k+1}}{(j+k+1)!}.$$

If $\sigma^2(n, k)$ denotes the variance of the number of nodes of altitude k in a random recursive tree T_n, then it follows from Theorem 7.1 and Corollary 7.2 that

$$\sigma^2(n, 1) = \sum_{j=1}^{n-1} j^{-1} - \sum_{j=1}^{n-1} j^{-2} = \log n + \left(\gamma - \frac{\pi^2}{6}\right) + O(n^{-1})$$

and

$$\sigma^2(n, k) = \{(k-1)!\}^{-2} \cdot \{(\log n)^{2k-1}/(2k-1) + O((\log n)^{2k-2})\}$$

if $k \geqq 2$ and $k = o(\log n)$ as $n \to \infty$.

THEOREM 7.2. *If $\mu(n)$ denotes the expected value of $\alpha(u, T_n)$ over all nodes u of all recursive trees T_n, then*

$$\mu(n) = \sum_{j=2}^{n} j^{-1} = \log n + (\gamma - 1) + O(n^{-1}) \quad \textit{for } n \geqq 2.$$

Proof. It follows from Theorems 2.3 and 7.1 and relation (7.1) that

$$M(x) = \sum_{n=1}^{\infty} \mu(n)x^n = \sum_{k=1}^{\infty} kF_k(x) = \sum_{k=1}^{\infty} ky^{k+1}/(k+1)!$$

$$= ye^y - (e^y - 1) = (1-x)^{-1} \cdot (y - x) = (1-x)^{-1} \cdot \sum_{n=2}^{\infty} x^n/n.$$

This implies the required result.

Similarly, it can be shown that

$$V(x) = y^2e^y - 2M(x) = (1-x)^{-1} \cdot y^2 - 2M(x).$$

If $\sigma^2(n)$ denotes the variance of the altitude $\alpha = \alpha(u, T_n)$ over all nodes u of all recursive trees T_n, then it follows from Theorems 7.1 and 7.2 and Corollary 7.1 that

$$\sigma^2(n) = f(n+1, 2) - \mu(n) - \mu^2(n) = \sum_{j=1}^{n} j^{-1} - \sum_{j=1}^{n} j^{-2}$$

$$= \log n + \left(\gamma - \frac{\pi^2}{6}\right) + O(n^{-1}).$$

This and Chebyshev's inequality imply that

$$\Pr\{|\alpha - \log n| < (\log n)^{1/2+\epsilon}\} \to 1$$

for any positive ϵ as $n \to \infty$.

We remark in closing that results equivalent to Theorems 7.1 and 7.2 were obtained in [**11**] by a different approach.

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