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INTEGRATION MAPS AND LOCAL EQUICONTINUITY OF SPECTRAL MEASURES

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Abstract

One of the useful features of spectral measures which happen to be equicontinuous is that their associated integration maps are bicontinuous isomorphisms of the corresponding L^1-space onto their ranges. It is shown here that equicontinuity is not necessary for this to be the case; a somewhat weaker property suffices. This is of some interest in practice since there are many natural examples of spectral measures which fail to be equicontinuous.

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Since its conception the notion of a normal operator T in a Hilbert space has been intimately connected with its resolution of the identity. This is a (unique) spectral measure P defined on the Borel subsets of $\mathbb{C}$, with support the spectrum of T, such that T is synthesized from the projections in the range of P via a suitable integral. In the 1950's N. Dunford initiated the study of scalar-type spectral operators in a general Banach space. These are the analogues of normal operators in Hilbert space. As in the Hilbert space setting, the fundamental concept is again that of a spectral measure from which the given operator is synthesized; see [5], for example.

In the 1960's the theory of scalar-type spectral operators was extended to the setting of locally convex Hausdorff spaces (briefly, lcHs). But, new phenomena soon emerged in the non-normable setting which are simply not present in Banach spaces. Something as basic as the uniform boundedness of the range of a spectral measure, interpreted as *equicontinuity* in a lcHs, fails to hold in general, [9]. Since the uniform boundedness of a spectral measure in Banach spaces played such a crucial role in many of the arguments, most of the initial developments in the lc-setting dealt almost exclusively

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with spectral measures which were assumed to be equicontinuous, [1–4, 6, 18–21]. As successful as the theory was for such spectral measures, it was also realized that it excluded many natural examples. Further investigations [12–15] revealed that in many arguments it is not so much the equicontinuity of the spectral measure P which is relevant, but rather that its associated *integration map* $I_P : f \mapsto \int_\Omega f \, dP$, defined on the space $\mathscr{L}^1(P)$ of all P-integrable functions, should be a bicontinuous isomorphism of $\mathscr{L}^1(P)$ onto its range. Of course, equicontinuous spectral measures always have this property. Associated with P is also the family of X-valued vector measures $\{Px : x \in X\}$, where Px is specified via evaluation of the projections $P(E)$ at x and X denotes the underlying lcHs. These vector measures in turn induce the associated family of X-valued integration maps (one for each $x \in X$) given by $I_{Px} : g \mapsto \int_\Omega g \, d(Px)$, defined on the space $\mathscr{L}^1(Px)$ of all Px-integrable functions. The importance of this family of maps $\{I_{Px} : x \in X\}$ lies with the fact that, under certain mild assumptions on the underlying lcHs X, their ranges can be identified with the important class of *cyclic subspaces* of X generated by P. It turns out that many features of the theory take on a simpler and more transparent form when these integration maps $\{I_{Px} : x \in X\}$ are bicontinuous isomorphisms onto their ranges in X, a property which again is automatic if P is equicontinuous.

The aim of this note is to investigate the connection between equicontinuity of P, the property of each integration map I_{Px}, for $x \in X$, being a bicontinuous isomorphism, and the property of the global integration map I_P being a bicontinuous isomorphism. It is shown that I_{Px} (for a given $x \in X$) is a bicontinuous isomorphism precisely when the restriction of P to the cyclic space generated by x is equicontinuous; we simply say that P is *locally equicontinuous at* x in this case. Moreover, it is shown that if P is locally equicontinuous at *every* $x \in X$, then also the global integration map I_P is necessarily a bicontinuous isomorphism. An example is given which shows that the converse of this statement is false in general. Examples of spectral measures P for which I_P fails to be a bicontinuous isomorphism have been known for some time, [15]. Of course, such a P cannot be equicontinuous. We also exhibit a *non*-equicontinuous spectral measure P which is both locally equicontinuous and for which I_P is a bicontinuous isomorphism.

1. Preliminaries

Let Y be a lcHs and Y' be its dual space, that is, the space of all $\mathbb{C}$-valued continuous linear functionals on Y. Given $y \in Y$ and $y' \in Y'$ we write $y'(y) = \langle y, y' \rangle$. Let $\mathscr{P}(Y)$ denote the family of all continuous seminorms on Y. The linear span of a subset W in Y is denoted by $\mathrm{sp}(W)$ and the closed linear span of W in Y by $\overline{\mathrm{sp}}(W)$. We denote the range of a linear operator T by $\mathscr{R}(T)$.

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Let Σ be a σ-algebra of subsets of a non-empty set Ω. The characteristic function of each set $E \in \Sigma$ is denoted by χ_E. By $\mathscr{L}^0(\Sigma)$ we denote the space of all $\mathbb{C}$-valued, Σ-measurable functions on Ω. The linear subspace of $\mathscr{L}^0(\Sigma)$ consisting of all Σ-simple functions on Ω is denoted by $\mathrm{sim}(\Sigma)$. A σ-additive set function $m : \Sigma \to Y$ is called a *vector measure*. The Orlicz-Pettis lemma [7, Theorem I.1.3] ensures that $m : \Sigma \to Y$ is σ-additive if and only if the set function $\langle m, y' \rangle : \Sigma \to \mathbb{C}$ given by $\langle m, y' \rangle(E) = \langle m(E), y' \rangle$ for every $E \in \Sigma$ is σ-additive for each $y \in Y'$.

Let $m : \Sigma \to Y$ be a vector measure. Let $[Y]_m$ denote the *sequential closure* of $\mathrm{sp}(m(\Sigma))$ in Y. It is always assumed that $[Y]_m$ has the relative topology from Y. A function $f \in \mathscr{L}^0(\Sigma)$ is called *m-integrable* if it is $\langle m, y' \rangle$-integrable for each $y' \in Y'$ and if, given any $E \in \Sigma$, there is a unique vector $\int_E f\, dm$ in Y satisfying $\langle \int_E f\, dm, y' \rangle = \int_E f\, d\langle m, y' \rangle$ for every $y' \in Y'$. Clearly every Σ-simple function is m-integrable. The vector space of all m-integrable functions is denoted by $\mathscr{L}^1(m)$. Let $q \in \mathscr{P}(Y)$. Define a seminorm $q(m)$ on $\mathscr{L}^1(m)$ by

$$q(m)(f) = \sup_{E \in \Sigma} q\left(\int_E f\, dm\right), \qquad f \in \mathscr{L}^1(m).$$

Equip $\mathscr{L}^1(m)$ with the locally convex topology $\tau(m)$ defined by the family of seminorms $\{q(m) : q \in \mathscr{P}(Y)\}$. This topology is the same as that defined in [7, Chapter II] and is called the mean convergence topology. The space $\mathrm{sim}(\Sigma)$ is *sequentially* $\tau(m)$-*dense* in $\mathscr{L}^1(m)$. In fact, this has been shown in [8, Theorem 2.2 and Theorem 2.4] with the additional assumption that Y is sequentially complete. But, we do not actually need this assumption; see [10, Proposition 1.2].

A function $f \in \mathscr{L}^1(m)$ is called *m-null* if $\int_E f\, dm = 0$ for every $E \in \Sigma$. We identify $\mathscr{L}^1(m)$ with its quotient space with respect to the closed subspace of all m-null functions. So, we can regard $\mathscr{L}^1(m)$ as a lcHs.

The vector measure m is called *closed* if the subset $\Sigma(m) = \{\chi_E : E \in \Sigma\}$ of $\mathscr{L}^1(m)$ is complete with respect to the topology induced by $\tau(m)$, [7, page 71]. Whenever $[Y]_m$ is sequentially complete, the vector measure m is closed if and only if $\mathscr{L}^1(m)$ is $\tau(m)$-complete [16, Theorem 2].

The *integration map* associated with m is the map $I_m : \mathscr{L}^1(m) \to Y$ defined by

$$I_m(f) = \int_\Omega f\, dm, \qquad f \in \mathscr{L}^1(m).$$

Clearly I_m is linear and continuous. Moreover, $\mathscr{R}(I_m) \subseteq [Y]_m$; see [10, page 347].

Let X be a lcHs. The vector space of all continuous linear operators from X into itself is denoted by $L(X)$. The space $L(X)$ equipped with the strong operator topology (that is, the topology of pointwise convergence on X) is a lcHs and is denoted by $L_s(X)$. The topology of $L_s(X)$ is generated by the family of seminorms

$$q_x : T \mapsto q(Tx), \qquad T \in L(X),$$

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for all $x \in X$ and $q \in \mathcal{P}(X)$.

Let $P : \Sigma \to L_s(X)$ be a *spectral measure*. In other words, P is σ-additive and multiplicative (that is, $P(E \cap F) = P(E)P(F)$ for all $E, F \in \Sigma$), and $P(\Omega)$ equals the identity operator $\mathbf{I}$ on X. The space $\mathcal{L}^1(P)$ is an algebra of functions (under pointwise operations) such that

$$\int_E fg\,dP = P(E)I_P(f)I_P(g) = P(E)I_P(g)I_P(f), \qquad E \in \Sigma,$$

for all $f, g \in \mathcal{L}^1(P)$, [13, Corollary 2.1]. Therefore, the integration map $I_P : \mathcal{L}^1(P) \to L_s(X)$ is a continuous, algebra homomorphism onto its range $\mathcal{R}(I_P)$, which is contained in $[L_s(X)]_P$. Moreover, I_P is always *injective* because each $f \in I_P^{-1}(\{0\})$ satisfies $\int_E f\,dP = P(E)I_P(f) = 0$ for every $E \in \Sigma$, that is, f is P-null.

Let $x \in X$. Define a *vector measure* $Px : \Sigma \to X$ by $Px(E) = P(E)x$ for every $E \in \Sigma$. The integration map $I_{Px} : \mathcal{L}^1(Px) \to X$ is also always *injective* because the multiplicativity of P implies that

$$(1) \qquad \int_E f\,d(Px) = P(E)I_{Px}(f), \qquad f \in \mathcal{L}^1(Px),\ E \in \Sigma.$$

Furthermore, $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(Px)$ and $\int_E g\,d(Px) = (\int_E g\,dP)x$ for every $g \in \mathcal{L}^1(P)$ and $E \in \Sigma$. The closed subspace $P(\Sigma)[x] = \overline{\mathrm{sp}}(Px(\Sigma))$ of X is called the *cyclic subspace* generated by x; it always has the relative topology from X. Then $\mathcal{R}(I_{Px}) \subseteq [X]_{Px} \subseteq P(\Sigma)[x]$. Moreover, $\mathcal{R}(I_{Px})$ is dense in $P(\Sigma)[x]$ as it contains $I_{Px}(\mathrm{sim}(\Sigma)) = \mathrm{sp}(Px(\Sigma))$.

2. Locally equicontinuous spectral measures

Throughout this section let X be a lcHs and P be an $L_s(X)$-valued spectral measure on a σ-algebra Σ of subsets of a non-empty set Ω.

The spectral measure P is called *equicontinuous* if its range $P(\Sigma)$ is an equicontinuous subset of $L(X)$. If X is quasi-barrelled, in particular, if X is metrizable, then P is necessarily equicontinuous [12, Proposition 2.5]. As noted in the introduction one of the most important features of equicontinuous spectral measures is the following one.

LEMMA 1. *Suppose that the spectral measure P is equicontinuous.*

(i) *For every $x \in X$, the integration map $I_{Px} : \mathcal{L}^1(Px) \to X$ is a bicontinuous isomorphism onto its range.*

(ii) *The integration map $I_P : \mathcal{L}^1(P) \to L_s(X)$ is a bicontinuous algebra isomorphism onto its range.*

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PROOF. (i) The first part of the proof of Proposition 2.1 in [4] is still valid in our setting and establishes (i).

(ii) See [14, Lemma 1.11]. □

In this section we introduce the notion of locally equicontinuous spectral measures; they always satisfy (i) and (ii) of Lemma 1. Equicontinuous spectral measures are always locally equicontinuous, but the converse is not valid in general; a counterexample is given (see Example 8).

Let $x \in X$. Fix $E \in \Sigma$. The subspaces $\mathscr{R}(I_{Px})$ and $P(\Sigma)[x]$ of X are invariant for the operator $P(E)$. The restriction $P(E)|_{\mathscr{R}(I_{Px})}$ of $P(E)$ to $\mathscr{R}(I_{Px})$ belongs to $L(\mathscr{R}(I_{Px}))$. Similarly, the restriction $P(E)|_{P(\Sigma)[x]}$ of $P(E)$ to $P(\Sigma)[x]$ is an operator belonging to $L(P(\Sigma)[x])$. Clearly the set functions $P_{\mathscr{R}(I_{Px})} : E \mapsto P(E)|_{\mathscr{R}(I_{Px})}$ and $P_{P(\Sigma)[x]} : E \mapsto P(E)|_{P(\Sigma)[x]}$ are spectral measures with values in $L_s(\mathscr{R}(I_{Px}))$ and $L_s(P(\Sigma)[x])$ respectively.

The following result characterizes (for a fixed $x \in X$) the property (i) of Lemma 1 in terms of equicontinuity of the restriction of P to certain invariant subspaces.

PROPOSITION 2. *Let P be a spectral measure and $x \in X$. The following statements are equivalent*:

(i) *The spectral measure $P_{P(\Sigma)[x]} : \Sigma \to L_s(P(\Sigma)[x])$ is equicontinuous.*

(ii) *The spectral measure $P_{\mathscr{R}(I_{Px})} : \Sigma \to L_s(\mathscr{R}(I_{Px}))$ is equicontinuous.*

(iii) *The integration map $I_{Px} : \mathscr{L}^1(Px) \to X$ is a bicontinuous isomorphism onto its range.*

(iv) *For each seminorm $q \in \mathscr{P}(X)$ there is a seminorm $r \in \mathscr{P}(X)$ such that*

$$q_x(P)(f) \le r_x(I_P f), \qquad f \in \mathscr{L}^1(P). \tag{2}$$

PROOF. (i) implies (iii). Given $q \in \mathscr{P}(X)$ there is $r \in \mathscr{P}(X)$ such that $q(P(E)y) \le r(y)$ for each $E \in \Sigma$ and $y \in P(\Sigma)[x]$. We have used here the fact that every continuous seminorm on a subspace of X is the restriction of some element (not necessarily unique) from $\mathscr{P}(X)$, whenever this subspace has the relative topology. From (1) and the inclusion $\mathscr{R}(I_{Px}) \subseteq P(\Sigma)[x]$, we have

$$\sup_{E\in\Sigma} q\left(\int_E f\, d(Px)\right) = \sup_{E\in\Sigma} q(P(E)I_{Px}f) \le r(I_{Px}f)$$

for every $f \in \mathscr{L}^1(Px)$. This implies that the continuous linear injection I_{Px} has a continuous inverse on its range $\mathscr{R}(I_{Px})$.

(ii) implies (iii). This was established in the proof that (i) implies (iii).

(iii) implies (i). Given $E \in \Sigma$ define a linear operator $S_E : \mathscr{L}^1(Px) \to \mathscr{L}^1(Px)$ by $S_E(f) = \chi_E f$ for every $f \in \mathscr{L}^1(Px)$. From the definition of the topology

$\tau(Px)$ on $\mathscr{L}^1(Px)$ it follows that $\{S_E : E \in \Sigma\}$ is an equicontinuous subset of $L(\mathscr{L}^1(Px))$. Again from (1) we see that $P(E) = I_{Px} S_E (I_{Px})^{-1}$ on $\mathscr{R}(I_{Px})$, for every $E \in \Sigma$. So, the family $\{P(E) : E \in \Sigma\}$ restricted to $\mathscr{R}(I_{Px})$ is an equicontinuous subset of $L(\mathscr{R}(I_{Px}))$. Since $\mathscr{R}(I_{Px})$ is dense in $P(\Sigma)[x]$ it follows that $P_{P(\Sigma)[x]}(\Sigma)$ is equicontinuous in $L(P(\Sigma)[x])$.

(iii) implies (ii). This was established in the proof that (iii) implies (i).

(iii) implies (iv). Given $q \in \mathscr{P}(X)$ there is $r \in \mathscr{P}(X)$ such that

$$q(Px)(g) \leq r(I_{Px} g), \qquad g \in \mathscr{L}^1(Px). \tag{3}$$

If $f \in \mathscr{L}^1(P) \subseteq \mathscr{L}^1(Px)$, then it is routine to verify (by (1) again) that

$$q(Px)(f) = q_x(P)(f) \quad \text{and} \quad r(I_{Px} f) = r_x(I_P f). \tag{4}$$

Thus (3) and (4) applied to f yield (2).

(iv) implies (iii). Fix $q \in \mathscr{P}(X)$ and let $r \in \mathscr{P}(X)$ be as in (iv). We only need to verify that $(I_{Px})^{-1}$ is continuous on $\mathscr{R}(I_{Px})$. Let $\xi \in \mathscr{R}(I_{Px})$ and take $g \in \mathscr{L}^1(Px)$ such that $\xi = I_{Px} g$. Choose a sequence $\{g_n\}_{n=1}^{\infty} \subseteq \operatorname{sim}(\Sigma)$ such that $g_n \to g$ in $\mathscr{L}^1(Px)$ as $n \to \infty$. Then $I_{Px} g_n \to I_{Px} g$ in X as $n \to \infty$. Since (4) holds for each Σ-simple function f, we have

$$\begin{aligned} q(Px)(g) &= \lim_{n\to\infty} q(Px)(g_n) = \lim_{n\to\infty} q_x(P)(g_n) \\ &\leq \lim_{n\to\infty} r_x(I_P g_n) = \lim_{n\to\infty} r(I_{Px} g_n) = r(I_{Px} g). \end{aligned}$$

That is, $q(Px)((I_{Px})^{-1}\xi) \leq r(\xi)$. Since $\xi \in \mathscr{R}(I_{Px})$ is arbitrary this establishes continuity of $(I_{Px})^{-1}$ on its range. □

The spectral measure P is said to be *locally equicontinuous* if the restricted spectral measure $P_{P(\Sigma)[x]} : \Sigma \to L_s(P(\Sigma)[x])$ is equicontinuous for *every* $x \in X$. In particular, equicontinuous spectral measures are locally equicontinuous. The following result follows immediately from Proposition 2.

THEOREM 3. *The spectral measure $P : \Sigma \to L_s(X)$ is locally equicontinuous if and only if for each $x \in X$ the integration map $I_{Px} : \mathscr{L}^1(Px) \to X$ is a bicontinuous isomorphism onto its range.*

Recall that a vector $x \in X$ is called a *cyclic vector* for P if $X = P(\Sigma)[x]$ or, equivalently, if $X = \overline{\mathscr{R}(I_{Px})}$.

COROLLARY 4. *Suppose that there is a cyclic vector $x \in X$ for the spectral measure P. Then P is equicontinuous if and only if the integration map I_{Px} is a bicontinuous isomorphism onto its range.*

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The following result makes the connection between local equicontinuity of P and continuity properties of the global integration map $I_P : \mathscr{L}^1(P) \to L_s(X)$.

THEOREM 5. *If the spectral measure P is locally equicontinuous, then the integration map $I_P : \mathscr{L}^1(P) \to L_s(X)$ is necessarily a bicontinuous linear and algebra isomorphism onto its range.*

PROOF. Fix $x \in X$ and $q \in \mathscr{P}(X)$, which specify a typical seminorm q_x generating the topology in $L_s(X)$. By Proposition 2(iv) there is $r \in \mathscr{P}(X)$ such that (2) is satisfied. Since r_x is a continuous seminorm on $L_s(X)$ and $\mathscr{R}(I_P)$ has the relative topology from $L_s(X)$, this shows that the continuous injection I_P has a continuous inverse on $\mathscr{R}(I_P)$. □

REMARK. The seminorm r given in the above proof may vary with x. Indeed, it is precisely this possible dependence of r on x (and q, of course) which allows for the possibility of $(I_P)^{-1}$ to be continuous *without* P being equicontinuous; see Example 8.

In view of Proposition 2 there arises the question of whether the inclusion $\mathscr{R}(I_{Px}) \subseteq P(\Sigma)[x]$ can be strict. Certainly if X is metrizable, then $\mathscr{R}(I_{Px})$ is closed in X and hence $R(I_{Px}) = P(\Sigma)[x]$ for every $x \in X$. To see this, let $\{f_n\}_{n=1}^{\infty} \subseteq \mathscr{L}^1(Px)$ be a sequence such that $I_{Px}(f_n) \to z$ as $n \to \infty$, for some $z \in X$. Then for each $E \in \Sigma$ it follows from (1) and the continuity of $P(E)$ that $I_{Px}(\chi_E f_n) \to P(E)z$ as $n \to \infty$. Accordingly, I_{Px} is Σ-*converging* in the sense of [11, page 516] and so Proposition 1.6 and Proposition 2.6 of [11] imply that $\mathscr{R}(I_{Px})$ is closed in the metrizable space X. However, the inclusion $\mathscr{R}(I_{Px}) \subseteq P(\Sigma)[x]$ may be strict in the non-metrizable setting.

EXAMPLE 6. Let $X = \mathbb{C}^{[0,1]}$ be equipped with the product topology. Let Σ be the σ-algebra of all Borel subsets of $\Omega = [0, 1]$ and $P : \Sigma \to L_s(X)$ be the equicontinuous spectral measure defined by $P(E)x = \chi_E x$ for every $E \in \Sigma$ and $x \in X$. For x the constant function one on Ω it is routine to check that $\mathscr{R}(I_{Px}) = \mathscr{L}^0(\Sigma) \subsetneqq X = P(\Sigma)[x]$.

Since $\Sigma(P) = \{\chi_E : E \in \Sigma\}$ is always a closed set in $\mathscr{L}^1(P)$ it follows that if I_P is a bicontinuous isomorphism onto its range, then $P(\Sigma)$ must be a closed subset in $\mathscr{R}(I_P) \subseteq L_s(X)$. In particular, if there exists $f \in \mathscr{L}^1(P)\backslash\Sigma(P)$ and a net $\{E_\alpha\} \subseteq \Sigma$ such that $P(E_\alpha) \to \int_\Omega f\, dP$ for the strong operator topology, then I_P *cannot* be an isomorphism onto its range. For instance, let $X_1 = L^2([0, 1])$ be equipped with its weak topology, in which case X_1 is a quasicomplete lcHs. Let $\Omega = [0, 1]$ and Σ be the σ-algebra of all Borel subsets of Ω. Then $P_1 : \Sigma \to L_s(X_1)$ defined by $P_1(E) : f \mapsto \chi_E f$, for each $f \in X_1$ and $E \in \Sigma$, is a (closed) spectral measure. It is shown in [17, pages 369–370] that there exists a sequence $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ such that $P_1(E_n) \to (1/2)\mathbf{I}$ in $\mathscr{R}(I_{P_1})$. Accordingly, $(I_{P_1})^{-1}$ is not continuous.

The following example shows that the converse of Theorem 5 is not valid in general, that is, I_P can be a bicontinuous isomorphism onto its range *without* being locally equicontinuous.

EXAMPLE 7. Let $P_1 : \Sigma \to L_s(X_1)$ be as in the previous paragraph. Let $X_2 = L^2([0,1])$ be equipped with its usual norm $u : f \mapsto (\int_0^1 |f(t)|^2\,dt)^{1/2}$. Define an equicontinuous spectral measure $P_2 : \Sigma \to L_s(X_2)$ by $P_2(E) : g \mapsto \chi_E g$, for each $g \in X_2$ and $E \in \Sigma$. Let X denote the direct sum $X_1 \oplus X_2$, equipped with the topology generated by the family of seminorms $\{\rho_\psi : \psi \in L^2([0,1])\}$ where

$$\rho_\psi(f_1 \oplus f_2) = |\langle f_1, \psi\rangle| + u(f_2), \qquad f_1 \oplus f_2 \in X.$$

Then X is a quasicomplete lcHs. Define a (closed) spectral measure $P : \Sigma \to L_s(X)$ by $P(E)(f_1 \oplus f_2) = P_1(E)f_1 \oplus P_2(E)f_2$, for each $E \in \Sigma$ and $f_1 \oplus f_2 \in X$. Then $\mathscr{L}^1(P) = L^\infty([0,1])$ as vector spaces. Indeed, if $\varphi \in L^\infty([0,1])$, then

$$\int_E \varphi\,dP : f_1 \oplus f_2 \mapsto \chi_E\varphi f_1 \oplus \chi_E\varphi f_2 = \left(\int_E \varphi\,dP_1\right) f_1 \oplus \left(\int_E \varphi\,dP_2\right) f_2$$

for every $E \in \Sigma$ and $f_1 \oplus f_2 \in X$, which shows that $L^\infty([0,1]) \subseteq \mathscr{L}^1(P)$. Conversely, if $\varphi \in \mathscr{L}^1(P)$, then $\varphi \in \mathscr{L}^1(Px)$ for each $x \in X$. By considering elements of the form $x = 0 \oplus f_2$, with $f_2 \in X_2$, and noting that $P(E)x = 0 \oplus P_2(E)f_2$ is an element of the closed P-invariant subspace $\{0\} \oplus X_2 \simeq X_2$, for each $E \in \Sigma$, it follows that $\varphi \in \mathscr{L}^1(P_2 f_2)$ for each $f_2 \in X_2$. But, $\mathscr{L}^1(P_2 f_2) = \{g \in \mathscr{L}^0(\Sigma) : g f_2 \in L^2([0,1])\}$ with $\int_E g\,d(P_2 f_2) = \chi_E g f_2$ for each $E \in \Sigma$. Choosing $E = [0,1]$ we see that $\varphi f_2 \in L^2([0,1])$ for all $f_2 \in L^2([0,1])$ which implies that $\varphi \in L^\infty([0,1])$.

The seminorms generating the topology $\tau(P_2)$ in $\mathscr{L}^1(P_2)$ are of the form

$$u_\xi(P_2) : \varphi \mapsto \sup_{E \in \Sigma} u\left(\left(\int_E \varphi\,dP_2\right)\xi\right), \quad \varphi \in L^\infty([0,1]) = \mathscr{L}^1(P_2),$$

for $\xi \in L^2([0,1])$. The seminorms generating $\tau(P_1)$ are of the form

$$q_{\psi,\xi}(P_1) : \varphi \mapsto \sup_{E \in \Sigma} \left|\left\langle\left(\int_E \varphi\,dP_1\right)\xi, \psi\right\rangle\right|, \quad \varphi \in L^\infty([0,1]) = \mathscr{L}^1(P_1),$$

for arbitrary $\psi, \xi \in L^2([0,1])$. By the Cauchy-Schwarz inequality it follows that

$$(5) \qquad q_{\psi,\xi}(P_1)(\varphi) \le u(\psi)\cdot u_\xi(P_2)(\varphi), \quad \varphi \in L^\infty([0,1]),$$

for all $\psi, \xi \in L^2([0,1])$.

Let $\{\varphi_\alpha\} \subseteq L^\infty([0,1])$ be a net such that $I_P(\varphi_\alpha) \to 0$ in $\mathscr{R}(I_P)$. To show $(I_P)^{-1}$ is continuous we need to verify that $\varphi_\alpha \to 0$ in $\mathscr{L}^1(P)$. By considering the elements

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$x = 0 \oplus g \in X$, where $g \in X_2$, it follows from $I_P(\varphi_\alpha) \to 0$ in $\mathscr{R}(I_P)$ that $I_{P_2}(\varphi_\alpha) \to 0$ in $\mathscr{R}(I_{P_2}) \subseteq L_s(X_2)$. A typical seminorm generating $\tau(P)$ is of the form (for $\varphi \in \mathscr{L}^1(P)$)

$$\begin{aligned}(\rho_\psi)_{\xi_1 \oplus \xi_2}(P)(\varphi) &= \sup_{E \in \Sigma} \rho_\psi\left(\left(\int_E \varphi\, dP_1\right)\xi_1 \oplus \left(\int_E \varphi\, dP_2\right)\xi_2\right) \\ &\le \sup_{E \in \Sigma}\left|\left\langle\left(\int_E \varphi\, dP_1\right)\xi_1, \psi\right\rangle\right| + \sup_{E \in \Sigma} u\left(\left(\int_E \varphi\, dP_2\right)\xi_2\right) \\ &= q_{\psi,\xi_1}(P_1)(\varphi) + u_{\xi_2}(P_2)(\varphi),\end{aligned}$$

for some $\psi \in L^2([0,1])$ and $\xi_1 \oplus \xi_2 \in X$. Then (5) implies that

$$(6) \qquad (\rho_\psi)_{\xi_1 \oplus \xi_2}(P)(\varphi) \le u(\psi) \cdot u_{\xi_1}(P_2)(\varphi) + u_{\xi_2}(P_2)(\varphi), \quad \varphi \in \mathscr{L}^1(P).$$

Since P_2 is equicontinuous it follows that $(I_{P_2})^{-1}$ is continuous (see Lemma 1). But, $I_{P_2}(\varphi_\alpha) \to 0$ in $\mathscr{R}(I_{P_2}) \subseteq L_s(X_2)$ and so $\varphi_\alpha \to 0$ in $\mathscr{L}^1(P_2)$. In particular, $u_{\xi_1}(P_2)(\varphi_\alpha) \to 0$ and $u_{\xi_2}(P_2)(\varphi_\alpha) \to 0$ for each $\xi_1, \xi_2 \in L^2([0,1])$, and we see from (6) that $(\rho_\psi)_{\xi_1 \oplus \xi_2}(P)(\varphi_\alpha) \to 0$. This shows that $\varphi_\alpha \to 0$ in $\mathscr{L}^1(P)$ and hence, $(I_P)^{-1}$ is continuous. Accordingly, I_P is a bicontinuous isomorphism of $\mathscr{L}^1(P)$ onto its range.

To see that P is not locally equicontinuous we argue as follows. As noted earlier $(I_{P_1})^{-1}$ is not continuous on $\mathscr{R}(I_{P_1}) \subseteq L_s(X_1)$ and so by Theorem 5 there must exist $f \in X_1$ such that $(I_{P_1 f})^{-1}$ is not continuous from $\mathscr{R}(I_{P_1 f}) \subseteq X_1$ onto $\mathscr{L}^1(P_1 f)$. Then $x = f \oplus 0 \in X$ has the property that $(I_{Px})^{-1}$ is not continuous from $\mathscr{R}(I_{Px}) \subseteq X$ onto $\mathscr{L}^1(Px)$, where we have used the easily verified facts that $\mathscr{L}^1(Px) \simeq \mathscr{L}^1(P_1 f)$ and $\mathscr{R}(I_{Px}) = \mathscr{R}(I_{P_1 f}) \oplus \{0\}$.

It may be of interest to note that $\mathscr{L}^1(P)$ is actually $\tau(P)$-complete. Indeed, from the various definitions and inequalities above we see easily that

$$u_\xi(P_2)(\varphi) = (\rho_\psi)_{0 \oplus \xi}(P)(\varphi), \quad \varphi \in L^\infty([0,1]),$$

for each $\xi \in L^2([0,1])$ and $\psi \in L^2([0,1])$. Since $\mathscr{L}^1(P) = L^\infty([0,1]) = \mathscr{L}^1(P_2)$ as vector spaces, the previous equality and (6) show that $\mathscr{L}^1(P)$ and $\mathscr{L}^1(P_2)$ are isomorphic as lcHs. But, $\mathscr{L}^1(P_2)$ is complete [14, Proposition 3.16], as P_2 is a closed measure [14, Proposition 3.9]) and $[L_s(X_2)]_{P_2}$ is sequentially complete. We know that $[L_s(X_2)]_{P_2}$ is sequentially complete because the space $L_s(X_2)$ is quasicomplete. Consequently, also $\mathscr{L}^1(P)$ is complete. Since I_P is a bicontinuous isomorphism of $\mathscr{L}^1(P)$ onto its range it follows that $\mathscr{R}(I_P)$ is a complete subspace of $L_s(X)$.

We conclude with an example of a spectral measure P which is not equicontinuous, but which is locally equicontinuous. In particular, I_P is then also a bicontinuous isomorphism onto its range (see Theorem 5).

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EXAMPLE 8. Let Ω be an infinite set and let X denote the space $c_{00}(\Omega)$ of all $\mathbb{C}$-valued functions x on Ω such that x vanishes outside a finite subset of Ω. Let Y denote the space $\ell^1(\Omega)$ of all $\mathbb{C}$-valued functions y on Ω satisfying $\Sigma_{\omega\in\Omega}|y(\omega)| < \infty$. Equip X with the weakest topology $\sigma(X, Y)$ making each functional $y \in Y$ continuous on X, where $\langle x, y\rangle = \sum_{\omega\in\Omega} x(\omega)y(\omega)$ for each $x \in X$. Let $\ell^\infty(\Omega)$ denote the space of all bounded $\mathbb{C}$-valued functions on Ω.

Let f be a $\mathbb{C}$-valued function on Ω. Define a linear operator $M_f : X \to X$ by $M_f : x \mapsto xf$ for every $x \in X$. Then $\langle M_f x, y\rangle = \sum_{\omega\in\Omega} x(\omega)(yf)(\omega)$ for every $x \in X$ and $y \in Y$. Hence, M_f is continuous if and only if $yf \in Y$ for every $y \in Y$, that is, if and only if $f \in \ell^\infty(\Omega)$.

Let Σ be a non-trivial σ-algebra of subsets of Ω (that is, Σ contains infinitely many elements). Define a set function $P : \Sigma \to L_s(X)$ by $P(E) = M_{\chi_E}$ for each $E \in \Sigma$. It is routine to verify that P is a spectral measure, that $\mathscr{L}^1(P) = \mathscr{L}^0(\Sigma) \cap \ell^\infty(\Omega)$ and that $\int_E f\, dP = P(E)M_f$ for every $f \in \mathscr{L}^1(P)$ and $E \in \Sigma$.

Fix $x \in X$ and let $x^{-1}(\mathbb{C}\backslash\{0\}) = \{\omega_1, \cdots, \omega_n\}$. A typical seminorm q generating the topology of X is of the form $q(z) = |\langle z, y\rangle|, z \in X$, for some $y \in Y$. Define $r \in \mathscr{P}(X)$ by

$$r(z) = \sum_{k=1}^{n} |\langle z, y(\omega_k)\chi_{\{\omega_k\}}\rangle|, \quad z \in X.$$

Then it is easily verified that

$$q_x(P)(f) = \sup_{E\in\Sigma} \left| \sum_{k=1}^{n} f(\omega_k)x(\omega_k)y(\omega_k)\chi_E(\omega_k) \right| \le r_x(I_P f),$$

for every $f \in \mathscr{L}^1(P)$. Proposition 2 shows that the integration map I_{Px} is a bicontinuous isomorphism onto its range. Since $x \in X$ is arbitrary it follows from Theorem 3 that P is locally equicontinuous. Then Theorem 5 ensures that I_P is a bicontinuous isomorphism of $\mathscr{L}^1(P)$ onto its range.

Finally, to see that P fails to be equicontinuous we refer to Example 1(iv) and Proposition 3 of [9].

It may be in interest to note that $\mathscr{L}^1(P)$ is typically *not* complete for this example. Indeed, suppose that Ω is uncountable, that Σ contains all singleton sets $\{\omega\}$, for $\omega \in \Omega$, but $\Sigma \ne 2^\Omega$. Then there exists an infinite subset $F \subseteq \Omega$ which is not an element of Σ. Let $\mathscr{F}$ denote the family of all finite subsets of F directed by inclusion. Then $\{P(E)\}_{E\in\mathscr{F}} \subseteq \mathscr{R}(I_P)$ is a net which converges to M_{χ_F} in $L_s(X)$. Since $\chi_F \notin \mathscr{L}^1(P)$ we see that $\{P(E)\}_{E\in\mathscr{F}}$ is a Cauchy net in $\mathscr{R}(I_P)$ having no limit in $\mathscr{R}(I_P)$. But, I_P is a bicontinuous isomorphism of $\mathscr{L}^1(P)$ onto its range and so $\mathscr{L}^1(P)$ cannot be complete.

REMARK. Example 7 and Example 8 both provide spectral measures P which are not equicontinuous, but such that I_P is a bicontinuous isomorphism of $\mathscr{L}^1(P)$ onto its range. This answers a question posed in [14, page 13].

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